Lineability in Multifractal Analysis

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Abstract

The multifractal behavior of generic functions belonging to Hölder, Sobolev or Besov spaces has been investigated by many authors, using the concepts of Baire residuality and of prevalence. This paper aims at obtaining the corresponding results in the framework supplied by the notion of lineability. Furthermore, we also study the question of algebrability, proving negative and positive results.

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1 Introduction

It is now common knowledge that a "generic" continuous functions is nowhere differentiable. The set formed by these particular functions has been thoroughly studied since the publication of their archetype, the Weierstraß function

$$W_{a,b}(x) = \sum_{k=0}^{+\infty} a^k \cos(b^k \pi x),$$

with $0 < a < 1$ and b any odd integer such that $ab > 1 + 3\pi/2$. A first result concerning the size of the set of nowhere differentiable functions in the space of continuous function has been obtained in 1931 as a nice application of the Baire

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category theorem: Banach [7] and Mazurkiewicz [37] proved that this set contains a countable union of dense open sets of $C([0,1])$; we say that such a set is residual in $C([0,1])$. In 1994, Hunt [27] extended this result to the generic setting of prevalence, a concept introduced in order to generalize the notion of Lebesgue almost everywhere to infinite dimensional spaces [12, 28]. In the meanwhile, the algebraic structure of this set has also been deeply investigated using the notions of lineability and algebrability, see e.g. [25, 19, 9, 34].

The regularity of nowhere differentiable functions can be studied through their Hölder pointwise regularity. While the regularity of the Weierstraß function is the same at every point [26], there exist functions whose regularity can change widely from a point to another; Multifractal analysis is concerned with the study of such irregular functions. Let us start by recalling some basic definitions: The starting point is the definition of pointwise regularity $C^\alpha(x_0)$.

Definition 1. *Let $x_0 \in \mathbb{R}^d$ and let $\alpha \geq 0$. A locally bounded function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $C^\alpha(x_0)$ if there exist a constant $C > 0$ and a polynomial P_{x_0} of degree less than $[\alpha]$ such that*

$$|f(x) - P_{x_0}(x)| \leq C|x - x_0|^\alpha \tag{1}$$

for every x in a neighborhood of x_0.

The *Hölder exponent* $h_f(x_0)$ of f at x_0 allows to quantify the local smoothness of f at x_0; it is defined as its maximal regularity at x_0, i.e.

$$h_f(x_0) = \sup\left\{\alpha \geq 0 : f \in C^\alpha(x_0)\right\},$$

possibly equal to $+\infty$. Observe that when $h_f(x_0) \leq 1$ (which is the case if f is not differentiable at x_0), the Hölder exponent is simply given by the formula

$$h_f(x_0) = \liminf_{x \to x_0} \frac{\log|f(x) - f(x_0)|}{\log|x - x_0|}.$$

The purpose of multifractal analysis is to determine the fractal dimension of the level sets of the function $x_0 \mapsto h_f(x_0)$. The *Hölder spectrum* d_f of f is defined by the function

$$d_f : h \in [0, +\infty] \mapsto \dim_{\mathcal{H}} \left\{x_0 : h_f(x_0) = h\right\}$$

where $\dim_{\mathcal{H}}$ stands for the Hausdorff dimension. The Hölder spectrum gives a geometrical idea of the diversity and the distribution of the local behaviors of the function under consideration.

While the pointwise reguarity is encapsulated by the Hölder exponent, the global Hölder regularity can be characterized by Hölder spaces: Let us recall that f belongs to $C^\alpha(I)$, with $I \subseteq \mathbb{R}^d$ and $\alpha \geq 0$, if (1) holds for every $x_0 \in I$, the constant C being uniform. Similarly to what was done in the case of nowhere differentiable functions, the multifractal behavior of generic functions belonging to Hölder spaces (but also Sobolev and Besov spaces, see Section 2.3) has been investigated via their Hölder spectrum by many authors, using the concepts of Baire residuality and of prevalence [30, 33, 22, 23, 20]. All these results have been obtained via wavelet decompositions of functions and thanks to a characterization of the regularity based on the wavelet coefficients. This paper aims at obtaining the corresponding results in the framework supplied by the notion of lineability. Introduced in [1], this concept has attracted the attention of many authors, see e.g. the review of Bernal-González, Pellegrino and Seoane-Sepúlveda [10]. Basically, a property is generic in the sense of lineability if this property holds for every non-zero function of a subspace of infinite dimension. More precisely, we have the following definition.

Definition 2. *Let X be a vector space, M a subset of X, and κ a cardinal number. The subset M is said to be* κ-lineable *if $M \cup \{0\}$ contains a vector subspace of dimension κ. The set M is simply* lineable *if the existing subspace is infinite dimensional. When X is a topological vector space and when the above vector space can be chosen to be dense in X, we say that M is* κ-dense-lineable *(or, simply, dense-lineable if κ is infinite).*

The paper is structured as follows. Section 2 is dedicated to some recalls about the definition of the Hausdorff dimension, wavelet basis and the characterization of the regularity using wavelets. In Section 3, we present classical results concerning the Hölder spectrum of a generic function in a given Sobolev or Besov spaces, and we prove that the same holds with the notion of lineability. Finally, in Section 4, we prove that this generic behavior cannot hold on an algebra. However, we get a positive result concerning Hölder spaces, allowing the multifractal behavior to differ on a set of Hausdorff dimension 0.

