Natural equivariant quantizations

Fabian Radoux

12th November 2008

Introduction

- At the origin : $qp \to QP$

$$qp \to \frac{1}{2}(QP + PQ)$$

with $P = \partial_x$; $Q = x$.

Introduction

- At the origin : $qp \rightarrow QP$

$$qp \rightarrow \frac{1}{2}(QP + PQ)$$

with $P = \partial_x$; $Q = x$.

Problems : It depends on the order

Introduction

- At the origin : $qp \rightarrow QP$

$$qp \rightarrow \frac{1}{2}(QP + PQ)$$

with $P = \partial_x$; $Q = x$.

Problems : It depends on the order

It depends on the coordinates system

- Quantization : $Q : \mathcal{S}(M) \mapsto \mathcal{D}_{\frac{1}{2}}(M)$

- Quantization : $Q : \mathcal{S}(M) \mapsto \mathcal{D}_{\frac{1}{2}}(M)$

$$Q : \text{linear bijection}$$

- Quantization : $Q : \mathcal{S}(M) \mapsto \mathcal{D}_{\frac{1}{2}}(M)$

  Q : linear bijection

  preserves the principal symbol

- Quantization : $Q : \mathcal{S}(M) \mapsto \mathcal{D}_{\frac{1}{2}}(M)$

  Q : linear bijection

  preserves the principal symbol

- $\nexists Q : Q(\Phi^* S) = \Phi^* Q(S)\ \forall$ local diffeomorphism Φ

- Quantization : $Q : \mathcal{S}(M) \mapsto \mathcal{D}_{\frac{1}{2}}(M)$

  Q : linear bijection

  preserves the principal symbol

- $\nexists Q : Q(\Phi^* S) = \Phi^* Q(S)\ \forall$ local diffeomorphism Φ

  $\nexists Q : Q(L_X S) = L_X Q(S)\ \forall$ vector field X

"Flat" case :

- $\exists Q : Q(\Phi_g^* S) = \Phi_g^* Q(S) \; \forall g \in G$

"Flat" case :

- $\exists Q : Q(\Phi_g^* S) = \Phi_g^* Q(S) \; \forall g \in G$

  $\exists Q : Q(L_{h^*} S) = L_{h^*} Q(S) \; \forall h \in \mathfrak{g}$

"Flat" case :

- $\exists Q : Q(\Phi_g^* S) = \Phi_g^* Q(S)\ \forall g \in G$

  $\exists Q : Q(L_{h^*} S) = L_{h^*} Q(S)\ \forall h \in \mathfrak{g}$

- $PGL(m+1, \mathbb{R})$ acts on $\mathbb{R}P^m$

"Flat" case :

- $\exists Q : Q(\Phi_g^* S) = \Phi_g^* Q(S) \; \forall g \in G$

  $\exists Q : Q(L_{h^*} S) = L_{h^*} Q(S) \; \forall h \in \mathfrak{g}$

- $PGL(m+1, \mathbb{R})$ acts on $\mathbb{R}P^m$

  $$X \in sl(m+1, \mathbb{R}) \mapsto X^* \text{ vector field on } \mathbb{R}^m$$

"Flat" case :

- $\exists Q : Q(\Phi_g^* S) = \Phi_g^* Q(S) \ \forall g \in G$

  $\exists Q : Q(L_{h^*} S) = L_{h^*} Q(S) \ \forall h \in \mathfrak{g}$

- $PGL(m+1, \mathbb{R})$ acts on $\mathbb{R}P^m$

  $X \in sl(m+1, \mathbb{R}) \mapsto X^*$ vector field on $\mathbb{R}^m$

  $\exists Q : L_X Q(S) = Q(L_X S) \ \forall X \in sl(m+1, \mathbb{R})$

"Flat" case :

- $\exists Q : Q(\Phi_g^* S) = \Phi_g^* Q(S) \; \forall g \in G$

  $\exists Q : Q(L_{h^*} S) = L_{h^*} Q(S) \; \forall h \in \mathfrak{g}$

- $PGL(m+1, \mathbb{R})$ acts on $\mathbb{R}P^m$

  $X \in sl(m+1, \mathbb{R}) \mapsto X^*$ vector field on $\mathbb{R}^m$

  $\exists Q : L_X Q(S) = Q(L_X S) \; \forall X \in sl(m+1, \mathbb{R})$

- $SO(p+1, q+1)$ acts on $S^p \times S^q$

"Flat" case :

