Natural and invariant quantizations

Fabian Radoux

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  - $Q(\{f, g\}) = \frac{i}{\hbar}[Q(f), Q(g)]$.

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$$Q(f) = \frac{\hbar}{i} X_f + f - \langle X_f, \alpha \rangle.$$

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- Geometric quantization Q_G: $Q_G = Q|_{\mathcal{A}}$.

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- Casimir operator corresponding to (V,β):

$$\sum_{i=1}^{n} \beta(u_i')\beta(u_i).$$

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- Generalization (F. Boniver, P. Mathonet): IFFT-algebras $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$

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- One adds terms whose order in p^*f is smaller than k ...
- One finds then:

$$Q_M(\nabla, S)(f) = {p^*}^{-1}(\sum_{l=0}^{k} C_{k,l} \langle Div^{\omega^l} p^*S, \nabla^{\omega^{k-l}}_s p^*f \rangle),$$

with

$$C_{k,l} = \frac{(\lambda + \frac{k-1}{m+1}) \cdots (\lambda + \frac{k-l}{m+1})}{\gamma_{2k-1} \cdots \gamma_{2k-l}} \begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1.$$

Other differential operators and conformal case (P. Mathonet, R.)

- "Flat" case: if S is a symbol, $S \in C^\infty(\mathbb{R}^m, \vee^k \mathbb{R}^m \otimes gl(V_1, V_2))$.

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  $C = -\frac{1}{2}\rho_*(\mathcal{E}) + \frac{1}{2m}\rho_*(\mathcal{E})^2 + \sum_j \rho_*(A_j)\rho_*(A_j^*)$,
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- If S is G_0-equivariant, $Q(S)$ is G_0-equivariant and $Q_\omega(Q(S))$ preserves the G_0-equivariance.

- Remark: this method allows to find natural maps

$$Q : \{\text{reductions of } P^2M \text{ to } H\} \to \{\text{quantizations on } M\},$$

where P^2M is the second-order frame bundle and where H is a Lie group corresponding to an IFFT-algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

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- Conformally invariant quantization (J. Silhan)

Explicit formulae

- Projective case, differential operators acting between densities:

$$Q(\nabla, S)(f) = \langle S, \nabla_s^{\omega^k} f\rangle$$
$$+\sum_{l=1}^{k} \frac{(\lambda + \frac{k-1}{m+1})\cdots(\lambda + \frac{k-l}{m+1})}{\gamma_{2k-1}\cdots\gamma_{2k-l}} \left(\begin{array}{c} k \\ l \end{array}\right) \langle Div^{\omega^l} S, \nabla_s^{\omega^{k-l}} f\rangle.$$

- Conformal case, differential operators acting between densities, trace-free symbols:

$$Q(g, S)(f) = \langle S, \nabla_s^{\omega^k} f\rangle$$
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Div^{\omega^l} S &\longrightarrow \pi_l\left(\sum_{j=0}^{l}(Div + T_2)^j\right)S, \\
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\nabla_s^{\omega^{k-l}} f &\longrightarrow \pi_{k-l}\Big(\sum_{j=0}^{k-l}(\nabla_s + T_1)^j\Big)f, \\
T_1|_{\Gamma(S^j T^*M)\otimes\mathcal{F}_\lambda(M)} &= (-\lambda(m+1)-j)(j+1)r \vee .
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- Explicit formulae for natural and conformally invariant quantizations thanks to tractor calculus (J. Silhan)

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- $W = \sum_\nu {\kappa_0}^k_{i_{\nu(1)} i_{\nu(2)} l} {\kappa_0}^l_{i_{\nu(3)} i_{\nu(4)} k} e_{i_{\nu(1)}} \vee e_{i_{\nu(2)}} \vee e_{i_{\nu(3)}} \vee e_{i_{\nu(4)}}$,

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- $\kappa(X, Y) = \Omega(\omega^{-1}(X), \omega^{-1}(Y))$,
- $\kappa_0 \in C^\infty(P, \wedge^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \otimes \mathfrak{g}_1)_H$,
- $W = \sum_\nu {\kappa_0}^k_{i_{\nu(1)} i_{\nu(2)} l} {\kappa_0}^l_{i_{\nu(3)} i_{\nu(4)} k} e_{i_{\nu(1)}} \vee e_{i_{\nu(2)}} \vee e_{i_{\nu(3)}} \vee e_{i_{\nu(4)}}$,
- $W \in C^\infty(P, \vee^4 \mathfrak{g}_{-1}^*)_H$,

Non-uniqueness of the quantizations

- $\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$,
- $\kappa(X, Y) = \Omega(\omega^{-1}(X), \omega^{-1}(Y))$,
- $\kappa_0 \in C^\infty(P, \wedge^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \otimes \mathfrak{g}_1)_H$,
- $W = \sum_\nu {\kappa_0}^k_{i_{\nu(1)} i_{\nu(2)} l} {\kappa_0}^l_{i_{\nu(3)} i_{\nu(4)} k} e_{i_{\nu(1)}} \vee e_{i_{\nu(2)}} \vee e_{i_{\nu(3)}} \vee e_{i_{\nu(4)}}$,
- $W \in C^\infty(P, \vee^4 \mathfrak{g}_{-1}^*)_H$,
- $S \mapsto \langle W, S \rangle$ natural and invariant map between $\mathcal{S}^k_\delta$ and $\mathcal{S}^{k-4}_\delta$.