2 Preliminaries

In this section, we recall some definitions and results useful for the sequel.

2.1 Hausdorff dimension

The notion of dimension which is mainly used in multifractal analysis is the Hausdorff dimension. Let us recall here its definition. For more information, we refer the reader to [17, 18].

Let $E \subseteq \mathbb{R}^d$. If $E \subseteq \bigcup_{i\in\mathbb{N}} E_i$ with $0 \leq \operatorname{diam}(E_i) \leq \delta$ for every $i \in \mathbb{N}$, we say that

$(E_i)_{i\in\mathbb{N}}$ is a countable δ-covering of E. For every $r \geq 0$ and $\delta > 0$, one sets

$$\mathcal{H}^r_\delta(E) = \inf \left\{ \sum_{i\in\mathbb{N}} \mathrm{diam}(E_i)^r : (E_i)_{i\in\mathbb{N}} \text{ countable } \delta\text{-covering of } E \right\}$$

Since $\mathcal{H}^r_\delta(E)$ is a decreasing function with respect to δ, one can define the r*-dimensional Hausdorff outer measure* of E by setting

$$\mathcal{H}^r(E) = \sup_{\delta>0} \mathcal{H}^r_\delta(E) = \lim_{\delta\to 0^+} \mathcal{H}^r_\delta(E).$$

Definition 3. *The* Hausdorff dimension *of E is the unique value* $\dim_{\mathcal{H}}(E)$ *such that*

$$\mathcal{H}^r(E) = \begin{cases} +\infty & \text{if } r < \dim_{\mathcal{H}}(E), \\ 0 & \text{if } r > \dim_{\mathcal{H}}(E). \end{cases}$$

Moreover, we use the convention that $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

The existence of this critical value is a consequence of the following fact: If $r < r'$, one has

$$\mathcal{H}^r(E) < +\infty \Rightarrow \mathcal{H}^{r'}(E) = 0 \quad \text{and} \quad \mathcal{H}^{r'}(E) > 0 \Rightarrow \mathcal{H}^{r'}(E) = +\infty.$$

2.2 Wavelet and Schauder bases

Orthonormal wavelet bases appeared to be a useful tool for the study of multifractal properties of functions: As we will see in the next subsection, classical function spaces, such as Hölder, Sobolev or Besov spaces, can be characterized by conditions on the wavelet coefficients (provided that the wavelets used are smooth enough). Moreover, the Hölder pointwise regularity can also be characterized by decay conditions on the wavelet coefficients. In this subsection, we recall the definition of orthonormal wavelet bases. We refer the reader for instance to [15, 38, 16, 36] for more details. Next, we present the Schauder basis which will be useful for the construction proposed in Section 4.

A wavelet basis of $L^2(\mathbb{R}^d)$ is composed of a function φ and $2^d - 1$ functions $\psi^{(i)}$, $1 \leq i < 2^d$, called wavelets, such that

$$\left\{\varphi(x-k) : k \in \mathbb{Z}^d\right\} \cup \left\{\psi^{(i)}(2^j x - k) : 1 \leq i < 2^d, k \in \mathbb{Z}^d, j \in \mathbb{N}\right\}$$

forms an orthogonal basis of $L^2(\mathbb{R}^d)$. Any function $f \in L^2(\mathbb{R}^d)$ can thus be written as

$$f(x) = \sum_{k\in\mathbb{Z}^d} C_k \varphi(x-k) + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}^d} \sum_{i=1}^{2^d-1} c^{(i)}_{j,k} \psi^{(i)}(2^j x - k),$$

where the convergence holds in $L^2(\mathbb{R}^d)$ and the wavelet coefficients are

$$C_k = \int_{\mathbb{R}^d} f(x)\varphi(x-k)\,dx \quad \text{and} \quad c^{(i)}_{j,k} = 2^{dj}\int_{\mathbb{R}^d} f(x)\psi^{(i)}(2^j x - k)\,dx. \tag{2}$$

Note that (2) make sense even if f does not belong to $L^2(\mathbb{R}^d)$; if f is a tempered distribution and if one uses smooth enough wavelets, these formulas can be interpreted as a duality products. It is in particular the case when considering Sobolev or Besov spaces, as in the next subsection. Let us also note that we do not choose the $L^2(\mathbb{R}^d)$ normalization for the wavelets, but rather an $L^\infty(\mathbb{R}^d)$ normalization, which is better fitted to the study of the Hölderian regularity. We say that the wavelet basis is N-smooth if for all i, $\psi^{(i)}$ belongs to C^N and if the partial derivatives $\partial^\alpha\psi^{(i)}$ have fast decay for every $|\alpha| \leq N$.