- $\exists Q : Q(\Phi_g^* S) = \Phi_g^* Q(S)\ \forall g \in G$

  $\exists Q : Q(L_{h^*} S) = L_{h^*} Q(S)\ \forall h \in \mathfrak{g}$

- $PGL(m+1, \mathbb{R})$ acts on $\mathbb{R}P^m$

  $X \in sl(m+1, \mathbb{R}) \mapsto X^*$ vector field on $\mathbb{R}^m$

  $\exists Q : L_X Q(S) = Q(L_X S)\ \forall X \in sl(m+1, \mathbb{R})$

- $SO(p+1, q+1)$ acts on $S^p \times S^q$

  $X \in so(p+1, q+1) \mapsto X^*$ vector field on $\mathbb{R}^{p+q}$

"Flat" case :

- $\exists Q : Q(\Phi_g^* S) = \Phi_g^* Q(S)\ \forall g \in G$

  $\exists Q : Q(L_{h^*} S) = L_{h^*} Q(S)\ \forall h \in \mathfrak{g}$

- $PGL(m+1, \mathbb{R})$ acts on $\mathbb{R}P^m$

  $X \in sl(m+1, \mathbb{R}) \mapsto X^*$ vector field on $\mathbb{R}^m$

  $\exists Q : L_X Q(S) = Q(L_X S)\ \forall X \in sl(m+1, \mathbb{R})$

- $SO(p+1, q+1)$ acts on $S^p \times S^q$

  $X \in so(p+1, q+1) \mapsto X^*$ vector field on $\mathbb{R}^{p+q}$

  $\exists Q : L_X Q(S) = Q(L_X S)\ \forall X \in so(p+1, q+1)$

- Method of the Casimir operator :

$$C : \mathcal{S}(\mathbb{R}^m) \mapsto \mathcal{S}(\mathbb{R}^m) \quad ; \quad \mathcal{C} : \mathcal{D}(\mathbb{R}^m) \mapsto \mathcal{D}(\mathbb{R}^m)$$

- Method of the Casimir operator :

$$C : \mathcal{S}(\mathbb{R}^m) \mapsto \mathcal{S}(\mathbb{R}^m) \quad ; \quad \mathcal{C} : \mathcal{D}(\mathbb{R}^m) \mapsto \mathcal{D}(\mathbb{R}^m)$$

Condition : $\exists Q$ such that if $C(S) = \alpha S$, then

- Method of the Casimir operator :

$$C : \mathcal{S}(\mathbb{R}^m) \mapsto \mathcal{S}(\mathbb{R}^m) \quad ; \quad \mathcal{C} : \mathcal{D}(\mathbb{R}^m) \mapsto \mathcal{D}(\mathbb{R}^m)$$

Condition : $\exists Q$ such that if $C(S) = \alpha S$, then

$$\mathcal{C}(Q(S)) = \alpha Q(S)$$

- Method of the Casimir operator :

$$C : \mathcal{S}(\mathbb{R}^m) \mapsto \mathcal{S}(\mathbb{R}^m) \quad ; \quad \mathcal{C} : \mathcal{D}(\mathbb{R}^m) \mapsto \mathcal{D}(\mathbb{R}^m)$$

Condition : $\exists Q$ such that if $C(S) = \alpha S$, then

$$\mathcal{C}(Q(S)) = \alpha Q(S) \text{ and } \sigma(Q(S)) = S$$

"Curved" case :

$\bullet Q(\nabla) : \mathcal{S}^3(M) \mapsto \mathcal{D}^3(M)$

$$Q(\nabla) = Q(\nabla') \text{ if } \nabla' = \nabla + \alpha \vee id$$

"Curved" case :

