

Natural and
projectively
equivariant
quantizations

Fabian Radoux

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Cartan fiber
bundles and
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The case of the
densities

Other differential
operators

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ULB

25 November 2006

Introduction

- At the origin : $qp \rightarrow QP$

$$qp \rightarrow \frac{1}{2}(QP + PQ)$$

with $P = \partial_x$; $Q = x$.

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Problems : It depends on the order

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It depends on the coordinates system

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- Quantization : $Q : \mathcal{S}(M) \mapsto \mathcal{D}_{\frac{1}{2}}(M)$

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- $\#Q : Q(\Phi^* S) = \Phi^* Q(S) \forall$ local diffeomorphism Φ
 $\#Q : Q(L_X S) = L_X Q(S) \forall$ vector field X

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- $\exists Q : L_X Q(S) = Q(L_X S) \forall X \in \mathfrak{sl}(m+1, \mathbb{R})$

- Method of the Casimir operator :

$$\mathcal{C} : \mathcal{S}(\mathbb{R}^m) \mapsto \mathcal{S}(\mathbb{R}^m) \quad ; \quad \mathcal{C} : \mathcal{D}(\mathbb{R}^m) \mapsto \mathcal{D}(\mathbb{R}^m)$$

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"Curved" case :

- $Q(\nabla) : \mathcal{S}^3(M) \mapsto \mathcal{D}^3(M)$

$$Q(\nabla) = Q(\nabla') \text{ if } \nabla' = \nabla + \alpha \vee id$$

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- Conjecture : $Q(\nabla) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$

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$$L_X Q(\nabla^0)(S) = Q(\nabla^0)(L_X S) \quad \forall X \in \mathfrak{sl}(m+1, \mathbb{R})$$

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- M. Bordemann method :

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- Questions :

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- Questions : Critical values of δ

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- Cartan fiber bundles and connections

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One finds then :

$$Q_M(\nabla, S)(f) = p^{*-1} \left(\sum_{l=0}^k C_{k,l} \langle \text{Div}^{\omega^l} p^* S, \nabla_s^{\omega^{k-l}} p^* f \rangle \right),$$

$$\text{with } C_{k,l} = \frac{(\lambda + \frac{k-1}{m+1}) \dots (\lambda + \frac{k-l}{m+1})}{\gamma_{2k-1} \dots \gamma_{2k-l}} \binom{k}{l}, \quad \forall l \geq 1, \quad C_{k,0} = 1$$

- Other differential operators (P. Mathonet, R.) : method of the Casimir operator :

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"Flat" case

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Affine quantization Q_{Aff} :
 ∂_i

"Curved" case

"Affine" quantization Q_ω :
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Affine quantization $Q_{Aff} :$
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Application $\gamma :$

$$\mathcal{L}_{X^h} Q_{Aff}(S)(f) = Q_{Aff}((L_{X^h} + \gamma(h))S)(f)$$

"Curved" case

"Affine" quantization $Q_\omega :$
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Quantization :

$Q_{Aff}(Q(S))$, $Q(S)$ such that

if $C(S) = \alpha S$, then

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Then :

$$\mathcal{L} \circ Q = Q \circ L$$

because

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$$(L_{h^*} + \gamma(h)) \circ Q = Q \circ L_{h^*}$$

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• Conclusion : "Flat" case
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"Curved" case
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