The prototype of wavelet bases in $L^2(\mathbb{R})$ is given by the Haar basis, defined by the functions

$$\varphi = 1_{[0,1)} \quad \text{and} \quad \psi = 1_{[0,1/2)} - 1_{[1/2,1)}.$$

Note that it is composed of non-continuous functions and therefore, it cannot give a basis of other function spaces such as the space of continuous functions. The Schauder basis was introduced in order to "regularize" the Haar basis. It is of the same kind of a wavelet basis since it is obtained by taking a primitive of the Haar wavelet ψ. More precisely, let Λ be the "hat function" defined by

$$\Lambda(x) = \begin{cases} \min(x, 1-x) & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that if f is a continuous function from $[0,1]$ to $\mathbb{R}$, then

$$f(x) = f(0) + \big(f(1) - f(0)\big)x + \sum_{j\in\mathbb{N}}\sum_{k=0}^{2^j-1} c_{j,k}\Lambda(2^j x - k)$$

where the convergence is uniform on $[0,1]$ and

$$c_{j,k} = f\left(\frac{2k+1}{2^{j+1}}\right) - \frac{1}{2}\left(f\left(\frac{k}{2^j}\right) + f\left(\frac{k+1}{2^j}\right)\right).$$

2.3 Characterization of global and pointwise regularity

An important result of wavelet theory is the fact that wavelets give unconditional bases of many function spaces, such as Sobolev spaces $L^{p,s}(\mathbb{R}^d)$ or Besov spaces $B^{s,q}_p(\mathbb{R}^d)$ (see [38]). Furthermore, even for spaces which do not have unconditional bases (such as the Hölder spaces $C^s(\mathbb{R}^d)$), wavelets supply a characterization and an equivalent norm which is given by a condition on the wavelet coefficients.

Let us assume that the wavelets used are N-smooth with $N \geq [s]+1$. As proved in [38], Sobolev spaces have the following characterization (which can be taken as definition) in terms of wavelet coefficients: for $p > 1$ and $s > 0$, one has

$$f \in L^{p,s}(\mathbb{R}^d) \Longleftrightarrow \left\{ \begin{array}{l} (C_k)_{k\in\mathbb{Z}^d} \in l^p(\mathbb{Z}^d) \\ \left(\displaystyle\sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}^d} \sum_{i=1}^{2^d-1} \left| c_{j,k}^{(i)} \right|^2 2^{2sj} \chi_{\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)}(x) \right)^{1/2} \in L^p(\mathbb{R}^d) \end{array} \right. \tag{3}$$

where

$$\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right) = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right).$$

Similarly, in [38], Besov spaces are characterized as follows: for $p, q > 0$ and $s \in \mathbb{R}$, one has

$$f \in B_p^{s,q}(\mathbb{R}^d) \Longleftrightarrow \left\{ \begin{array}{l} (C_k)_{k\in\mathbb{Z}^d} \in l^p(\mathbb{Z}^d) \\ \left(\displaystyle\sum_{k\in\mathbb{Z}^d} \sum_{i=1}^{2^d-1} \left| c_{j,k}^{(i)} 2^{(s-\frac{d}{p})j} \right|^p \right)^{1/p} = \varepsilon_j \quad \text{with} \quad (\varepsilon_j)_{j\in\mathbb{N}} \in l^q(\mathbb{N}). \end{array} \right. \tag{4}$$

Moreover, in both cases, the characterizations (3) and (4) give an equivalent norm on the considered space. We use obvious adaptations in the case $p = +\infty$. In particular, when $p = q = +\infty$, $B_p^{s,q}(\mathbb{R}^d)$ is the Hölder space $C^s(\mathbb{R}^d)$. Therefore, one has

$$f \in C^s(\mathbb{R}^d) \Longleftrightarrow (C_k)_{k\in\mathbb{Z}^d} \in l^\infty(\mathbb{Z}^d) \quad \text{and} \quad \sup_{j\in\mathbb{N}} \sup_{k\in\mathbb{Z}^d} \sup_{1\leq i<2^d} \left| c_{j,k}^{(i)} 2^{sj} \right| < +\infty. \tag{5}$$

The pointwise Hölder regularity can also be expressed in terms of a condition on wavelet coefficients [29].

Proposition 1. *Let $x_0 \in \mathbb{R}^d$ and let $\alpha \geq 0$. Let us assume that the wavelets are N-smooth with $N \geq [\alpha]+1$. If f belongs to $C^\alpha(x_0)$, then*

$$\exists C > 0 \quad \textit{such that} \quad |c_{j,k}^{(i)}| \leq C 2^{-\alpha j} \big(1 + |2^j x - k|\big)^\alpha \quad \forall i, j, k. \tag{6}$$

Conversely, if (6) holds and if there is $\varepsilon > 0$ such that $f \in C^\varepsilon(\mathbb{R}^d)$, then $f \in C^\beta(x_0)$ for every $\beta < \alpha$.

Let us end this section by mentioning that this criteria is also valid if one replaces to wavelet coefficients by the coefficients of f in the Schauder basis [14].

