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OPTION PRICING MODELS: A BINOMIAL APPROACH

par

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1. Introduction

This paper sets out to show that the Two-State Option Pricing Model, developed by Rendleman and Bartter (1979) and Cox, Ross and Rubinstein (1979), is the starting point of the derivation of any option pricing model, the Binomial Option Pricing Model and the Black and Scholes Model being only particular cases. Starting from a two-state process, this paper presents simple proofs of option pricing formulae in continuous time such as the Black and Scholes and the absolute process models. The approach is different from that followed by Cox, Ross and Rubinstein and Rendleman and Bartter who considered models in continuous time as limiting cases of the binomial option pricing formula. To derive option pricing formulae in continuous time we use here the binomial distribution as an approximation of continuous distributions as, for example, the normal and the lognormal distributions.

The main point is that only two elements are necessary to derive any option valuation model: a two-state process which allows the construction of a hedge portfolio, and the appropriate assumptions concerning the distribution function of the price of the underlying asset. Concerning the process of the stock price, various diffusion processes can be considered, but in order to avoid negative asset prices, most models in option valuation assume that the process of the asset price is multiplicative or, expressed in continuous time, follows a geometric Brownian motion. The preference is given to the multiplicative process over the other processes even if, as Cox and Rubinstein (1976) noted, it appears in various studies that the variance of the rate of return varies inversely with the asset price.[1]

The basic idea of option pricing is valuation by arbitrage. In discrete time it is always possible to value options by arbitrage when the underlying asset follows a two-state process. This is in fact a necessary and sufficient condition. Considering a single period, it is possible to construct a hedge portfolio composed of the underlying asset and the option which replicates the cash flows of the two-states of nature, whatever the price of the asset may be at the end of the period. To avoid riskless arbitrage the value of the option must be such that the hedge portfolio yields the riskless interest rate. As for the multiperiod case, the solution is achieved recursively by constructing for each period the hedge portfolio, starting with the last period and working backward to the initial period.

Although this two-state process does not permit the derivation of any simple option valuation formula, its advantage is that it can encompass any option pricing formula. To derive any particular option pricing model from this two-state process, one only needs to define certain additional assumptions concerning the distribution function of the underlying asset price. So, for example, a standard assumption is that the asset price has either a multiplicative binomial distribution, or in its limiting case, a log-normal distribution. These are the so-called Binomial Option Pricing Model and the Black and Scholes Model. The additive binomial distribution and the normal distribution are straightforward alternative distribution functions which lead to option valuation models comparable to the multiplicative binomial and Black and Scholes models. But these models are only particular cases. Although this is beyond the scope of this paper, it could be possible to derive option pricing formulae when the asset price has another probability distribution.

The structure of the present paper is as follows : in the next section the basic principles leading to the valuation by arbitrage are presented and the two-state option pricing process is developed and discussed. The third and fourth sections are then devoted to the derivation of some models in discrete time and in continuous time. Among them the Multiplicative and Additive Binomial Option Pricing Model and their limiting cases, the Black and Scholes and the Absolute Process or Additive Normal Option Pricing Model.

2. The Two-State Option Pricing Process

In discrete time a necessary and sufficient condition to value an option on an asset is that the price of this asset follows a two-state process. This idea has already been developed by Rendleman and Bartter (1979), Cox, Ross and Rubinstein (1979) and Cox and Rubinstein (1983). But while these studies mainly concentrated on a multiplicative two-state process, which states that the return of the underlying asset follows a geometric progression, it is demonstrated here that this is also valid when one considers any process.

The basic concept used to value an option is arbitrage. It implies that the value of two assets with the same payoff and the same risk must be equal, otherwise it is possible to make immediate riskless profit by buying the asset which has the lower value and selling the other one.

Let us describe the process, illustrated by the tree diagram in figure 1, for a one-period European call option, assuming a perfect market and no dividend. Let us also define S as the initial asset price, K as the exercise price of the option, r as one plus the riskless interest rate R of the period to expiration, U and D as the change in the asset price if it increases or decreases, C as the value of the call option at the beginning of the period, and C_u and C_d as the value of the option at the end of the period when the asset price increases or decreases respectively.

The value of the option at the end of the period is a linear function of the asset price, it is therefore possible to construct a portfolio, called hedge portfolio, combining one call option and a proportion Δ of the asset, which has a certain payoff, no matter what the price of the asset may be at the end of the period.

(1) $$\Delta = \frac{C_u - C_d}{U - D}$$

To preclude arbitrage, the payoff of the hedge portfolio must be equal to the payoff of an investment B of the same value in the riskless asset, that is:

(2) $$C_u - \Delta(S+U) = C_d - \Delta(S+D) = rB$$

Therefore the value of the call option C which results from these arbitrage arguments is after some arrangements:

(3) $$C = \frac{1}{r}\left(pC_u + (1-p)C_d\right)$$

where $p=(SR - D)/(U - D)$ and $(1-p)=(U - SR)/(U - D)$ are the risk adjusted probabilities to obtain C_u and C_d , and where the inequalities $U>SR>D$ hold in order to preclude arbitrage.

Since at the end of the period the price of the European call option is equal to the maximum of zero and the difference between the asset price and the exercise price, then (3) becomes,

$$(4) \qquad C=\frac{1}{r}\left(p \max(0,S+U-K) + (1-p) \max(0,S+D-K)\right)$$

Suppose now that one wants to valuate an option for a multiperiod case. If the market is perfect and if one can trade at the beginning of each period, then the hedge ratio can be adjusted and the procedure can be applied to each period. Let us consider the case where the successive trials generate sequences of ups and downs in the asset price for two discrete periods, as it is illustrated in figure 2, referring to U_1 and D_1 as the upward and downward moves of the asset price for the first period, U_{21} (U_{22}) as the upward move for the second period conditional to an upward (downward) move for the first period, D_{21} (D_{22}) as the downward move for the second period conditional to an upward (downward) move for the first period, and r_1 and r_2 as one plus the riskless rate of interest (R_1 and R_2) for the first and the second period.