$\bullet Q(\nabla) : \mathcal{S}^3(M) \mapsto \mathcal{D}^3(M)$

$$Q(\nabla) = Q(\nabla') \text{ if } \nabla' = \nabla + \alpha \vee id$$

$\bullet Q(g) : \mathcal{S}^2(M) \mapsto \mathcal{D}^2(M)$

$$Q(g) = Q(g') \text{ if } g = fg' \text{ with } f > 0$$

- Conjecture : $Q(g) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$

- Conjecture : $Q(g) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural

- Conjecture : $Q(g) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

  $$Q(g) = Q(g') \text{ if } g = fg' \text{ with } f > 0$$

- Conjecture : $Q(g) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

$$Q(g) = Q(g') \text{ if } g = fg' \text{ with } f > 0$$

$$\varphi_t^* Q(g^0)(S) = Q(\varphi_t^* g^0)(\varphi_t^* S)$$

- Conjecture : $Q(g) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

$$Q(g) = Q(g') \text{ if } g = fg' \text{ with } f > 0$$

$$\varphi_t^* Q(g^0)(S) = Q(\varphi_t^* g^0)(\varphi_t^* S)$$

$$L_X Q(g^0)(S) = Q(g^0)(L_X S)\ \forall X \in so(p+1, q+1)$$

- Conjecture : $Q(g) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

$$Q(g) = Q(g') \text{ if } g = fg' \text{ with } f > 0$$

$$\varphi_t^* Q(g^0)(S) = Q(\varphi_t^* g^0)(\varphi_t^* S)$$

$$L_X Q(g^0)(S) = Q(g^0)(L_X S) \; \forall X \in so(p+1, q+1)$$

- Conjecture : $Q(\nabla) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$

- Conjecture : $Q(g) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

$$Q(g) = Q(g') \text{ if } g = fg' \text{ with } f > 0$$

$$\varphi_t^* Q(g^0)(S) = Q(\varphi_t^* g^0)(\varphi_t^* S)$$

$$L_X Q(g^0)(S) = Q(g^0)(L_X S) \; \forall X \in so(p+1, q+1)$$

- Conjecture : $Q(\nabla) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural

- Conjecture : $Q(g) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

$$Q(g) = Q(g') \text{ if } g = fg' \text{ with } f > 0$$

$$\varphi_t^* Q(g^0)(S) = Q(\varphi_t^* g^0)(\varphi_t^* S)$$

$$L_X Q(g^0)(S) = Q(g^0)(L_X S)\ \forall X \in so(p+1, q+1)$$

- Conjecture : $Q(\nabla) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

$$Q(\nabla) = Q(\nabla') \text{ if } \nabla' = \nabla + \alpha \vee id$$

- Conjecture : $Q(g) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

$$Q(g) = Q(g') \text{ if } g = fg' \text{ with } f > 0$$

$$\varphi_t^* Q(g^0)(S) = Q(\varphi_t^* g^0)(\varphi_t^* S)$$

$$L_X Q(g^0)(S) = Q(g^0)(L_X S) \ \forall X \in so(p+1, q+1)$$

- Conjecture : $Q(\nabla) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

$$Q(\nabla) = Q(\nabla') \text{ if } \nabla' = \nabla + \alpha \vee id$$

$$\phi_t^* Q(\nabla^0)(S) = Q(\phi_t^* \nabla^0)(\phi_t^* S)$$

- Conjecture : $Q(g) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

$$Q(g) = Q(g') \text{ if } g = fg' \text{ with } f > 0$$

$$\varphi_t^* Q(g^0)(S) = Q(\varphi_t^* g^0)(\varphi_t^* S)$$

$$L_X Q(g^0)(S) = Q(g^0)(L_X S) \ \forall X \in so(p+1, q+1)$$

- Conjecture : $Q(\nabla) : \mathcal{S}(M) \mapsto \mathcal{D}(M)$ natural and

$$Q(\nabla) = Q(\nabla') \text{ if } \nabla' = \nabla + \alpha \vee id$$

$$\phi_t^* Q(\nabla^0)(S) = Q(\phi_t^* \nabla^0)(\phi_t^* S)$$

$$L_X Q(\nabla^0)(S) = Q(\nabla^0)(L_X S) \ \forall X \in sl(m+1, \mathbb{R})$$

•M. Bordemann method :

•M. Bordemann method : $M \mapsto \tilde{M}$;

•M. Bordemann method : $M \mapsto \tilde{M}$;

$$\nabla \mapsto \tilde{\nabla};$$

•M. Bordemann method : $M \mapsto \tilde{M}$;

$$\nabla \mapsto \tilde{\nabla};$$

$$S \mapsto \tilde{S};$$

•M. Bordemann method : $M \mapsto \tilde{M}$;

$$\nabla \mapsto \tilde{\nabla};$$

$$S \mapsto \tilde{S};$$

$$Q(\widetilde{\nabla)(S})(f) = \tau(\tilde{\nabla})(\tilde{S})(\tilde{f})$$

- Questions :