Proposition 2. *Let $x_0 \in [0,1]$ and let $\alpha \geq 0$. If f belongs to $C^\alpha(x_0)$, then its coefficients $c_{j,k}$ in the Schauder basis satisfy*

$$\exists C > 0 \quad \textit{such that} \quad |c_{j,k}| \leq C 2^{-\alpha j} \big(1 + |2^j x - k|\big)^\alpha \quad \forall j, k. \tag{7}$$

Conversely, if (7) holds and if there is $\varepsilon > 0$ such that $f \in C^\varepsilon([0,1])$, then $f \in C^\beta(x_0)$ for every $\beta < \alpha$.

3 Lineability in the Hölder setting

Baire-type and prevalent results concerning the pointwise regularity of functions in either Sobolev or Besov spaces were investigated in [30, 23]. If $s < d/p$, those spaces share the following property: In each of them, a generic function is nowhere locally bounded. If $s > d/p$, we have the following result.

Theorem 1. *Let $p > 0$, $q > 0$ and $s > \frac{d}{p}$.*

1. *The set of functions f of $B_p^{s,q}(\mathbb{R}^d)$ satisfying*

$$h_f(x) \in \left[s - \frac{d}{p}, s\right], \quad \forall x \in \mathbb{R}^d \tag{8}$$

   and

$$d_f(h) = p(h - s) + d, \quad \forall h \in \left[s - \frac{d}{p}, s\right] \tag{9}$$

   is residual and prevalent in $B_p^{s,q}(\mathbb{R}^d)$.

2. *For any $x \in \mathbb{R}^d$, the set of functions of $B_p^{s,q}(\mathbb{R}^d)$ satisfying*

$$h_f(x) = s - \frac{d}{p} \tag{10}$$

   is residual and prevalent in $B_p^{s,q}(\mathbb{R}^d)$.

3. *If $p > 1$, the same results hold for $L^{p,s}(\mathbb{R}^d)$.*

This theorem shows that the generic functions in a given Besov or Sobolev space are multifractal, except when $p = +\infty$ where the spectrum is reduced to one point. Let us mention that the critical case $s = d/p$ has been treated in [33, 23]. Depending on the value taken by q, one obtains results similar to the case $s > d/p$ or to the case $s < d/p$. Let us now prove the equivalent of Theorem 1 in the lineability setting.

Theorem 2. *Let $p > 0$, $q > 0$ and $s > \frac{d}{p}$.*

1. *The set of functions f of $B_p^{s,q}(\mathbb{R}^d)$ satisfying (8) and (9) is $\mathfrak{c}$-lineable.*
2. *For any $x \in \mathbb{R}^d$, the set of functions of $B_p^{s,q}(\mathbb{R}^d)$ satisfying (10) is $\mathfrak{c}$-lineable.*
3. *If $p > 1$, the same results hold for $L^{p,s}(\mathbb{R}^d)$.*

Proof. We will only prove the first point: the two others can be studied in a similar way. Let us fix a wavelet basis N-smooth, with $N \geq [s]+1$. Let us also fix a function f in $B^{s,q}_p(\mathbb{R}^d)$ which satisfies (8) and (9) and let us denote by $c^{(i)}_{j,k}$ its wavelet coefficients. For every $a>0$, let us consider the function f_a whose wavelet coefficients are given by

$$\frac{1}{j^a} c^{(i)}_{j,k}.$$

Using the characterizations (4) of Besov spaces, it is clear that f_a belongs to $B^{s,q}_p(\mathbb{R}^d)$. Let $\mathcal{V}$ denote the subspace of $B^{s,q}_p(\mathbb{R}^d)$ spanned by the functions f_a, $a>0$. Let us note that the wavelet coefficients of any non-zero linear combination of the functions f_a are of the order of magnitude of its "largest" component: More precisely, if $n \geq 1$, $a_n > \cdots > a_1 > 0$, and $\beta_1, \ldots, \beta_n \neq 0$, and if

$$g = \sum_{i=1}^{n} \beta_i f_{a_i},$$

then, the wavelet coefficients $d^{(i)}_{j,k}$ of g satisfy

$$\frac{|\beta_1|}{2j^{a_1}} |c^{(i)}_{j,k}| \leq |d^{(i)}_{j,k}| \leq \frac{2|\beta_1|}{j^{a_1}} |c^{(i)}_{j,k}| \tag{11}$$

for every i,j,k, with j large enough. The characterization of the Hölder exponent given in Proposition 1 together with (11) give that for any $x_0 \in \mathbb{R}^d$, one has $h_g(x_0) = h_f(x_0)$. Consequently, g is not identically zero and satisfies (8) and (9). The conclusion follows. □

Let us note that in the separable case, one can slightly modify the above construction in order to get the dense-lineability, as stated in the following result.