The initial value of a call is determined recursively by applying iteratively the relationship (4) to the nodes of each trial or period, starting with the last period and working backward to the initial period. So, the values of the option at the nodes of the tree diagram are:

$$(5) \qquad C_u=\frac{1}{r_2}\left(p_{21} \max(0,S+U_1+U_{21}-K) + (1-p_{21}) \max(0,S+U_1+D_{21}-K)\right)$$

$$(6) \qquad C_d=\frac{1}{r_2}\left(p_{22} \max(0,S+D_1+U_{22}-K) + (1-p_{22}) \max(0,S+D_1+D_{22}-K)\right)$$

$$(7) \qquad C=\frac{1}{r_1}\frac{1}{r_2}\Big(p_1p_{21}\max(0,S+U_1+U_{21}-K) + p_1(1-p_{21})\max(0,S+U_1+D_{21}-K) + (1-p_1)p_{22}\max(0,S+D_1+U_{22}-K) + (1-p_1)(1-p_{22})\max(0,S+D_1+D_{22}-K)\Big)$$

where $p_{21}=((S+U_1)R_2-D_{21})/(U_{21}-D_{21})$ and $p_{22}=((S+D_1)R_2-D_{22})/(U_{22} - D_{22})$.

This procedure can be used to calculate the option value whatever the number of periods or moves is. But as the number of periods becomes large, the computation is tedious and it is often convenient to make some additional assumptions about the probability distribution of the asset price. These assumptions allow us to derive a simple option pricing formula which is easy to apply. In this respect, let us note that the fundamental element to derive a simple option pricing model is to define a certain relationship between the changes of the asset price, which allows the transformation of equation (7) into

$$(8) \qquad C = e^{-Rt} \underset{p}{E} \left(\max(0,S(t)-K)\right)$$

where E is the expected value operator using a particular distribution with parameter p.

As an example, Cox and Ross (1976) and Cox and Rubinstein (1983) give a complex formula for valuing options in continuous time when the variance of the asset price depends on both the price and the time but has a constant elasticity. In this case, the diffusion process of the asset price is the following:

$$(9) \qquad dS = S\mu dt + S^{\rho}\sigma dz$$

where ρ is the elasticity of the variance, μ the instantaneous drift of the asset price and σ its standard deviation.

The complex option valuation formula, Cox and Ross called constant elasticity-of-variance formula, can be simplified by assuming a particular diffusion process. So, when the elasticity of the variance ρ is equal to one, the diffusion process is a geometric Brownian motion process and the solution obtained is the Black and Scholes formula. Other processes that also lead to a simplified formula are, for example, the square-root process and the absolute process, called also here additive normal process, with elasticity of variance equal respectively to a half and zero. The formula for these two processes can be found in Cox and Ross (1976).

Two types of process are distinguished here: the multiplicative process and the additive or absolute process, and for each of them three simple valuation formulae, corresponding to three different distributions of the asset price, will be derived. To prove option pricing formulae in continuous time, the stochastic process of the underlying asset price is approximated with a n-stage binomial process with the same mean and variance. The use of a binomial process requires the definition of an upward and downward move, but when we go to the limit of the process, the option can be valued without knowing these moves which are replaced by the standard deviation of the asset price.

3. Multiplicative Stochastic Processes.

3.1. The Binomial Option Pricing Formula

The Binomial Option Pricing Formula, developed by Cox, Ross and Rubinstein (1979) and Rendleman and Bartter (1979), assumes that the price of the underlying asset of a European call option follows a multiplicative binomial process. At each period, the asset price S_t can either increase to the amount $S_t+u'S_t$ with probability q or decrease to the amount $S_t+d'S_t$ with probability (1-q), u' and d' being constant over time. In both cases, u' and d' can be interpreted as the rates of return of the asset. The riskless interest rate for one discrete period is also assumed constant across the periods.

In that case, the movements in the tree diagram of figure 2 are such that $U_1=Su'$, $D_1=Sd'$, $U_{21}=S(1+u')u'$, $U_{22}=S(1+d')u'$, $D_{21}=S(1+u')d'$, $D_{22}=(S+D_1)d=S(1+d')d'$.

It can be easily shown in this case that $p_1=p_{21}=p_{22}=p$, with $p=(R-d')/(u'-d')$.[2] This process of the asset price can then be illustrated by the tree diagram represented in figure 3. Substituting the new values of U_1, D_1, U_{21}, U_{22}, D_{21} and D_{22} in (7) and rearranging yields:

$$(10)\quad C=\frac{1}{r^2}\Big(p^2\max(0,S(1+u')^2-K)+2p(1-p)\max(0,S(1+u')(1+d')-K)$$
$$+(1-p)^2\max(0,S(1+d')^2-K)\Big)$$

Hence, generalizing for n periods, the expression of C becomes :

(11) $$C = \frac{1}{r^n}\left(\sum_{x=0}^{n} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \max\left(0; S(1+u')^x(1+d')^{n-x} - K\right)\right)$$

Some sequences in the tree diagram have a number of downward movements so that at expiration the asset price is lower than the exercise price. The value of the option for these sequences is negative and the option will not be exercised. A sequence in the tree will give a positive value to the option if its number α of upward moves is such that $Su^{\alpha}d^{n-\alpha}-K>0$. Taking the logarithm and replacing 1+u' and 1+d' with u and d respectively, this means that $\alpha>(\log(K/Sd^n)/\log(u/d))$, x positive. If the sequences for which the option will never be exercised are eliminated, the value of the call is after splitting and rearranging,

(12) $$C = S \sum_{x=\alpha}^{n} \frac{n!}{x!(n-x)!} \left(\frac{pu}{r}\right)^x \left(\frac{(1-p)d}{r}\right)^{n-x} - \frac{K}{r^n} \sum_{x=\alpha}^{n} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

This expression is the sum of two binomial expansions, therefore the Binomial Option Pricing formula can be written :

(13) $$\boxed{C = S\, B[\alpha;n;p'] - \frac{K}{r^n} B[\alpha;n;p]}$$

where:
- $p=(r-d)/(u-d)$, $(1-p)=(u-r)/(u-d)$, $p'=up/r$, $(1-p')=(d(1-p))/r$,
- u and d are equal to one plus the rate of return of the asset if it increases or decreases respectively,
- α is the smallest positive integer such that $u^{\alpha}d^{n-\alpha}S-K>0$,
- $B[\alpha;n;p]$ and $B[\alpha;n,p']$ are the cumulative probabilities of the complementary binomial distribution of obtaining a number of upward moves equal to or lower than a, for n trials and with a probability of occurrence of one upward move in the asset price for a trial equal to p and p'.