- Questions : Critical values of δ

- Questions : Critical values of δ

  Explicit formula

- Questions : Critical values of δ

  Explicit formula

  Uniqueness

- Questions : Critical values of δ

  Explicit formula

  Uniqueness

  Other differential operators

- Questions : Critical values of δ

  Explicit formula

  Uniqueness

  Other differential operators

  Conformal case

•Cartan fiber bundles and connections

•Cartan fiber bundles and connections

$$[\nabla] \to P \mapsto M$$

• Cartan fiber bundles and connections

$$[\nabla] \to P \mapsto M$$

$$[\nabla] \to \omega : TP \mapsto \mathfrak{g}$$

•Cartan fiber bundles and connections

$$[\nabla] \to P \mapsto M$$

$$[\nabla] \to \omega : TP \mapsto \mathfrak{g}$$

$$\omega_u : T_uP \mapsto \mathfrak{g} \text{ bijection } \forall u \in P$$

•Cartan fiber bundles and connections

$$[\nabla] \to P \mapsto M$$

$$[\nabla] \to \omega : TP \mapsto \mathfrak{g}$$

$$\omega_u : T_uP \mapsto \mathfrak{g} \text{ bijection } \forall u \in P$$

$$T \to p^*T \in \mathcal{C}^\infty(P, V)_H$$

•Cartan fiber bundles and connections

$$[\nabla] \to P \mapsto M$$

$$[\nabla] \to \omega : TP \mapsto \mathfrak{g}$$

$$\omega_u : T_uP \mapsto \mathfrak{g} \text{ bijection } \forall u \in P$$

$$T \to p^*T \in \mathcal{C}^\infty(P, V)_H$$

$$\text{with } H = G_0 \rtimes G_1,\ \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

•Cartan fiber bundles and connections

$$[\nabla] \to P \mapsto M$$

$$[\nabla] \to \omega : TP \mapsto \mathfrak{g}$$

$$\omega_u : T_uP \mapsto \mathfrak{g} \text{ bijection } \forall u \in P$$

$$T \to p^*T \in \mathcal{C}^\infty(P, V)_H$$

$$\text{with } H = G_0 \rtimes G_1, \ \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{h}, \ \mathfrak{g}_{-1} \cong \mathbb{R}^m$$

•Cartan fiber bundles and connections

$$[\nabla] \rightarrow P \mapsto M$$

$$[\nabla] \rightarrow \omega : TP \mapsto \mathfrak{g}$$

$$\omega_u : T_u P \mapsto \mathfrak{g} \text{ bijection } \forall u \in P$$

$$T \rightarrow p^* T \in \mathcal{C}^\infty(P, V)_H$$

$$\text{with } H = G_0 \rtimes G_1, \ \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{h}, \ \mathfrak{g}_{-1} \cong \mathbb{R}^m$$

$$\omega \rightarrow \nabla^\omega(e_i) = L_{\omega^{-1}(e_i)}, \ e_i \in \mathfrak{g}_{-1}$$

•Cartan fiber bundles and connections

$$[\nabla] \to P \mapsto M$$

$$[\nabla] \to \omega : TP \mapsto \mathfrak{g}$$

$$\omega_u : T_uP \mapsto \mathfrak{g} \text{ bijection } \forall u \in P$$

$$T \to p^*T \in \mathcal{C}^\infty(P, V)_H$$

$$\text{with } H = G_0 \rtimes G_1,\ \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{h},\ \mathfrak{g}_{-1} \cong \mathbb{R}^m$$

$$\omega \to \nabla^\omega(e_i) = L_{\omega^{-1}(e_i)},\ e_i \in \mathfrak{g}_{-1}$$

$$f\ G_0\text{-equivariant} \Rightarrow \nabla^\omega f$$

G_0-equivariant

•Cartan fiber bundles and connections

$$[\nabla] \to P \mapsto M$$

$$[\nabla] \to \omega : TP \mapsto \mathfrak{g}$$

$$\omega_u : T_u P \mapsto \mathfrak{g} \text{ bijection } \forall u \in P$$

$$T \to p^* T \in \mathcal{C}^\infty(P, V)_H$$

$$\text{with } H = G_0 \rtimes G_1,\ \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{h},\ \mathfrak{g}_{-1} \cong \mathbb{R}^m$$

$$\omega \to \nabla^\omega(e_i) = L_{\omega^{-1}(e_i)},\ e_i \in \mathfrak{g}_{-1}$$

$$f \ G_0\text{-equivariant} \Rightarrow \nabla^\omega f$$

G_0-equivariant

$$f \ G_1\text{-equivariant} \not\Rightarrow \nabla^\omega f$$

G_1-equivariant

•The case of the densities (P. Mathonet, R.) :

