Projectively equivariant quantization over $\mathbb{R}^{p|q}$

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  - $Q(\{f, g\}) = \frac{i}{\hbar}[Q(f), Q(g)]$.

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$$Q(f) = \frac{\hbar}{i} X_f + f - \langle X_f, \alpha \rangle.$$

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- Geometric quantization Q_G: $Q_G = Q|_{\mathcal{A}}$.

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- Casimir operator corresponding to (V,β):

$$\sum_{i=1}^{n} \beta(u_i')\beta(u_i).$$

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- $f \in C^{\infty p|q}$:
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- Space of tensor fields of type $(V, \rho) = C^{\infty p|q} \otimes V$.

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- Quantization on $\mathbb{R}^{p|q}$: linear bijection $Q : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$ s.t. $\sigma_k(Q(S)) = S$ for all $S \in \mathcal{S}_\delta^k$.

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- Casimir operator of (V, β):

$$\sum_{i=1}^{n} (-1)^{\tilde{u}_i} \beta(u_i)\beta(u'_i) = \sum_{i=1}^{n} \beta(u'_i)\beta(u_i).$$

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- The Casimir operator C of $\mathfrak{pgl}(p+1|q) \cong \mathfrak{sl}(p+1|q)$ on $(\mathcal{S}^k_\delta, L)$ is equal to $\alpha(k,\delta)\mathrm{Id}$, where

$$\alpha(k,\delta) = \frac{p-q}{2}\delta^2 - \frac{2k+p-q}{2}\delta + \frac{k(k+(p-q))}{(p-q+1)}.$$

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$$N : \mathcal{S}^k_\delta \to \mathcal{S}^{k-1}_\delta : S \mapsto 2\sum_i \gamma(\epsilon^i) L_{X^{e_i}} S,$$

where $\epsilon^r = \frac{(-1)^{\tilde{r}}}{2(p-q+1)} \varepsilon^r$.

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  2. $Q(S) := \hat{S}$.
  3. If $S \in \mathcal{S}^k_\delta$, $Q(L_{X^h}S) = \mathcal{L}_{X^h}(Q(S))$ because they are eigenvectors of $\mathcal{C}$ of eigenvalue $\alpha(k,\delta)$ and because their term of degree k is exactly $L_{X^h}S$.

Divergence operator:

$$\mathrm{div} : \mathcal{S}^k_\delta \to \mathcal{S}^{k-1}_\delta : S \mapsto \sum_{j=1}^{p+q} (-1)^{\tilde{y^j}} i(\varepsilon^j)\partial_{y^j} S.$$

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Theorem

If δ is not critical, then the map $Q : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$ defined by

$$Q(S)(f) = \sum_{r=0}^{k} C_{k,r} Q_{\mathrm{Aff}}(\mathrm{div}^r S)(f), \quad \textit{for all } S \in \mathcal{S}_\delta^k$$

is the unique $\mathfrak{sl}(p+1|q)$-equivariant quantization if

$$C_{k,r} = \frac{\prod_{j=1}^{r}((p-q+1)\lambda + k - j)}{r! \prod_{j=1}^{r}(p-q+2k-j-(p-q+1)\delta)} \textit{ for all } r \geqslant 1.$$

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- Killing form of $\mathfrak{psl}(p+1|p+1)$ vanishes, but K defined by
  $$K([A],[B]) = \mathrm{str}AB$$
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- The $\mathfrak{psl}(p+1, p+1)$-equivariant quantizations are $\mathfrak{pgl}(p+1, p+1)$-equivariant (equivariance with respect to the Euler vector field corresponding to Id).

•If $q = p + 1$, Q does not depend on δ and λ.