3.2. The Black and Scholes Option Pricing Formula

Let us assume that the asset price follows the stationary stochastic process:

(14) $$\frac{dS}{S} = \mu dt + \sigma dz$$

where μ is the instantaneous drift of the asset price or change in per cent, σ^2 its variance and where $dz=\xi\sqrt{dt}$, ξ being a normal time independent random variable with mean zero and unit variance. The mean E[dS] and variance var[dS] of the change in the asset price are equal to $S\mu dt$ and $S^2\sigma^2 dt$.

This process can be represented by a n-stage binomial process whose one stage for a time subinterval is illustrated in figure 4. The upward and downward moves in this tree for each time subinterval of length Δt are such that their mean and variance are equal to the mean and variance of the asset price, E[dS] and var[dS], the probability of the binomial distribution being arbitrarily set equal to 1/2.

(15) $$\Delta u = S\mu\Delta t + S\sigma\sqrt{\Delta t}$$

(16) $$\Delta d = S\mu\Delta t - S\sigma\sqrt{\Delta t}$$

with $\Delta t=t/n$ $(n\to\infty)$, t being the length of the whole period taken into consideration. As for the riskless interest rate of each subinterval, it is set equal to:

(17) $$\Delta R = \frac{1}{n}\left(e^{Rt} - 1\right)$$

where R is the continuously compounded annual riskless rate of interest.

The risk adjusted probability p of each subinterval is for n large:

(18) $$p = \frac{S\Delta R - \Delta d}{\Delta u - \Delta d} = \frac{\Delta R - \mu\Delta t + \sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}} = \frac{S\frac{1}{n}(e^{Rt} - 1) - S\mu\frac{1}{n} + S\sigma\sqrt{\frac{t}{n}}}{2S\sigma\sqrt{\frac{t}{n}}} = \frac{1}{2}$$

Using (8), the value of the option is

(19) $$C = e^{-Rt}\underset{p}{E}\left(\max(0, S(1+ \Delta S/S) - K)\right)$$

where $(1+\Delta S/S) = (1+\Delta u/S)^x\ (1+\Delta d/S)^{n-x}$ follows a logbinomial distribution with parameter p. If we now denote $\ln(1+\Delta S/S)$ with Y, then e^Y follows a binomial distribution and we obtain,

$$(20)\qquad C = e^{-Rt} \underset{p}{E}\left(\max(0, Se^Y - K)\right)$$

The central limit theorem guarantees that at the limit the normal distribution is a good approximation of the binomial distribution for large n, even when p is not equal to 1/2. Hence the equation (20) can be transformed into the following integral :[3]

$$(21)\qquad C = e^{-Rt} \int_{-\infty}^{\infty} f(Y)\max(0, Se^Y - K)\, dY$$

where the asset price $S(t)=Se^Y$ is lognormally distributed with mean $S(1+\hat{\mu})$, $\hat{\mu}$ being equal to $E_p[\Delta S/S]$, and variance $S^2\hat{\sigma}^2$, $\hat{\sigma}^2$ being equal to $\mathrm{var}_p[\Delta S/S]$.

In order to eliminate the sequences which give the option a value lower than or equal to zero, the lower bound of the integral must be evaluated. The value Y_α of Y such that $Se^{Y_\alpha}-K=0$ is, if we take the logarithm, $Y_\alpha=\ln(K/S)$. Hence,

$$(22)\qquad C = e^{-Rt} \int_{Y_\alpha}^{\infty} f(Y)\, Se^Y\, dY - e^{-Rt} \int_{Y_\alpha}^{\infty} f(Y)\, K\, dY$$

$$(23)\qquad C = e^{-Rt} \int_{Y_\alpha}^{\infty} S \frac{1}{\sigma_Y \sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_Y^2}(Y-\mu_Y)^2 + Y\right] dY$$

$$- e^{-Rt} \int_{Y_\alpha}^{\infty} K \frac{1}{\sigma_Y \sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_Y^2}(Y-\mu_Y)^2\right] dY$$

Then[4]

$$(24)\quad C = e^{-Rt}\, S e^{\mu_Y + \frac{1}{2}\sigma_Y^2} \int_{Y_\alpha}^{\infty} \frac{1}{\sigma_Y \sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_Y^2}\left(Y-\mu_Y-\sigma_Y^2\right)^2\right] dY$$

$$- e^{-Rt}\, K \int_{Y_\alpha}^{\infty} \frac{1}{\sigma_Y \sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_Y^2}(Y-\mu_Y)^2\right] dY$$

The first integral is the cumulative normal distribution of Y with mean $\mu_Y+\sigma_Y^2$ and variance σ_Y^2 , and the second integral is the cumulative normal distribution of Y with mean μ_Y and variance σ_Y^2. If we convert the two normal distributions into standard normal variable, the lower bounds Z_1 and Z_2 of the two integrals are respectively equal to $(Y_a - \mu_Y - \sigma_Y^2)/\sigma_Y$ and $(Y_a - \mu_Y)/\sigma_Y$

From the binomial distribution we have for n large, [5 6]

$$(25)\quad \hat{\mu} = E[\Delta S/S] = n\left(p\,\frac{\Delta u}{S} + (1-p)\,\frac{\Delta d}{S}\right) = e^{Rt} - 1$$

$$(26)\quad \hat{\sigma}^2 = var[\Delta S/S] = np(1-p)\left(\frac{\Delta u}{S} - \frac{\Delta d}{S}\right)^2 = \sigma^2 t - \frac{1}{n}\left(e^{Rt} - 1 - \mu t\right) = \sigma^2 t$$

We also know from the moment generating function of the lognormal distribution that $S(1+\hat{\mu}) = S\, e^{\mu_Y+\sigma_Y^2/2}$ and that σ_Y^2 is equal to the variance of the logarithm of $(1+\Delta S/S)$. [7] Since $\hat{\sigma}^2$,the variance of $\Delta S/S$, is equal to $\sigma^2 t$, this means that σ_Y^2 corresponds to the variance of the continuously compounded return of the asset per unit of time var[ln(1+(ΔS/S))], denoted hereafter $\sigma_c^2 t$. Hence,

$$(27)\quad \mu_Y = \ln(1+\hat{\mu}) - \frac{1}{2}\sigma_Y^2 = \ln(e^{Rt}) - \frac{1}{2}\sigma_Y^2 = Rt - \frac{1}{2}\sigma_c^2$$