•The case of the densities (P. Mathonet, R.) :

$$S \to p^*S \in \mathcal{C}^\infty(P, S^k_\delta(\mathbb{R}^m))_H$$

•The case of the densities (P. Mathonet, R.) :

$$S \to p^*S \in \mathcal{C}^\infty(P, S^k_\delta(\mathbb{R}^m))_H$$

$$f \to p^*f \in \mathcal{C}^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$$

•The case of the densities (P. Mathonet, R.) :

$$S \to p^*S \in \mathcal{C}^\infty(P, S^k_\delta(\mathbb{R}^m))_H$$

$$f \to p^*f \in \mathcal{C}^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$$

$$\omega \to Div^\omega = \sum_i i(\epsilon^i) L_{\omega^{-1}(e_i)}$$

•The case of the densities (P. Mathonet, R.) :

$$S \to p^*S \in \mathcal{C}^\infty(P, S^k_\delta(\mathbb{R}^m))_H$$

$$f \to p^*f \in \mathcal{C}^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$$

$$\omega \to Div^\omega = \sum_i i(\epsilon^i) L_{\omega^{-1}(e_i)}$$

Condition : $L_{h^*} Q(p^*S)(p^*f) = 0 \; \forall h \in \mathfrak{g}_1$

•The case of the densities (P. Mathonet, R.) :

$$S \to p^*S \in \mathcal{C}^\infty(P, S^k_\delta(\mathbb{R}^m))_H$$

$$f \to p^*f \in \mathcal{C}^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$$

$$\omega \to Div^\omega = \sum_i i(\epsilon^i)L_{\omega^{-1}(e_i)}$$

Condition : $L_{h^*}Q(p^*S)(p^*f) = 0 \; \forall h \in \mathfrak{g}_1$

$\langle p^*S, \nabla_s^{\omega^k} p^*f\rangle$ not G_1-equivariant !

•The case of the densities (P. Mathonet, R.) :

$$S \to p^*S \in \mathcal{C}^\infty(P, S^k_\delta(\mathbb{R}^m))_H$$

$$f \to p^*f \in \mathcal{C}^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$$

$$\omega \to Div^\omega = \textstyle\sum_i i(\epsilon^i) L_{\omega^{-1}(e_i)}$$

Condition : $L_{h^*} Q(p^*S)(p^*f) = 0 \; \forall h \in \mathfrak{g}_1$

$\langle p^*S, \nabla^{\omega^k}_s p^*f \rangle$ not G_1-equivariant !

One adds terms of lower orders in p^*f...

•The case of the densities (P. Mathonet, R.) :

$$S \rightarrow p^*S \in \mathcal{C}^\infty(P, S^k_\delta(\mathbb{R}^m))_H$$

$$f \rightarrow p^*f \in \mathcal{C}^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$$

$$\omega \rightarrow Div^\omega = \sum_i i(\epsilon^i) L_{\omega^{-1}(e_i)}$$

Condition : $L_{h^*} Q(p^*S)(p^*f) = 0 \ \forall h \in \mathfrak{g}_1$

$\langle p^*S, \nabla^{\omega^k}_s p^*f \rangle$ not G_1-equivariant !

One adds terms of lower orders in p^*f...

One finds then :

$$Q_M(\nabla, S)(f) = {p^*}^{-1}\left(\sum_{l=0}^k C_{k,l} \langle Div^{\omega^l} p^*S, \nabla^{\omega^{k-l}}_s p^*f \rangle\right),$$

with $C_{k,l} = \frac{(\lambda + \frac{k-1}{m+1}) \cdots (\lambda + \frac{k-l}{m+1})}{\gamma_{2k-1} \cdots \gamma_{2k-l}} \begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1$

•Other differential operators (P. Mathonet, R.) : method of the Casimir operator :

•Other differential operators (P. Mathonet, R.) : method of the Casimir operator :

"Flat" case

"Curved" case

•Other differential operators (P. Mathonet, R.) : method of the Casimir operator :

"Flat" case	"Curved" case
Affine quantization Q_{Aff} :	"Affine" quantization Q_ω :
∂_i	$L_{\omega^{-1}(e_i)}$

•Other differential operators (P. Mathonet, R.) : method of the Casimir operator :

"Flat" case	"Curved" case
Affine quantization Q_{Aff} :	"Affine" quantization Q_ω :
∂_i	$L_{\omega^{-1}(e_i)}$
Application γ :	Application γ :
$\mathcal{L}_{X^h} Q_{Aff}(S)(f) = Q_{Aff}((L_{X^h} + \gamma(h))S)(f)$	$\mathcal{L}_{h^*} Q_\omega(S)(f) = Q_\omega((L_{h^*} + \gamma(h))S)(f), \; h \in \mathfrak{g}_1$