Corollary 1. *Let $p,q \in (0,+\infty)$ and $s > \frac{d}{p}$. The sets considered in Theorem 2 are $\mathfrak{c}$-dense-lineable in $B^{s,q}_p(\mathbb{R}^d)$ and $L^{p,s}(\mathbb{R}^d)$ respectively.*

Proof. Again, let us only treat the first point and let us fix a wavelet basis N-smooth, with $N \geq [s]+1$. Since $p,q < +\infty$, the wavelets give also a basis of $B^{s,q}_p(\mathbb{R}^d)$. Consequently, finite wavelet series with rational coefficients form a dense subspace of this space; Let $(F_n)_{n\in\mathbb{N}}$ denote the sequence of these functions. Let us also choose a sequence $(a_n)_{n\in\mathbb{N}}$ of different positive numbers. For every $n \in \mathbb{N}$, we fix $\varepsilon_n > 0$ such that

$$\|\varepsilon_n f_{a_n}\|_{B^{s,q}_p(\mathbb{R}^d)} < \frac{1}{n},$$

where the functions f_a, $a>0$, are defined as in the proof of Theorem 2. Let us define

$$g_n = F_n + \varepsilon_n f_{a_n}.$$

By construction, the functions g_n, $n \in \mathbb{N}$, form a dense subspace of $B^{s,q}_p(\mathbb{R}^d)$. Finally, we consider the subspace $\mathfrak{D}$ generated by

$$\{g_n : n \in \mathbb{N}\} \cup \{f_a : a \in A\}$$

where $A = \{a > 0 : a \neq a_n \ \forall n \in \mathbb{N}\}$. Since it contains the functions g_n, $n \in \mathbb{N}$, it is clear that the subspace $\mathfrak{D}$ is dense in $B^{s,q}_p(\mathbb{R}^d)$. Moreover, it has maximal dimension since it contains the linearly independent functions f_a, $a \in A$. Finally, any non-zero element of $\mathfrak{D}$ has, for large scales, the same wavelet coefficients as a non-zero linear combination of the functions f_a, $a > 0$. The conclusion follows then with the same arguments as in Theorem 2. $\square$

Remark 1. Let us mention that similar results of Baire and prevalence genericity have been obtained in intersection of Besov spaces in order to prove the validity of the multifractal formalism (we refer the reader to [30, 20] for more information about this subject). The proofs presented above can easily be adapted in this case. The same remark applies also for the so-called $\mathcal{S}^\nu$ spaces, introduced in order to propose another formalism adapted to non-concave spectra [5, 4].

Remark 2. Different lineability results can hold simultaneously in a given space: If s', p' are such that $s' - d/p' > s - d/p$ and $p \geq p'$, classical Besov embeddings give $B^{s',q}_{p'}(\mathbb{R}^d) \subseteq B^{s,q}_p(\mathbb{R}^d)$. Then, starting from a generic function of $B^{s',q}_{p'}(\mathbb{R}^d)$ instead of $B^{s,q}_p(\mathbb{R}^d)$, the subspaces constructed in Theorem 2 or in Corollary 1 for $B^{s',q}_{p'}(\mathbb{R}^d)$ give the dense-lineability in $B^{s,q}_p(\mathbb{R}^d)$ of the set of functions f such that $d_f(h) = p'(h - s') + d$. This is a contradistinction with the Baire or prevalence case, since these notions are stable under intersection.

Let us end this section with the study of another type of pointwise reguarlity: Indeed, a drawback of the Hölder exponent is that, by definition, it cannot take negative values: This is a severe restriction for real-world applications since several signals and images cannot be modeled by locally bounded functions. In [31], alternative regularity exponents, the *p-exponents*, have been introduced, motivated by the necessity of introducing regularity exponents that could be defined even for non-locally bounded functions. They can be defined with the help of local L^p-conditions introduced by Calderón and Zygmund [11]. They have the advantage of only making the assumption that f locally belongs to $L^p(\mathbb{R}^d)$, and therefore allows to obtain extensions of Theorem 1 when $s - d/p < 0$.

Definition 4. *Let $x_0 \in \mathbb{R}^d$, $\alpha \in \mathbb{R}$ and $p \geq 1$. Assume that $f \in L^p_{loc}(\mathbb{R}^d)$; the function f belongs to $T^p_\alpha(x_0)$ if there exist a constant $C > 0$ and a polynomial P_{x_0} of degree less than $[\alpha]$ such that, for r small enough,*

$$\left(\frac{1}{r^d}\int_{B(x_0,r)} \left|f(x) - P_{x_0}(x)\right|^p dx\right)^{1/p} \leq C r^\alpha, \tag{12}$$

where $B(x_0, r)$ denotes the ball of center x_0 and radius r.

The *p-exponent* of f at x_0 is defined as

$$h^p_f(x_0) = \sup\{\alpha \in \mathbb{R} : f \in T^p_\alpha(x_0)\}. \tag{13}$$

The condition that f locally belongs to $L^p(\mathbb{R}^d)$ implies that (12) holds for $\alpha = -d/p$, so that $h^p_f(x_0) \geq -d/p$. The *p-spectrum* $d^p_f(h)$ of f is defined as the Hausdorff dimension of the set of points where the p-exponent takes the value h. Remark that the usual Hölder regularity corresponds to the case $p = +\infty$. Let us also mention that recently, using the wavelet framework, the definition of the p-exponents has been extended to the case $p \in (0, 1)$, see [32].