As we need the probabilities of the complementary cumulative normal distribution, that is $1-N(Z_1)$ and $1-N(Z_2)$, and given the symmetry property of the normal distribution which states 1-N(Z)=N(-Z), the lower bounds of the integrals are after substituting μ_Y and σ_Y into their expression:

(28) $$Z_1 = \frac{\ln(S/Ke^{-Rt})}{\sigma_c\sqrt{t}} + \frac{1}{2}\sigma_c\sqrt{t}$$

(29) $$Z_2 = \frac{\ln(S/Ke^{-RT})}{\sigma_c\sqrt{t}} - \frac{1}{2}\sigma_c\sqrt{t} = Z_1 - \sigma_c\sqrt{t}$$

And the Black and Scholes Option Pricing Model is obtained :

(30) $$\boxed{C = SN[Z_1] - Ke^{-Rt}\, N[Z_1 - \sigma_c\sqrt{t}\ ]}$$

where
- σ_c is the standard deviation of the continuously compounded annual return of the asset price,
- R is the continuously compounded riskless rate of interest,
- N[.] is the cumulative normal distribution,
- t is the period to expiration expressed in years.

3.3. The Log-Poisson Option Pricing Formula

Another process often used to describe changes in asset prices is the log-poisson or jump process :

(31) $$\frac{dS}{S} = \mu dt + (k-1)\, d\pi$$

where μdt is the instantaneous drift and (k-1) the amplitude of the jump of the asset price, and $d\pi$ a continuous time poisson process with probability of jump λdt. The mean and variance of the asset price when the asset follows this process are $S\mu dt + S\lambda(k-1)dt$ and $S^2\lambda E\left[(k-1)^2\right]dt$. This process can again be represented by a binomial tree whose upward and downward moves for an infinitesimal subperiod, illustrated in figure 5, are :

(32) $$\Delta u = S\mu\Delta t + S(k-1)$$

(33) $$\Delta d = S\mu\Delta t$$

The asset price can jump for an amount S(k-1) at any time during the period t with probability λt. Therefore, considering a very small period of time, this means that the probability of a jump becomes very small, zero at the limit, while the probability of no jump becomes higher, one at the limit, otherwise the asset price will become infinite.

Let us consider this process as a special case of the log-binomial process with $u'=\Delta u/S$, $d'=\Delta d/S$ and $R=\Delta R$. The values of the risk adjusted probabilities p and p' of the binomial option pricing formula become :

$$(34) \quad p = \frac{R-d'}{u'-d'} = \frac{\Delta R - \Delta d/S}{\Delta u/S - \Delta d/S} = \frac{\Delta R - \mu \Delta t}{k-1} = \frac{1}{n}\left(\frac{e^{Rt} - 1 - t\mu}{k-1}\right)$$

$$(35) \quad p' = \frac{1+u'}{1+\Delta R}\, p = \frac{(\mu \Delta t + k)\,(e^{Rt} - 1 - t\mu)}{n(k-1)(1+\Delta R)} = \frac{(\mu t/n + k)\,(e^{Rt} - 1 - t\mu)}{n(k-1)(1+\frac{(e^{Rt}-1)}{n})}$$

As expected, both p and p' converge to zero as n increases. Therefore, since the probability p is small and the number of trial, i.e. n, is large, the Poisson probability distribution can be used to approximate the binomial probabilities. The cumulative probabilities of the binomial distributions of the BOP will then be replaced by the cumulative probabilities of two Poisson distributions with parameters λ' and λ.

$$(36) \quad \lambda' = np' = \frac{e^{Rt} - 1 - t\mu}{k-1}$$

$$(37) \quad \lambda = np = \frac{(\mu t/n + k)\,(e^{Rt} - 1 - t\mu)}{(k-1)(1+\frac{(e^{Rt}-1)}{n})} = \frac{k(e^{Rt} - 1 - t\mu)}{k-1}$$

As for the minimum number α of upward moves such that the call has a positive value, it is equal for large n to $(\log(K/S) - \mu t)/\log(k)$. [8]

Consequently, the jump option pricing formula can be written :

$$(38) \quad \boxed{C = S\,\Psi[a;\lambda'] - \frac{K}{r^n}\Psi[a;\lambda]}$$

where $\Psi(a;\lambda)$ and $\Psi(a;\lambda')$ are the cumulative probability of the complementary Poisson distribution with parameters p and p', that is, $\Psi(a;\lambda)=\sum_{x=a}^{\infty}(e^{-\lambda}\lambda^{x})/x!)$ and $\Psi(a;\lambda')=\sum_{x=a}^{\infty}(e^{-\lambda'}\lambda'^{x})/x!)$

4. Additive Stochastic Processes.

4.1. The Additive Binomial Option Pricing Formula

The Binomial Option Pricing Formula derived in the preceding section assumes that the asset price follows a multiplicative binomial distribution, or in the limit a lognormal distribution. It will be now assumed that the asset price follows an additive or absolute process, which means that at each discrete period the asset price can only either increase by the amount U=Su' or decrease by the amount D=Sd', S being the asset price at time 0. Both u' and d' can be considered as absolute rates of return, since the true rates of return at time t are equal to Su'/S_t and Sd'/S_t respectively. As to the riskless interest rate, it is kept constant. This process has, on the one hand, two disadvantages. Firstly the probability associated with a negative price of the asset is greater than zero when d' is negative and n>1/d'.[9] The relevance of this disadvantage can be questioned, even if the exclusion of negative prices for an asset is largely accepted in the financial literature as a constraint. Furthermore, this problem is easily overcome by restricting the process of the asset price with an absorbing barrier, as it has been done in Cox and Ross (1976) and Corhay (1989) for the continuous case. The second disadvantage is that p, the pseudo probability used in the binomial case, is not constant from one period to another, as it will be proved further. A simplification about the value of p across the periods will be necessary to derive a simple formula.

Under this set of assumptions, the risk adjusted probabilities in equation (7) are $p_1=(R-d')/(u'-d')$, $p_{21}=p_1+(u'R)/(u'-d')$ and $p_{22}=p_1+(d'R)/(u'-d')$. Denoting p_1 with p, and replacing $u'R/(u'-d')$ and $d'R/(u'-d')$ with u^* and d^*, the expression of the call for two periods (7) becomes:

(39) $$C \approx \frac{1}{r^2}\left\{p^2 C_{2u} + p(1-p)C_{u+d} + p(1-p)C_{d+u} + (1-p)^2 C_{2d}\right\} + \frac{1}{r^2}\left\{pu^*\left(C_{2u} - C_{u+d}\right) + (1-p)d^*\left(C_{d+u} - C_{2d}\right)\right\}$$

Given the definitions of u^* and d^* the second term of this expression, which becomes rapidly complex when the number of periods increases, has little impact on the value of C in comparison with the first term. So let us consider that the second term is insignificant and let us only examine the form of the first term of the expression.