•Other differential operators (P. Mathonet, R.) : method of the Casimir operator :

"Flat" case

Affine quantization Q_{Aff} :

∂_i

Application γ :

$\mathcal{L}_{X^h} Q_{Aff}(S)(f) = Q_{Aff}((L_{X^h} + \gamma(h))S)(f)$

Casimirs C and $\mathcal{C}$

∂_i

"Curved" case

"Affine" quantization Q_ω :

$L_{\omega^{-1}(e_i)}$

Application γ :

$\mathcal{L}_{h^*} Q_\omega(S)(f) = Q_\omega((L_{h^*} + \gamma(h))S)(f), \; h \in g_1$

"Casimirs" C^ω and $\mathcal{C}^\omega$

$L_{\omega^{-1}(e_i)}$

Quantization :

$Q_{Aff}(Q(S))$, $Q(S)$ such that

if $C(S) = \alpha S$, then

$\mathcal{C}(Q(S)) = \alpha Q(S)$ and

"head" of $Q(S) = S$

Quantization :

$Q_\omega(Q(S))$, $Q(S)$ such that

if $C^\omega(S) = \alpha S$, then

$\mathcal{C}^\omega(Q(S)) = \alpha Q(S)$ and

"head" of $Q(S) = S$

Quantization :

$Q_{Aff}(Q(S))$, $Q(S)$ such that

if $C(S) = \alpha S$, then

$\mathcal{C}(Q(S)) = \alpha Q(S)$ and

"head" of $Q(S) = S$

Then :
$\mathcal{L} \circ Q = Q \circ L$

because

$[\mathcal{C}, \mathcal{L}] = 0$ and $[C, L] = 0$

Quantization :

$Q_{\omega}(Q(S))$, $Q(S)$ such that

if $C^{\omega}(S) = \alpha S$, then

$\mathcal{C}^{\omega}(Q(S)) = \alpha Q(S)$ and

"head" of $Q(S) = S$

Then :
$(L_{h^*} + \gamma(h)) \circ Q = Q \circ L_{h^*}$

because $[\mathcal{C}^{\omega}, L_{h^*} + \gamma(h)] = 0$

and $[C^{\omega}, L_{h^*}] = 0$

Quantization :	Quantization :
$Q_{Aff}(Q(S))$, $Q(S)$ such that	$Q_\omega(Q(S))$, $Q(S)$ such that
if $C(S) = \alpha S$, then	if $C^\omega(S) = \alpha S$, then
$\mathcal{C}(Q(S)) = \alpha Q(S)$ and	$\mathcal{C}^\omega(Q(S)) = \alpha Q(S)$ and
"head" of $Q(S) = S$	"head" of $Q(S) = S$
Then :	Then :
$\mathcal{L} \circ Q = Q \circ L$	$(L_{h^*} + \gamma(h)) \circ Q = Q \circ L_{h^*}$
because	because $[\mathcal{C}^\omega, L_{h^*} + \gamma(h)] = 0$
$[\mathcal{C}, \mathcal{L}] = 0$ and $[C, L] = 0$	and $[C^\omega, L_{h^*}] = 0$
•Conclusion : "Flat" case	"Curved" case
∂_i	$L_{\omega^{-1}(e_i)}$

•Conformal case (P. Mathonet, R.) :

•Conformal case (P. Mathonet, R.) :

"Affine" quantization Q'_ω : Q_ω "not symmetrized"

•Conformal case (P. Mathonet, R.) :

"Affine" quantization Q'_ω : Q_ω "not symmetrized"

Application γ' : γ "not symmetric"

•Conformal case (P. Mathonet, R.) :

"Affine" quantization Q'_ω : Q_ω "not symmetrized"

Application γ' : γ "not symmetric"

$\mathcal{L}_{h^*} Q'_\omega(S)(f) = Q'_\omega((L_{h^*} + \gamma'(h))S)(f),\ h \in g_1$

•Conformal case (P. Mathonet, R.) :

"Affine" quantization Q'_ω : Q_ω "not symmetrized"

Application γ' : γ "not symmetric"

$\mathcal{L}_{h^*} Q'_\omega(S)(f) = Q'_\omega((L_{h^*} + \gamma'(h))S)(f), \; h \in g_1$

"Casimirs" $C^{\omega\prime}$ " = " C^ω and $\mathcal{C}^{\omega\prime}$

•Conformal case (P. Mathonet, R.) :