When they are both defined, the Hölder spectrum and the p-spectrum can differ, see e.g. [13]. However, the next result of [21] proves that generically (in the sense of prevalence) in a given Sobolev or Besov space, the Hölder spectrum and the p-spectrum coincide.

Theorem 3. *Let $s \geq 0$ and $p, q \in (0, +\infty)$.*

1. *For all $p_0 \geq 1$ such that $s - \frac{d}{p} > -\frac{d}{p_0}$, the set of functions f of $B^{s,q}_p(\mathbb{R}^d)$ satisfying*

$$h^{p_0}_f(x) \in \left[s - \frac{d}{p}, s\right], \quad \forall x \in \mathbb{R}^d \tag{14}$$

and

$$d^{p_0}_f(h) = p(h - s) + d, \quad \forall h \in \left[s - \frac{d}{p}, s\right] \tag{15}$$

is prevalent in $B^{s,q}_p(\mathbb{R}^d)$.

2. *If $p > 1$, the same result holds for $L^{p,s}(\mathbb{R}^d)$.*

Similarly to the classical Hölder case, a wavelet characterization of the p-exponent is proved in [31]: As in Proposition 1, the exact values of the wavelet coefficients are not crucial, and other values, satisfying (11) actually lead to the same p-exponent. Therefore, it is straightforward to adapt the proof of Theorem 2 to our present setting of p-exponent to get the equivalent of Theorem 3 with the notion of lineability.

Theorem 4. *Let $s \geq 0$ and $p, q \in (0, +\infty)$.*

1. *For all $p_0 \geq 1$ such that $s - \frac{d}{p} > -\frac{d}{p_0}$, the set of functions f of $B^{s,q}_p(\mathbb{R}^d)$ satisfying (14) and (15) is $\mathfrak{c}$-dense-lineable in $B^{s,q}_p(\mathbb{R}^d)$.*

2. *If $p > 1$, the same result holds for $L^{p,s}(\mathbb{R}^d)$.*

4 Algebrability

In this section, we consider the notion of algebrability: Besides asking for vector subspaces one could also study other structures, such as algebras, which motivated the following concept [3, 2].

Definition 5. *Let $\mathcal{A}$ be an algebra and $\mathcal{B}$ be a subset of $\mathcal{A}$. The set $\mathcal{B}$ is* κ-algebrable *if $\mathcal{B} \cup \{0\}$ contains a κ-generated subalgebra $\mathcal{C}$ of $\mathcal{A}$. The set $\mathcal{B}$ is simply* algebrable *if the cardinality of any system of generators of the existing subalgebra is infinite .*

Of course, any algebrable set is, automatically, lineable as well. In general, the converse is false: An example of this fact is given by our present setting of Hölder spectrum or p-spectrum, as stated in the next proposition.

Proposition 3. *The lineable sets considered in Theorem 2 and Theorem 4 are not* 1*-algebrable.*

Proof. Let us first consider the problem of the Hölder setting. Let f be a function satifying (9). Let x_0 be an arbitrary point in $\mathbb{R}^d$. Then, there exists a function g in the algebra generated by f such that $g(x_0) = 0$; indeed, if it is not the case for f, it suffices to consider the function $g(x) = f(x)^2 - f(x_0)f(x)$. For n large enough, the Hölder exponent of g^n will be arbitrarily large, so that g will not have the generic spectrum (9). Note that the same negative result holds for the p-spectrum, since the p-exponent is always larger than the Hölder exponent. □

One can conclude from this proof that the spectra will considerably be modified if the level sets $f^{-1}(\{y\})$ of f are "large". But one can hope to get a modification of the spectra only on a set of Hausdorff dimension zero. This is the objective of the last part of our paper. In order to present a positive result of algebrability, we consider the case of Hölder spaces $C^\alpha([0,1])$, with $\alpha \in (0,1)$. Clearly, these spaces are algebra.

First, we present a technique, the so-called *exponential-like function method*, which allows to get the algebrability of some sets of functions defined on $[0,1]$, see [24, 6].

Definition 6. *We say that a function $f : \mathbb{R} \to \mathbb{R}$ is* exponential-like (of range m) *whenever f is given by*

$$f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x}, \ x \in [0,1]$$

for some distinct non-zero real numbers $\beta_1, \dots, \beta_m$ and some non-zero real numbers $a_1, \dots, a_m$.

In [6], the authors proved a very useful property of exponential-like functions. Let us recall it here.

Lemma 1. *For every $m \in \mathbb{N}$, every exponential-like function f of range m and every $c \in \mathbb{R}$, the level set $f^{-1}(\{c\})$ has at most m elements.*

Let us also recall the following strengthened notion of algebrability introduced in [8].