The tree diagram of the different values for the asset for two periods is shown in figure 6. If we replace C_{2u}, C_{u+d} and C_{2d} in (39) by the expression of the value of the option at expiration, that is, $C_{2u}=\max(0,S+2Su'-K)$, $C_{u+d}=C_{d+u}=\max(0,S+Su'+Sd'-K)$ and $C_{2d}=\max(0,S+2d-K)$, then the first term of (39) is written

(40) $$C \approx \frac{1}{r^2}\left\{p^2\max(0,S+2Su'-K)+2p(1-p)\max(0,S+Su'+Sd'-K) +(1-p)^2\max(0,S+2Sd'-K)\right\}$$

The coefficients of (40) have the form of a binomial expression, therefore the relationship for n periods is, after eliminating the sequences which give the call option a value lower than or equal to zero:

(41) $$C \approx \frac{1}{r^n}\left(\sum_{x=\alpha}^{n} \frac{n!}{x!(n-x)!}\, p^x\,(1-p)^{(n-x)}\,(S + xSu' + (n-x)Sd' - K)\right)$$

where α is the smallest positive integer so that $\alpha Su'+(n-\alpha)Sd'+S-K > 0$.

This expression can be split into two binomial expansions. The first one is the cumulative binomial distribution and the second one is the expected value of the truncated binomial distribution. Consequently the approximated Additive Binomial Option Pricing formula can be written :

(42) $$\boxed{C \approx \frac{1}{r^n}\left[S(1+nd') - K\right]B[\alpha;n;p] + S(u'-d')\sum_{x=\alpha}^{n}\frac{1}{r^n}x\, b[x;n;p]}$$

where - $p = (R\text{-}d')/(u'\text{-}d')$, u' and d' being the rates of return of an upward (downward) move, and R being the riskless interest for one discrete period,

- α is the smallest positive integer so that $\alpha Su'+(n-\alpha)Sd'+S-K > 0$, that is $\alpha > (K-S(1+nd'))/(S(u'-d'))$
- $B[\alpha;n;p]$ is the cumulative probability of the complementary binomial distribution of obtaining a number of upward moves equal to or lower than α, for n trials and with a probability of occurrence of one upward move equal to p.
- $b[x;n;p]$ is the binomial probability to have x upward moves out of n moves with probability p to have an upward move.

4.2. The Additive Normal or Absolute Process Model

Let us now assume that the asset price follows an additive process whose changes dS is a stochastic variable with instantaneous drift μ and variance σ^2 per unit of time, σ^2 being independent of the price. The stochastic differential equation is written :

(43) $dS = S\mu dt + \sigma dz$

where $dz = \xi\sqrt{dt}$, ξ being a normal time independent variable with mean zero and unit variance. This process is called the Ornstein-Uhlenbeck process, and its mean and variance are equal to $Se^{\mu dt}$ and $\sigma^2 (e^{2\mu dt}-1)/2\mu$. [10]

The upward and downward moves of the binomial tree, illustrated in figure 7, which represents this process for an infinitesimal subperiod with the same mean and variance and q=1/2 are: [11]

(44) $$\Delta u = S\frac{1}{n}(e^{\mu t}-1) + \sigma\sqrt{\frac{e^{2\mu t}-1}{n(2\mu)}}$$

(45) $$\Delta d = S\frac{1}{n}(e^{\mu t}-1) - \sigma\sqrt{\frac{e^{2\mu t}-1}{n(2\mu)}}$$

As in the additive binomial case, Δu and Δu can be considered as absolute rates of returns. As for the riskless interest rate ΔR, $\Delta R=(1/n)(e^{Rt}-1)$, R being the continuously compounded annual riskless rate of interest. Of course, this model encounters the same problems as the additive binomial model concerning the positive probability associated with a negative asset price. But, unlike the discrete additive call valuation formula described in the preceding subsection, it will be shown that the normal additive option pricing formula is not an approximation.

The value of the risk adjusted probability for a subinterval is for n large:

$$(46) \qquad p=\frac{S\Delta R-\Delta d}{\Delta u-\Delta d}=\frac{\frac{1}{n}S(e^{Rt}-1)-S\frac{1}{n}(e^{\mu t}-1)+\sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}}}{2\sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}}}=\frac{1}{2}$$

Using (8), the value of the option is:

$$(47) \qquad C=e^{-Rt}\underset{p}{E}\left(\max(0,S+\Delta S-K)\right)$$

where $\Delta S = x\Delta u + (n-x)\Delta d$ follows a binomial distribution with $p=1/2$. For $n\rightarrow\infty$, we can use the central limit theorem, and the probability distribution of the stochastic variable ΔS can be represented by a normal distribution with mean and variance respectively equal to $n(p\Delta u+(1-p)\Delta d)$ and $np(1-p)(\Delta u-\Delta d)^2$.

Denoting ΔS with Y, the expression of the call value becomes:

$$(48) \qquad C=e^{-Rt}\int_{-\infty}^{\infty} f(Y)\max\{0;S+Y-K\}\,dY$$

The lower bound of the integral is such that the minimum return on the asset price involves enough upward moves so that the price is at least equal to the exercise price, that is in this case $Y_a=K-S$.In order to convert the values of the normal variable Y into the standard normal value, let us evaluate the mean μ_Y and the standard deviation σ_Y of the normal distribution of the returns. [12 13]

(49) $\mu_Y = E[\Delta S] = n(p\Delta u + (1-p)\Delta d) = S(e^{\mu t}-1) = S(e^{Rt}-1)$

(50) $\sigma_Y^2 = var(\Delta S) = np(1-p)(\Delta u - \Delta d)^2] = \sigma^2 \left(\frac{e^{2Rt}-1}{2R}\right)$

Let us now convert the normal variable Y into the standard normal variable Z, and evaluate the lower limit zl of the cumulative standard normal distribution, replacing Y_a , μ_Y and σ_Y^2 with their value:

(51) $Y = Z\sigma\sqrt{\frac{e^{2Rt}-1}{2R}} + S(e^{Rt} - 1)$

(52) $zl = \frac{Y_a - \mu_Y}{\sigma_Y} = \frac{K - Se^{Rt}}{\sigma\sqrt{\frac{e^{2Rt}-1}{2R}}}$

Then splitting the integral of (48) into two parts and replacing the normal variable Y with the standard normal variable yields:

(53) $C = e^{-Rt}(Se^{Rt} - K)\int_{zl}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\left(\frac{-Z^2}{2}\right)}dZ + e^{-Rt}\sigma\sqrt{\frac{e^{2Rt}-1}{2R}}\int_{zl}^{\infty}Z\frac{1}{\sqrt{2\pi}}e^{\left(\frac{-Z^2}{2}\right)}dZ$

Taking the probability of the complementary normal distribution, that is 1-Nc(zl), the expression of the additive normal or absolute process option pricing formula is [14]

(54) $\boxed{C = (S-Ke^{-Rt})\,Nc\left[\frac{Se^{Rt} - K}{\sigma\sqrt{\frac{e^{2Rt}-1}{2R}}}\right] + e^{-Rt}\sigma\sqrt{\frac{e^{2Rt}-1}{2R}}\,N\left[\frac{Se^{Rt} - K}{\sigma\sqrt{\frac{e^{2Rt}-1}{2R}}}\right]}$

where
- S is the initial asset price,
- K is the exercise price,
- R is the continuously compounded annual riskless rate of interest,
- σ is the standard deviation of the asset price,
- t is the time to maturity period expressed in years,
- Nc[.] is the cumulative standard normal distribution,

- N[.] is the standard normal distribution.

4.3. The Additive Poisson Option pricing Formula

As in the multiplicative case, one can consider an additive jump process as a special case of the additive binomial process. The stochastic jump process can be written:

$$(55) \quad dS = S\mu dt + (k\text{-}1)d\pi$$

where μdt is the instantaneous drift, (k-1) the amplitude of the jump of the asset price, and $d\pi$ a continuous time Poisson process with probability λdt. The mean and variance of this process are $S\mu dt+\lambda(k\text{-}1)dt$ and $\lambda E[(k\text{-}1)^2]dt$.

This process is represented by a n-stage binomial process whose one stage for an infinitesimal subperiod is illustrated in figure 8. The expression of the upward and downward moves in the binomial tree is :

$$(56) \quad \Delta u = S\mu\Delta t + (k\text{-}1)$$

$$(57) \quad \Delta d = S\mu\Delta t$$

As for the risk adjusted probability p, it is:

$$(58) \quad p = \frac{SR\text{-}\Delta d}{\Delta u\text{-}\Delta d} = \frac{S\Delta R\ \text{-}S\Delta\mu\Delta t}{k\text{-}1} = \frac{S(e^{Rt}\text{-}1\text{-}\mu t)}{n(k\text{-}1)}$$

Its value converges to zero for large n. Therefore the cumulative probability of the binomial distribution can be replaced by the cumulative probability of the Poisson distribution with parameter λ.

$$(59) \quad \lambda = np = \frac{S(e^{Rt}\text{-}1\text{-}\mu t)}{k\text{-}1}$$

Hence the call option valuation formula is:

$$(60) \quad \boxed{C = e^{-Rt}(S - K + nd)\Psi[\alpha;\lambda] + \sum_{x=\alpha}^{n} e^{-Rt}\, x(u\text{-}d)\psi[x;\alpha]}$$

where ψ and Ψ are respectively the probability and the cumulative probability of the Poisson distribution, $\psi = (e^{-p}p^x)/x!$ and $\sum_{x=\alpha}^{n}(e^{-p}p^x)/x!)$, and α, the minimum of upward moves which gives the option a positive value, is equal to $(K\text{-}S\text{-}\mu t)/(k\text{-}1)$

6. Conclusion

It is shown in this paper that whenever asset prices movements conform to a two-state process, any option on this asset can be valued by arbitrage. It has also been demonstrated that under some appropriate assumptions concerning the probability distribution of the underlying asset price, simple option valuation formulae can be derived. As examples, option models were derived in the cases of the asset price probability distribution conforms to a multiplicative or to an additive process.

Another interesting feature of the two-state approach is that it also allows us to present simple proofs of option pricing formulae in continuous time such as, for example, the Black and Scholes and the absolute process models.

If one looks at option pricing formulae for both stochastic processes, one can observe that they are all determined by the same variables, that is the underlying asset price, the exercise price of the option, the time to maturity, the riskless interest rate and the standard deviation of the rates of return or return of the asset. All these formulae can easily be tested and in order to use them, one only needs to manipulate tables of the binomial and standard normal distributions and, in the case of the continuous models, to estimate the standard deviation.

Up to now most of the option pricing formulae generally assume a multiplicative process of the underlying asset price. The main argument, which is debatable, against the use of an additive process being that it gives a positive probability of negative prices to the asset.

What could then be the reasons for using an additive option valuation formula? The first is that it is appropriate in cases where the asset return is normal. A second reason is that it can be more appropriate as far as options on multivariate assets are concerned. Considering particularly the case of a portfolio which is a linear combination of assets, the addition property of the binomial and normal distribution means that if an additive model is relevant to each of the asset forming a portfolio, then it is also relevant to valuate calls on the portfolio.

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1 See, for example, A. Christie (1982).

2 $p_1 = \frac{SR-D_1}{U_1-D_1} = \frac{SR-Sd'}{Su'-Sd'} = \frac{R-d'}{u'-d'}$

$p_{21} = \frac{(S+U_1)R-D_{21}}{U_{21}-D_{21}} = \frac{(S+Su')R-(S+Su')d'}{(S+Su')u'-(S+Su')d'} = \frac{R-d'}{u'-d'}$

$p_{22} = \frac{(S+D_1)R-D_{22}}{U_{22}-D_{22}} = \frac{(S+Sd')R-(S+Sd')d'}{(S+Sd')u'-(S+Sd')d'} = \frac{R-d'}{u'-d'}$

3 Another easy way to derive the Black and Scholes model consists in considering it as a special limiting case of the Binomial Option Pricing Formula, splitting the whole period into an infinite number of subperiods and defining the upward and downward moves of the limiting distribution so as to keep the same mean and variance as the initial binomial distribution. This is the approach followed by Cox, Ross and Rubinstein (1979) and Rendleman and Bartter(1979).