"Affine" quantization Q'_ω : Q_ω "not symmetrized"

Application γ' : γ "not symmetric"

$\mathcal{L}_{h^*} Q'_\omega(S)(f) = Q'_\omega((L_{h^*} + \gamma'(h))S)(f), \; h \in g_1$

"Casimirs" $C^{\omega\prime}$ " = " C^ω and $\mathcal{C}^{\omega\prime}$

Quantization : $Q'_\omega(Q'(S)), \; Q'(S)$ such that if $C'^\omega(S) = \alpha S$, then $\mathcal{C}^{\omega\prime}(Q'(S)) = \alpha Q'(S)$ and "head" of $Q'(S) = S$

$\bullet$Conformal case (P. Mathonet, R.) :

"Affine" quantization Q'_ω : Q_ω "not symmetrized"

Application γ' : γ "not symmetric"

$\mathcal{L}_{h^*} Q'_\omega(S)(f) = Q'_\omega((L_{h^*} + \gamma'(h))S)(f), \; h \in g_1$

"Casimirs" $C^{\omega\prime}$ " $=$ " C^ω and $\mathcal{C}^{\omega\prime}$

Quantization : $Q'_\omega(Q'(S)), \; Q'(S)$ such that if $C'^\omega(S) = \alpha S$, then $\mathcal{C}^{\omega\prime}(Q'(S)) = \alpha Q'(S)$ and "head" of $Q'(S) = S$

Then : $(L_{h^*} + \gamma'(h)) \circ Q' = Q' \circ L_{h^*}$ because $[\mathcal{C}^{\omega\prime}, L_{h^*} + \gamma'(h)] = 0$ and $[C^{\omega\prime}, L_{h^*}] = 0$

Remark : allows to find natural applications

$$Q : \{\mathrm{reductions\,of\,P^2M\,to\,H}\} \to \{\mathrm{quantizations\,on\,M}\},$$

where P^2M is the second order frame bundle and where H is a group linked to an AHS structure

•Quantizations and foliations (N. Poncin, R. Wolak, R.) :

•Quantize $M/G \leftrightarrow$ Quantize $M/\bar{\mathcal{F}}$

•Quantizations and foliations (N. Poncin, R. Wolak, R.) :

•Quantize $M/G \leftrightarrow$ Quantize $M/\bar{\mathcal{F}}$

$$\begin{array}{rccc} Q_M(\nabla_M)(S_M) : & M & \longrightarrow & M \\ & \pi \downarrow & & \downarrow \pi \\ Q_{M/\bar{\mathcal{F}}}(\nabla_{M/\bar{\mathcal{F}}})(S_{M/\bar{\mathcal{F}}}) : & M/\bar{\mathcal{F}} & \longrightarrow & M/\bar{\mathcal{F}} \end{array}$$

•Quantizations and foliations (N. Poncin, R. Wolak, R.) :

•Quantize $M/G \leftrightarrow$ Quantize $M/\bar{\mathcal{F}}$

$$\begin{array}{rcl} Q_M(\nabla_M)(S_M) : M & \longrightarrow & M \\ \pi \downarrow & & \downarrow \pi \\ Q_{M/\bar{\mathcal{F}}}(\nabla_{M/\bar{\mathcal{F}}})(S_{M/\bar{\mathcal{F}}}) : M/\bar{\mathcal{F}} & \longrightarrow & M/\bar{\mathcal{F}} \end{array}$$

Quantization commutes with reduction :

$$\pi(Q_M(\nabla_M)(S_M)) = Q_{M/\bar{\mathcal{F}}}(\pi\nabla_M)(\pi S_M)$$

•Quantizations and foliations (N. Poncin, R. Wolak, R.) :

•Quantize $M/G \leftrightarrow$ Quantize $M/\bar{\mathcal{F}}$

$$\begin{array}{rcl} Q_M(\nabla_M)(S_M) : M & \longrightarrow & M \\ \pi \downarrow & & \downarrow \pi \\ Q_{M/\bar{\mathcal{F}}}(\nabla_{M/\bar{\mathcal{F}}})(S_{M/\bar{\mathcal{F}}}) : M/\bar{\mathcal{F}} & \longrightarrow & M/\bar{\mathcal{F}} \end{array}$$

Quantization commutes with reduction :

$\pi(Q_M(\nabla_M)(S_M)) = Q_{M/\bar{\mathcal{F}}}(\pi\nabla_M)(\pi S_M)$

∇_M, S_M, $Q_M(\nabla_M)(S_M) \leftrightarrow$ adapted objects

•Quantizations and foliations (N. Poncin, R. Wolak, R.) :