Definition 7. *Given a commutative algebra $\mathcal{A}$ and a cardinal number κ, a subset $\mathcal{B} \subseteq \mathcal{A}$ is* strongly κ-algebrable *if there exists a κ-generated free algebra $\mathcal{C}$ contained in $\mathcal{B} \cup \{0\}$. A subset $\mathcal{B} \subseteq \mathcal{A}$ is* strongly algebrable *if it is strongly κ-algebrable for an infinite κ.*

We remind that a subset X of a commutative algebra generates a *free subalgebra* if for each polynomial P without a constant term and any $x_1, \dots, x_n \in X$, we have $P(x_1, \dots, x_n) = 0$ if and only if $P = 0$ (that is, the set of all elements of the form $x_1^{k_1} \dots x_n^{k_n}$ where $x_1, \dots, x_n \in X$ and where $k_1, \dots, k_n \in \mathbb{N}_0$ are not all equal to 0, is linearly independent).

The technique developed in [6] is presented in Proposition 4. Let us recall that a Hamel basis of $\mathbb{R}$ is a basis of $\mathbb{R}$ while considered as a $\mathbb{Q}$-vector space.

Proposition 4. *Let $\mathcal{F} \subseteq \mathbb{R}^{[0,1]}$ and assume that there exists $F \in \mathcal{F}$ such that $f \circ F \in \mathcal{F} \setminus \{0\}$ for every exponential-like function f. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More precisely, if $\mathcal{H}$ is a Hamel basis of $\mathbb{R}$, then the functions $\exp \circ (rF)$, $r \in \mathcal{H}$, are free generators of an algebra contained in $\mathcal{F} \cup \{0\}$.*

The following lemma is inspired by some constructions presented in [35].

Lemma 2. *Assume that $\alpha \in (0,1)$. There exists a function $F \in C^{\alpha}([0,1])$ whose Hölder exponents $h_F(x_0)$ equal α at every $x_0 \in [0,1]$ and whose level sets satisfy $\dim_{\mathcal{H}} F^{-1}(\{y\}) = 0$ for every $y \in \mathbb{R}$.*

Proof. We will construct F via its coefficients in the Schauder basis. First, let us consider a sequence $(r_n)_{n\in\mathbb{N}}$ of strictly positive real numbers which decreases to 0. By recurrence, we construct a strictly increasing sequence $(j_n)_{n\in\mathbb{N}}$ of natural numbers as follows: we fix $j_0 = 0$ and, assuming that j_l has been constructed for every $j < n$, we choose j_n large enough so that

$$\sum_{l=0}^{n-1} 2^{(1-\alpha)j_l} < \frac{1}{2} 2^{(1-\alpha)j_n} \tag{16}$$

and

$$\alpha r_{n-1} j_n > \big(1 - r_{n-1}(1-\alpha)\big) j_{n-1} + n \,. \tag{17}$$

Then, we set

$$c_{j,k} = \begin{cases} 2^{-\alpha j} & \text{if there exists } n \in \mathbb{N} \text{ such that } j = j_n \\ 0 & \text{otherwise} \end{cases}$$

and we consider the function F defined by

$$F(x)=\sum_{j\in\mathbb{N}}\sum_{k=0}^{2^j-1}c_{j,k}\Lambda(2^jx-k)\,.$$

From the definition of the coefficients $c_{j,k}$ and the localization of the support of Λ, it is clear that this series is uniformly convergent on $[0,1]$, hence F is well defined. Let us first show, using standard arguments, that F belongs to $C^\alpha([0,1])$. Let us fix $x,y\in[0,1]$ and let us consider $J\in\mathbb{N}$ such that $2^{-J-1}<|x-y|\leq 2^{-J}$. We have

$$\begin{aligned}|F(x)-F(y)|\leq&\left|\sum_{j<J}\sum_{k=0}^{2^j-1}c_{j,k}\big(\Lambda(2^jx-k)-\Lambda(2^jy-k)\big)\right|\\&+\left|\sum_{j\geq J}\sum_{k=0}^{2^j-1}c_{j,k}\Lambda(2^jx-k)\right|+\left|\sum_{j\geq J}\sum_{k=0}^{2^j-1}c_{j,k}\Lambda(2^jy-k)\right|\end{aligned}\tag{18}$$

Let us estimate the first term of (18). Since $|x-y|\leq 2^{-J}$, for every $j<J$, x and y belongs to the same dyadic interval $[\frac{k_0}{2^j},\frac{k_0+1}{2^j})$ of scale j. Moreover,

$$\big|\Lambda(2^jx-k_0)-\Lambda(2^jy-k_0)\big|=2^j|x-y|.$$

Consequently,

$$\begin{aligned}\left|\sum_{j<J}\sum_{k=0}^{2^j-1}c_{j,k}\big(\Lambda(2^jx-k)-\Lambda(2^jy-k)\big)\right|&\leq\sum_{j<J}2^{-\alpha j}2^j|x-y|\\&\leq C2^{(1-\alpha)J}|x-y|\\&\leq C|x-y|^\alpha\end{aligned}\tag{19}$$

for some constant $C>0$ independent of x,y. For the second term of (18), we use the fact that $2^{-J-1}<|x-y|$ to get

$$\left|\sum_{j\geq J}\sum_{k=0}^{2^j-1}c_{j,k}\Lambda(2^jx-k)\right|\leq\sum_{j\geq J}2^{-\alpha j}\leq C'|x-y|^\alpha\tag{20}$$

for some constant $C'>0$ independent of x and y. Similarly, the third term of (18) can be estimated by

$$\left|\sum_{j\geq J}\sum_{k=0}^{2^j-1}c_{j,k}\Lambda(2^jy-k)\right|\leq C'|x-y|^\alpha.\tag{21}$$

Putting together (18), (19), (20) and (21), we obtain that $F \in C^{\alpha}([0,1])$.

