Complexity results and exact algorithms for robust knapsack problems

F. Talla Nobibon[1], R. Leus[2]

[1] QuantOM, HEC Liège, Rue Louvrex 14, B-4000 Liège, Belgium.
Fabrice.TallaNobibon@ulg.ac.be
[2] ORSTAT, K.U.Leuven, Naamsestraat 69, B-3000 Leuven, Belgium
Roel.Leus@econ.kuleuven.be

Mots-clés : *knapsack problem, complexity, dynamic programming, scenario-relaxation algorithm.*

1 Introduction

Many real-life problems can be modeled either as a knapsack problem or as one of its variants [4]. The outputs of these deterministic models, however, suffer from imprecisions that make their practical implementation almost impossible in some cases [5]. These imprecisions usually stem from the lack of full information about the parameters of the problem and/or the dependence of these parameters on some uncontrolled events [5]. In this paper we examine procedures for producing *robust* solutions to knapsack problems, meaning that the solutions are immune to data uncertainty [3]. For short, we will speak of *robust knapsack problems*. We consider the case where uncertainty can affect both the profits and the weights of the items. We investigate both *discrete scenarios* as well as *interval scenarios* – in the former case, the possible values for the profits and the weights are in a discrete set [5], while the latter case assumes the values to be in a given interval [1]. For evaluation of the quality of a solution, three different criteria are considered : the *absolute robustness criterion*, the *min-max regret criterion* and the *min-max relative regret criterion*.

Given is a set $N = \{1,\dots,n\}$ of items, a set S of scenarios affecting the items and a capacity b of the knapsack. Each scenario $s \in S$ is a $2n$-vector (V^s, A^s), where $V^s = (v_1^s,\dots,v_n^s)$ and $A^s = (a_1^s,\dots,a_n^s)$ and v_i^s (a_i^s) is the profit (the weight) of item i under scenario s. We assume that for every $s \in S$, $0 \le a_i^s \le b$ for $i = 1,\dots,n$ and $\sum_{i=1}^n a_i^s > b$. With each scenario $s \in S$ corresponds a knapsack problem KP$_s$ defined as : $\max\{F_s(X) = \sum_{i=1}^n v_i^s x_i : \sum_{i=1}^n a_i^s x_i \le b,\ x_i \in \{0,1\},\ i = 1,\dots,n\}$. Let F_s^* be the optimal objective value of KP$_s$. For $X = (x_1,\dots,x_n) \in \{0,1\}^n$ satisfying $\sum_{i=1}^n a_i^s x_i \le b$, the regret of X under the scenario s is the value $F_s^* - F_s(X)$ and the *maximum regret* $Z(X)$ is defined by : $Z(X) = \max\{F_s^* - F_s(X) : s \in S\}$. In what follows, we define $K = \{X \in \{0,1\}^n : \sum_{i=1}^n a_i^s x_i \le b, \forall s \in S\}$.

2 Absolute robustness

We define problem AbKP (with absolute robustness objective) as : $\max_{X\in K} \min_{s\in S} F_s(X)$. We observe that if S is a cartesian product, AbKP can be solved in pseudo-polynomial time by considering only the scenario $\bar{s}$ given by $v_i^{\bar{s}} = \min\{v_i^s : s \in S\}$ and $a_i^{\bar{s}} = \max\{a_i^s : s \in S\}$. Next, we assume that S is not a cartesian product. The case where uncertainty affects only the profits of items is well studied in the literature ; confer [1, 5]. When uncertainty affects only the weights of items and the size of S is bounded, AbKP is a special case of multi-dimensional knapsack problem [4], which can be solved in pseudo-polynomial time. We obtain the following results.

Proposition 1 *When uncertainty affects only the weights of items and S is bounded, the problem AbKP has a PTAS but does not have a FPTAS.*

Proposition 2 *The problem AbKP is strongly NP-hard for an unbounded scenario set S.*

For the general problem AbKP with uncertainty regarding both the weights and the profits, we describe a pseudo-polynomial-time dynamic programming (DP) algorithm for solving the problem [6]. Further, we prove that AbKP has a PTAS but not a FPTAS. When $|S|$ is unbounded we show that the problem is strongly NP-hard and there is no approximation scheme. Further, in [6] we propose a scenario-relaxation algorithm [2] for solving the problem. This algorithm first solves a restricted problem with few scenarios. Next, the set of scenarios for which the obtained solution violates either the feasibility or the optimality condition is identified. If that set is empty then we stop ; otherwise one or more scenarios from that set are added to the restricted problem and the procedure is repeated.

3 Min-max regret

The min-max regret knapsack problem RgKP is $\min_{X\in K} Z(X) = \min_{X\in K} \max_{s\in S}\{F_s^* - F_s(X)\}$. Results about the special case where uncertainty affects only the profits of items can be found in [1, 5]. When uncertainty affects only the weights of items, the problem is equivalent to AbKP with the same restriction. For the general problem RgKP, we find that if S is bounded then the problem can be solved in pseudo-polynomial time using a DP algorithm and there is no approximation scheme. When S is unbounded, the problem is strongly NP-hard and does not have an approximation scheme, and we also derive a scenario-relaxation algorithm.

4 Min-max relative regret

We define ReKP as : $\min_{X\in K} \max_{s\in S} \left\{ \frac{F_s^* - F_s(X)}{F_s^*} \right\}$. When uncertainty affects only the profits of items, we prove that ReKP is equivalent to AbKP, and can be solved in pseudo-polynomial time if S is bounded and is strongly NP-hard otherwise. Further, there are no approximation algorithms in either case. When uncertainty affects only the weights of items, ReKP reduces to AbKP. For the general case, if $|S|$ is bounded, the problem is solvable in pseudo-polynomial time using DP and is strongly NP-hard otherwise ; we also derive a scenario-relaxation algorithm.

5 Interval scenarios

In this case, the profit v_i (the weight a_i) of item i can take any value between a lower bound v_i^L (respectively a_i^L) and an upper bound v_i^U (respectively a_i^U). We prove that the robust knapsack problem with interval scenarios is equivalent to the classic knapsack problem for the first criterion, and to a discrete-scenarios problem for the other two objectives.

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