4 The expression of the first term is obtained by incorporating e^Y into the exponent and by rearranging. See Rubinstein (1976).

5 $\hat{\mu} = n\left(p\frac{\Delta u}{S} + (1-p)\frac{\Delta d}{S}\right) = n\left(\frac{S\Delta R-\Delta d}{\Delta u-\Delta d}\frac{\Delta u}{S} + \frac{\Delta u-S\Delta R}{\Delta u-\Delta d}\frac{\Delta d}{S}\right) = n\Delta R = e^{Rt} - 1$

6 $\hat{\sigma}^2 = np(1-p)\left(\frac{1+\Delta u}{S} - \frac{1+\Delta d}{S}\right)^2 = n\frac{S\Delta R-\Delta d}{\Delta u-\Delta d}\frac{\Delta u-S\Delta R}{\Delta u-\Delta d}\left(\frac{\Delta u-\Delta d}{S}\right)^2$

$\hat{\sigma}^2 = \frac{n}{S^2}(S\Delta R\Delta u - S\Delta RS\Delta R - \Delta d\Delta u + \Delta dS\Delta R)$

Replacing Δu and Δd with $S\mu\Delta t+S\sigma\sqrt{\Delta t}$ and $S\mu\Delta t-S\sigma\sqrt{\Delta t}$,

$\hat{\sigma}^2 = n\left(\sigma^2\Delta t - (\Delta R-\mu\Delta t)^2\right) = n\left(\sigma^2\frac{t}{n} - \left(\frac{1}{n}(e^{Rt}-1) - \mu\frac{t}{n}\right)^2\right) = \sigma^2 t - \frac{1}{n}\left(e^{Rt} - 1 - \mu t\right)^2$

7 If X has a lognormal distribution and Y=ln(X), then Y is normally distributed and we have var[ln(X)]= σ_Y^2 . See Mood, Graybill and Boes (1985), page 117.

8 $\alpha = \frac{\log\left(K/S(1+d')^n\right)}{\log(1+u')-\log(1+d')} = \frac{\log(K/S) - \log\left((1+\mu t/n)^n\right)}{\log\left(\frac{\mu T/n + k}{1+\mu T/n}\right)} = \frac{\log(K/S)-\mu t}{\log(K)}$ for large n.

9 The asset price becomes negative after n periods if $S\text{-}nd'S<0$, that is, if $n>1/d'$

10 See Uhlenbeck and Ornstein (1930), or Cox and Miller (1985), page 226.

11 $E[\Delta S] = q\Delta u + (1-q)\Delta d = 1/2\,\Delta u + 1/2\,\Delta d = \frac{1}{n} S(e^{2\mu t}-1)$

$$\mathrm{var}[dS] = q(1-q)(\Delta u-\Delta d)^2 = 1/4\left(2\sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}}\right)^2 = \sigma^2\,\frac{e^{2\mu t}-1}{n2\mu}$$

12 $\mu_Y = n(p\Delta u + (1-p)\Delta d) = n\left(\frac{S\Delta R-\Delta d}{\Delta u-\Delta d}\Delta u + \frac{\Delta u-S\Delta R}{\Delta u-\Delta d}\Delta d\right) = nS\Delta R = S(e^{Rt}-1)$

$$\mu_Y = n(p\Delta u + (1-p)\Delta d) = n(1/2\,\Delta u + 1/2\,\Delta d) = S(e^{\mu t}-1)$$

13 $\sigma_Y^2 = np(1-p)(\Delta u-\Delta d)^2 = n(\Delta u-\Delta d)^2\,\frac{S\Delta R-\Delta d}{\Delta u-\Delta d}\,\frac{\Delta u-S\Delta R}{\Delta u-\Delta d}$

$$\sigma_Y^2 = n\left(S\Delta R\left(\frac{S}{n}(e^{\mu t}-1)+\sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}}\right) - S\Delta R\,S\Delta R - \left(\frac{S}{n}(e^{\mu t}-1)+\sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}}\right)\left(\frac{S}{n}(e^{\mu t}-1)-\sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}}\right) + S\Delta R\left(\frac{S}{n}(e^{\mu t}-1)-\sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}}\right)\right)$$

from (49) we have $S(e^{Rt}-1) = S(e^{\mu t}-1)$, therefore $\mu = R$ and $S\,\frac{1}{n}(e^{\mu t}-1) = S\Delta R$, hence

$$\sigma_Y^2 = n\left(S\Delta R\,S\Delta R + S\Delta R\,\sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}} - S\Delta R\,S\Delta R - S\Delta R\,S\Delta R + S\Delta R\,\sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}} - \sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}}\,S\Delta R + \sigma^2\,\frac{e^{2\mu t}-1}{n2\mu} + S\Delta R\,S\Delta R - S\Delta R\,\sigma\sqrt{\frac{e^{2\mu t}-1}{n2\mu}}\right)$$

$$\sigma_Y^2 = \sigma^2\left(\frac{e^{2Rt}-1}{2R}\right)$$

14 See Winkler R.L., Roodman G.M. and Britney R.R., 1972.

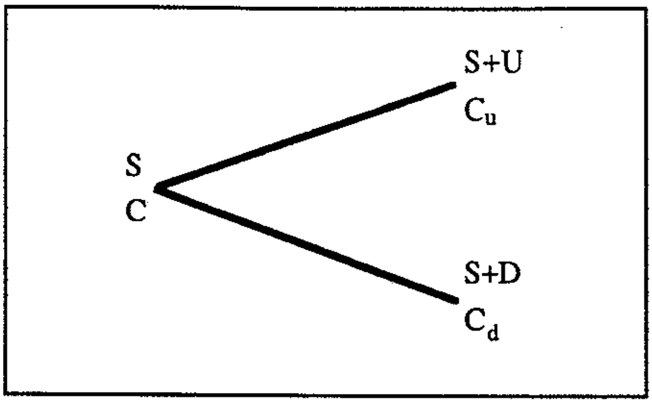

FIGURE 1—Two-state process for one period.

