Natural and projectively equivariant quantizations

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Introduction

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$$qp \to \frac{1}{2}(QP + PQ)$$

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- Quantization : $Q : \mathcal{S}(M) \mapsto \mathcal{D}_{\frac{1}{2}}(M)$

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  $\nexists Q : Q(L_X S) = L_X Q(S)$ $\forall$ vector field X

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- Method of the Casimir operator :

$$C : \mathcal{S}(\mathbb{R}^m) \mapsto \mathcal{S}(\mathbb{R}^m) \quad ; \quad \mathcal{C} : \mathcal{D}(\mathbb{R}^m) \mapsto \mathcal{D}(\mathbb{R}^m)$$

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$$\mathcal{C}(Q(S)) = \alpha Q(S) \text{ and } \sigma(Q(S)) = S$$

"Curved" case :

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One finds then :

$$Q_M(\nabla, S)(f) = {p^*}^{-1}\left(\sum_{l=0}^k C_{k,l} \langle Div^{\omega^l} p^*S, \nabla^{\omega^{k-l}}_s p^*f\rangle\right),$$

with $C_{k,l} = \dfrac{(\lambda + \frac{k-1}{m+1})\cdots(\lambda + \frac{k-l}{m+1})}{\gamma_{2k-1}\cdots\gamma_{2k-l}} \begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1$

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"Flat" case	"Curved" case
Affine quantization Q_{Aff} :	"Affine" quantization Q_ω :
∂_i	$L_{\omega^{-1}(e_i)}$
Application γ :	Application γ :
$\mathcal{L}_{X^h} Q_{Aff}(S)(f) =$	$\mathcal{L}_{h^*} Q_\omega(S)(f) =$
$Q_{Aff}((L_{X^h} + \gamma(h))S)(f)$	$Q_\omega((L_{h^*} + \gamma(h))S)(f),\ h \in g_1$

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Casimirs C and $\mathcal{C}$	"Casimirs" C^ω and $\mathcal{C}^\omega$
∂_i	$L_{\omega^{-1}(e_i)}$

Quantization :	Quantization :
$Q_{Aff}(Q(S))$, $Q(S)$ such that	$Q_\omega(Q(S))$, $Q(S)$ such that
if $C(S) = \alpha S$, then	if $C^\omega(S) = \alpha S$, then
$\mathcal{C}(Q(S)) = \alpha Q(S)$ and	$\mathcal{C}^\omega(Q(S)) = \alpha Q(S)$ and
"head" of $Q(S) = S$	"head" of $Q(S) = S$

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Then :
$\mathcal{L} \circ Q = Q \circ L$

because

$[\mathcal{C}, \mathcal{L}] = 0$ and $[C, L] = 0$

Quantization :

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if $C^\omega(S) = \alpha S$, then

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Then :
$(L_{h^*} + \gamma(h)) \circ Q = Q \circ L_{h^*}$

because $[\mathcal{C}^\omega, L_{h^*} + \gamma(h)] = 0$

and $[C^\omega, L_{h^*}] = 0$

Quantization :	Quantization :
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if $C(S)=\alpha S$, then	if $C^\omega(S)=\alpha S$, then
$\mathcal{C}(Q(S))=\alpha Q(S)$ and	$\mathcal{C}^\omega(Q(S))=\alpha Q(S)$ and
"head" of $Q(S)=S$	"head" of $Q(S)=S$
Then : $\mathcal{L}\circ Q=Q\circ L$	Then : $(L_{h^*}+\gamma(h))\circ Q=Q\circ L_{h^*}$
because $[\mathcal{C},\mathcal{L}]=0$ and $[C,L]=0$	because $[\mathcal{C}^\omega, L_{h^*}+\gamma(h)]=0$ and $[C^\omega, L_{h^*}]=0$
•Conclusion : "Flat" case ∂_i	"Curved" case $L_{\omega^{-1}(e_i)}$