About generic properties of "nowhere analyticity"

Céline Esser
Celine.Esser@ulg.ac.be

Joint work with Françoise Bastin
F.Bastin@ulg.ac.be

and Samuel Nicolay
S.Nicolay@ulg.ac.be

University of Liege – Institute of Mathematics

Functional Analysis:
Applications to Complex Analysis and Partial Differential Equations

Będlewo – 6-12 May 2012

Introduction

If f is a C^∞ function on an open interval containing x_0, its Taylor series at x_0 is denoted by

$$T(f, x_0)(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

This function f is analytic at x_0 if $T(f, x_0)$ converges to f on an open neighbourhood of x_0; if this is not the case, we say that f has a singularity at x_0.

A function with a singularity at each point of an interval is nowhere analytic on the interval.

Many examples of infinitely differentiable nowhere analytic functions exist. An example was given by Cellérier (1890) by the function defined for all $x \in \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{+\infty} \frac{\sin(a^n x)}{n!}$$

where a is a positive integer larger than 1. This function belongs to $C^\infty(\mathbb{R})$ by Weierstrass theorem but is nowhere analytic.

Introduction

We want to obtain generic properties about nowhere analytic functions from two points of view:

In topology: A generic property is a property that holds on a residual (or comeager) set (i.e. that contains a countable intersection of dense open sets).

In measure theory: A generic property is a property that holds on a prevalent set.

Content of the talk:

1. Prevalence
2. Generic properties of nowhere analytic functions
3. More with Gevrey classes

Prevalence

B. R. Hunt, T. Sauer, et J. A. Yorke, Prevalence: a translation-invariant “almost every" on infinite-dimensional spaces, *Bulletin of the American Mathematical Society*, **27** (1992), 2, 217-238

The prevalence is a notion introduced by Hunt, Sauer and Yorke to generalize the concept of “almost everywhere” in the case of infinite dimensional spaces keeping some properties of neglibigle sets:

- A negligible set has empty interior (ie.“almost every” implies density).
- Every subset of a negligible set is negligible.
- A countable union of negligible sets is negligible.
- Every translate of a negligible set is negligible.

It appears that it was impossible to define this notion in terms of a specific measure.

Proposition

Let B be a Borel set of $\mathbb{R}^n$. Then $\mathcal{L}(B) = 0$ if and only if there exists a probability measure μ with compact support such that

$$\mu(B + x) = 0 \ \ \forall x \in \mathbb{R}^n .$$

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Prevalence

Let E be a complete metric linear space.

Definition

A Borel subset B of E is shy if there exists a Borel probability measure μ on E with compact support such that $\mu(B+e)=0$ for all $e \in E$.

More generally, a subset V is called shy if it is contained in a shy Borel set. The complement of a shy set is called a prevalent set. A prevalent property is a property which holds on a prevalent set.

Proposition

A subset of $\mathbb{R}^n$ is shy if and only if it is negligible.

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- A shy set has empty interior (ie. all prevalent sets are dense).
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Generic properties of “nowhere analytic” functions

We consider the Frechet space $C^\infty([0,1])$ endowed with the sequence $(p_k)_{k\in\mathbb{N}_0}$ of semi-norms

$$p_k(f) = \sup_{j\leq k} \sup_{x\in[0,1]} |f^{(j)}(x)|.$$

Question: Generic properties concerning nowhere analytic functions of $C^\infty([0,1])$?

Remark that if f is not analytic at x_0, there are two kinds of singularity at x_0:

- Pringsheim singularity : the radius of convergence of the series is 0.
- Cauchy singularity : the series converges in some neighbourhood of x_0 but the limit does not represent f, as small as you take the neighbourhood of x_0.

Generic properties of “nowhere analytic” functions

Some results are known:

- Examples of functions with a Pringsheim singularity at each point.

  W. Rudin, Real and Complex Analysis, London: McGraw-Hill, 1970.

- The set of functions in $C^\infty([0,1])$ with a Pringsheim singularity at each point of $[0,1]$ is a residual (or comeager) subset of $C^\infty([0,1])$.

