Natural and projectively equivariant quantizations

Fabian Radoux

Nancy

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Introduction

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$$qp \to \frac{1}{2}(QP + PQ)$$

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- Quantization : $Q : \mathcal{S}(M) \mapsto \mathcal{D}_{\frac{1}{2}}(M)$

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- Method of the Casimir operator :

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"Curved" case :

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$$f\ G_0\text{-equivariant} \Rightarrow \nabla^\omega f$$

G_0-equivariant

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One finds then :

$$Q_M(\nabla, S)(f) = {p^*}^{-1}\left(\sum_{l=0}^k C_{k,l}\langle {Div^\omega}^l p^*S, \nabla^{\omega^{k-l}}_s p^*f\rangle\right),$$

with $C_{k,l} = \frac{(\lambda+\frac{k-1}{m+1})\cdots(\lambda+\frac{k-l}{m+1})}{\gamma_{2k-1}\cdots\gamma_{2k-l}}\begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1$

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Affine quantization Q_{Aff} :

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∂_i	$L_{\omega^{-1}(e_i)}$
Application γ :	Application γ :
$\mathcal{L}_{X^h} Q_{Aff}(S)(f) = Q_{Aff}((L_{X^h} + \gamma(h))S)(f)$	$\mathcal{L}_{h^*} Q_\omega(S)(f) = Q_\omega((L_{h^*} + \gamma(h))S)(f), \; h \in g_1$

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Casimirs C and $\mathcal{C}$ | "Casimirs" C^ω and $\mathcal{C}^\omega$

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Quantization :

$Q_{Aff}(Q(S))$, $Q(S)$ such that

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$\mathcal{L} \circ Q = Q \circ L$

because

$[\mathcal{C}, \mathcal{L}] = 0$ and $[C, L] = 0$

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Then :
$(L_{h^*} + \gamma(h)) \circ Q = Q \circ L_{h^*}$

because $[\mathcal{C}^\omega, L_{h^*} + \gamma(h)] = 0$

and $[C^\omega, L_{h^*}] = 0$

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if $C(S) = \alpha S$, then	if $C^\omega(S) = \alpha S$, then
$\mathcal{C}(Q(S)) = \alpha Q(S)$ and	$\mathcal{C}^\omega(Q(S)) = \alpha Q(S)$ and
"head" of $Q(S) = S$	"head" of $Q(S) = S$
Then : $\mathcal{L} \circ Q = Q \circ L$	Then : $(L_{h^*} + \gamma(h)) \circ Q = Q \circ L_{h^*}$
because	because $[\mathcal{C}^\omega, L_{h^*} + \gamma(h)] = 0$
$[\mathcal{C}, \mathcal{L}] = 0$ and $[C, L] = 0$	and $[C^\omega, L_{h^*}] = 0$
•Conclusion : "Flat" case ∂_i	"Curved" case $L_{\omega^{-1}(e_i)}$