Secondly, using Proposition 2, it is clear that the Hölder exponent of F is equal to α at every point of $[0,1]$.

In order to conclude, we still have to prove that the level sets of F have Hausdorff dimension zero. Let E denotes a set on which F is constant, and let us fix $r > 0$. For every $n \in \mathbb{N}$, we have

$$E = \bigcup_{k=0}^{2^{j_n+1}-1} E_{n,k} \quad \text{where} \quad E_{n,k} = \left[\frac{k}{2^{j_n+1}}, \frac{k+1}{2^{j_n+1}}\right).$$

Since $\operatorname{diam}(E_{n,k}) \leq \frac{1}{2^{j_n+1}}$, if we show that

$$\sum_{k=0}^{2^{j_n+1}-1} \operatorname{diam}(E_{n,k})^r \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

then we will get that $\mathcal{H}^r(E) = 0$. Let us consider $x, y \in E_{n,k}$. Similarly to what was done previously, for every $j < j_n$, we have

$$\begin{aligned} \left| \sum_{j<j_n} \sum_{k=0}^{2^j-1} c_{j,k} \big(\Lambda(2^j x - k) - \Lambda(2^j y - k)\big) \right| &\leq \sum_{l<n} 2^{(1-\alpha)j_l} |x-y| \\ &< \frac{1}{2} 2^{(1-\alpha)j_n} |x-y| \end{aligned} \tag{22}$$

where we have used condition (16). Moreover, we have

$$\left| \sum_{k=0}^{2^{j_n}-1} c_{j,k} \big(\Lambda(2^{j_n} x - k) - \Lambda(2^{j_n} y - k)\big) \right| = 2^{(1-\alpha)j_n} |x-y|, \tag{23}$$

and finally,

$$\begin{aligned} \left| \sum_{j>j_n} \sum_{k=0}^{2^j-1} c_{j,k} \big(\Lambda(2^j x - k) - \Lambda(2^j y - k)\big) \right| &\leq 2 \cdot \sum_{l>n} 2^{-\alpha j_l} \\ &< 4 \cdot 2^{-\alpha j_{n+1}}. \end{aligned} \tag{24}$$

We get from (22), (23) and (24) that

$$0 = \big|F(x) - F(y)\big| \geq \frac{1}{2} 2^{(1-\alpha)j_n} |x-y| - 4 \cdot 2^{-\alpha j_{n+1}},$$

hence

$$|x-y| \leq 8 \cdot 2^{-\alpha j_{n+1} + (\alpha-1)j_n}.$$

We obtain then that

$$\sum_{k=0}^{2^{j_{n+1}}-1} \operatorname{diam}(E_{n,k})^r \leq 8^r \cdot 2 \cdot 2^{-r\alpha j_{n+1}+(1-r(1-\alpha))j_n}.$$

The sequence $(2^{-r\alpha j_{n+1}+(1-r(1-\alpha))j_n})_{n\in\mathbb{N}}$ converges to 0 as n tends to infinity; indeed, for n large enough, we have $r_n \leq r$ and using condition (17), we get

$$r\alpha j_{n+1} \geq \alpha r_n j_{n+1} > \big(1 - r_n(1-\alpha)\big) j_n + n > \big(1 - r(1-\alpha)\big) j_n + n + 1\,.$$

The conclusion follows. □

Theorem 5. *Assume that $\alpha \in (0,1)$. The set of functions $f \in C^\alpha([0,1])$ for which there exists $E \subseteq [0,1]$ such that $\dim_{\mathcal{H}}(E) = 0$ and $h_f(x_0) = \alpha$ for every $x_0 \in [0,1] \setminus E$ is strongly $\mathfrak{c}$-algebrable.*

Proof. We will use the technique described in Proposition 4. Let us consider the function F constructed in Lemma 2 and any exponential-like function f. Let us also fix $x_0 \in [0,1]$. For every $x \in [0,1]$, we have

$$f \circ F(x) - f \circ F(x_0) = Df(z)\big(F(x) - F(x_0)\big)$$

for some z between $F(x)$ and $F(x_0)$. If $Df(F(x_0)) \neq 0$, one directly gets that

$$h_{f\circ F}(x_0) = h_F(x_0) = \alpha.$$

Let us note that the derivative Df of f is also an exponential-like function. Therefore, using Lemma 1, we know that its preimage $(Df)^{-1}(\{0\})$ has finitely many elements. From the construction of F, we get that the set of points x_0 for which $Df(F(x_0)) = 0$ has Hausdorff dimension 0, hence the conclusion. □

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