  L. Bernal-Gonzalez, Lineability of sets of nowhere analytic functions, *J. Math. Anal. Appl.*, **340** (2008), 1284-1295

  T.I. Ramsamujh, Nowhere Analytic C^∞ Functions, *J. Math. Anal. Appl.*, **160** (1991), 263-266

- There doesn’t exist functions with a Cauchy singularity at each point.

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Generic properties of “nowhere analytic” functions

Proposition

The set of nowhere analytic functions on $[0,1]$ is a prevalent subset of $C^\infty([0,1])$.

Remark

If A is a non-empty Borel subset of E such that the complement of A is a linear subspace of E, then A is prevalent.

For any closed subinterval I of $[0,1]$, we denote x_I the center point of I and

$$A(I,x_I) = \{f \in C^\infty([0,1]) : T(f,x_I) \text{ converges to } f \text{ on } I\}.$$

Since a function which is analytic at a point is analytic in a neighbourhood of this point, the set of nowhere analytic functions is the complement of the union of all $A(I,x_I)$ over rational subintervals $I \subset [0,1]$.

Open question: What about the prevalence of the set of functions with a Pringsheim singularity at each point?

More with Gevrey classes

Definition

For a real number $s > 0$ and an open subset Ω of $\mathbb{R}$, an infinitely differentiable function f in Ω is said to be Gevrey differentiable of order s at $x_0 \in \Omega$ if there exist a compact neighbourhood I of x_0 and constants $C, h > 0$ such that

$$\sup_{x \in I} \left| f^{(n)}(x) \right| \leq C h^n (n!)^s, \quad \forall n \in \mathbb{N}_0.$$

Remark that the case $s = 1$ corresponds to analyticity.

A nowhere Gevrey differentiable function on $[0,1]$ is a function that is not Gevrey differentiable of order s at x_0, for any $x_0 \in [0,1]$ and $s \geq 1$.

Question: Existence of nowhere Gevrey differentiable functions?

More with Gevrey classes

Lemma

Let $\lambda_k,\ k \in \mathbb{N}$, be a sequence of strictly positive numbers such that

$$\lambda_k \geq (k+1)^{(k+1)^2} \quad \& \quad \lambda_{k+1} \geq 2\sum_{j=1}^{k} \lambda_j^{2+k-j}, \qquad \forall k \in \mathbb{N}$$

and let f be the function defined on $\mathbb{R}$ by

$$f(x) = \sum_{k=1}^{+\infty} c_k e^{i\lambda_k x} \text{ with } c_k = \lambda_k^{1-k},\ k \in \mathbb{N}.$$

This function belongs to the class $C^\infty(\mathbb{R})$ and it is not Gevrey differentiable of order s at x_0, for any $x_0 \in \mathbb{R}$ and $s \geq 1$.

More with Gevrey classes

Proposition

The set of nowhere Gevrey differentiable functions is a prevalent subset of $C^\infty([0,1])$.

The set of nowhere Gevrey differentiable functions of $C^\infty([0,1])$ is the complement of

$$\bigcup_{s\in\mathbb{N}} \bigcup_{I\subset[0,1]} B(s,I)$$

where I denotes a rational subinterval of $[0,1]$ and

$$B(s,I) = \left\{ f \in C^\infty([0,1]) :\ \exists C, h > 0 \ \ \sup_{x\in I} |f^{(n)}(x)| \leq C h^n (n!)^s \ \forall n \in \mathbb{N}_0 \right\}.$$

Proposition

The set of nowhere Gevrey differentiable functions is a residual subset of $C^\infty([0,1])$.

More with Gevrey classes

Open question: Given an order $s > 1$, is the set of functions that are not Gevrey differentiable at any point of $[0, 1]$ and for any order $r < s$ a prevalent subset of the space $G^s([0, 1])$ of functions that are Gevrey differentiable of order s at every point of $[0, 1]$?

The first step is to find an appropriate topology on the space $G^s([0, 1])$ which makes it a complete metric linear space.

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