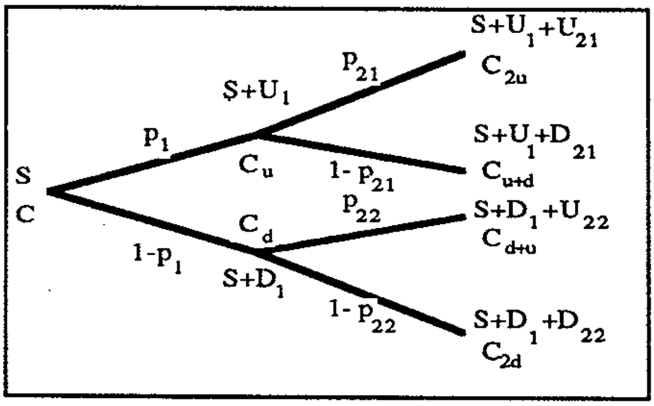

FIGURE 2—Two-state process for two periods.

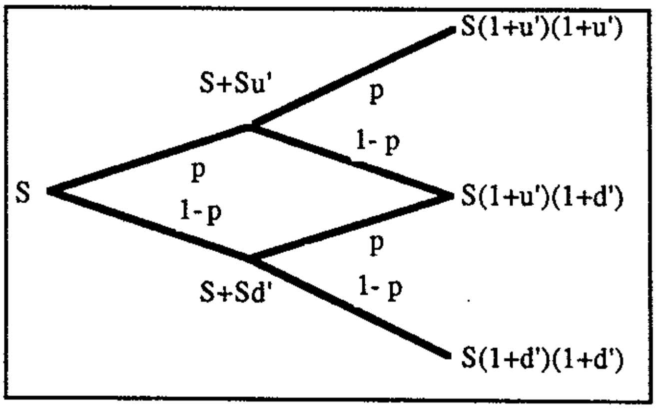

FIGURE 3—Log-binomial process.

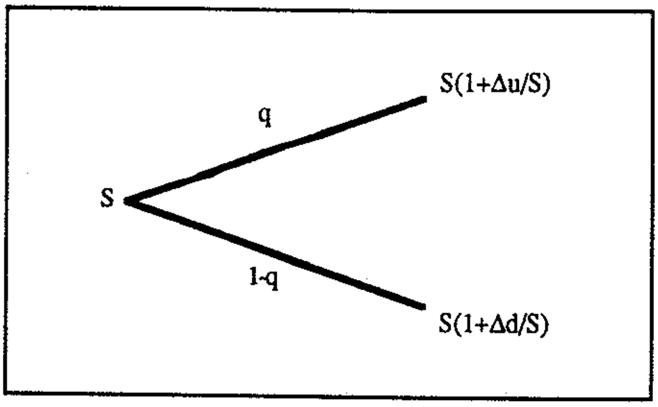

FIGURE 4 —Lognormal Process

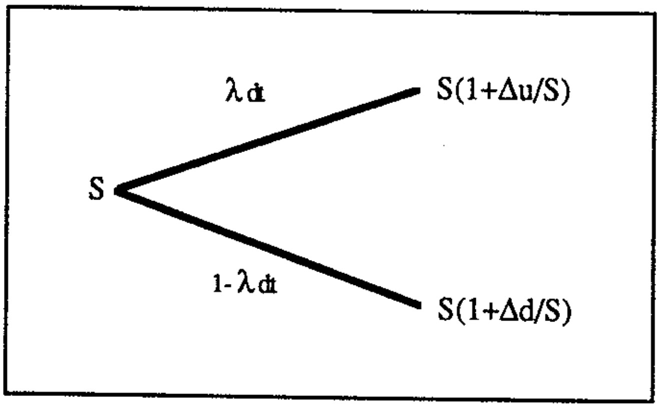

FIGURE 5—Log-Poisson Process

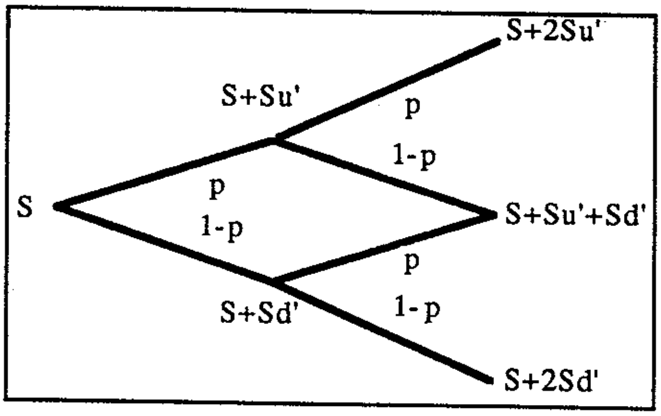

FIGURE 6—Additive Binomial Process

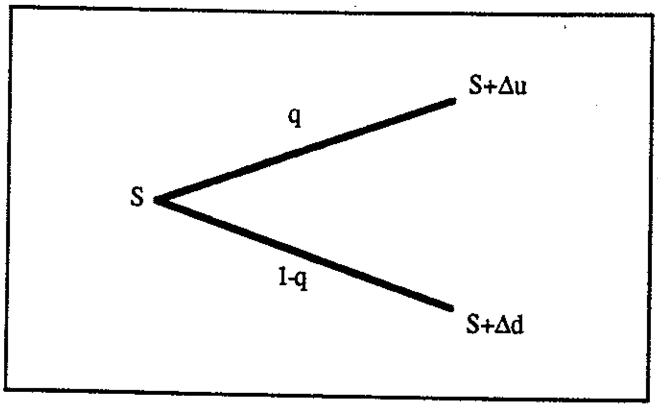

FIGURE 7—Normal or Absolute Process

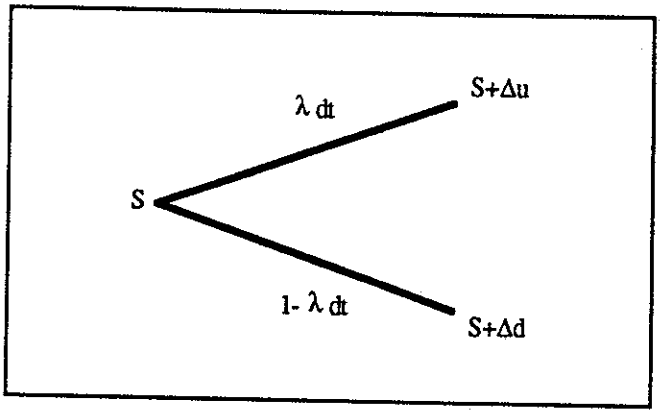

FIGURE 8—Additive Poisson Process