•Quantize $M/G \leftrightarrow$ Quantize $M/\bar{\mathcal{F}}$

$$\begin{array}{rcl} Q_M(\nabla_M)(S_M) : M & \longrightarrow & M \\ \pi \downarrow & & \downarrow \pi \\ Q_{M/\bar{\mathcal{F}}}(\nabla_{M/\bar{\mathcal{F}}})(S_{M/\bar{\mathcal{F}}}) : M/\bar{\mathcal{F}} & \longrightarrow & M/\bar{\mathcal{F}} \end{array}$$

Quantization commutes with reduction :

$\pi(Q_M(\nabla_M)(S_M)) = Q_{M/\bar{\mathcal{F}}}(\pi\nabla_M)(\pi S_M)$

∇_M, S_M, $Q_M(\nabla_M)(S_M) \leftrightarrow$ adapted objects

$\nabla_{M/\bar{\mathcal{F}}}$, $S_{M/\bar{\mathcal{F}}}$, $Q_{M/\bar{\mathcal{F}}}(\nabla_{M/\bar{\mathcal{F}}})(S_{M/\bar{\mathcal{F}}}) \approx$ foliated objects

Adapted vector field : X such that $[X, Y] \in T\mathcal{F}\ \forall Y \in T\mathcal{F}$ $(\to Vect_{\mathcal{F}}(M))$

Adapted vector field : X such that $[X, Y] \in T\mathcal{F}\ \forall Y \in T\mathcal{F}$ $(\rightarrow Vect_{\mathcal{F}}(M))$

In adapted coordinates (x, y) :
$X = \sum_i X^i(x, y)\partial_{x^i} + \sum_i X^i(y)\partial_{y^i}$

Adapted vector field : X such that $[X, Y] \in T\mathcal{F}\ \forall Y \in T\mathcal{F}$ $(\rightarrow Vect_{\mathcal{F}}(M))$

In adapted coordinates (x, y) :
$X = \sum_i X^i(x, y)\partial_{x^i} + \sum_i X^i(y)\partial_{y^i}$

Foliated vector field : $[X]$ with X adapted and with $[\quad] : TM \rightarrow TM/T\mathcal{F}$ $(\rightarrow Vect(M, \mathcal{F}))$

Adapted vector field : X such that $[X, Y] \in T\mathcal{F}\ \forall Y \in T\mathcal{F}$ $(\to Vect_{\mathcal{F}}(M))$

In adapted coordinates (x, y) :
$X = \sum_i X^i(x, y)\partial_{x^i} + \sum_i X^i(y)\partial_{y^i}$

Foliated vector field : $[X]$ with X adapted and with $[\quad] : TM \to TM/T\mathcal{F}$ $(\to Vect(M, \mathcal{F}))$

In adapted coordinates (x, y) : $[X] = \sum_i X^i(y)[\partial_{y^i}]$

Adapted vector field : X such that $[X, Y] \in T\mathcal{F}\ \forall Y \in T\mathcal{F}$ $(\to Vect_{\mathcal{F}}(M))$

In adapted coordinates (x, y) :
$X = \sum_i X^i(x, y)\partial_{x^i} + \sum_i X^i(y)\partial_{y^i}$

Foliated vector field : $[X]$ with X adapted and with $[\quad] : TM \to TM/T\mathcal{F}$ $(\to Vect(M, \mathcal{F}))$

In adapted coordinates (x, y) : $[X] = \sum_i X^i(y)[\partial_{y^i}]$

Projection : $\sum_i X^i(x, y)\partial_{x^i} + \sum_i X^i(y)\partial_{y^i} \mapsto \sum_i X^i(y)[\partial_{y^i}]$

Adapted connection :

$$\nabla_{\mathcal{F}} : Vect_{\mathcal{F}}(M) \times Vect_{\mathcal{F}}(M) \to Vect_{\mathcal{F}}(M)$$
$$: Vect_{\mathcal{F}}(M) \times T\mathcal{F} \to T\mathcal{F}$$

Adapted connection :

$$\nabla_{\mathcal{F}} : Vect_{\mathcal{F}}(M) \times Vect_{\mathcal{F}}(M) \to Vect_{\mathcal{F}}(M)$$
$$: Vect_{\mathcal{F}}(M) \times T\mathcal{F} \to T\mathcal{F}$$

Foliated connection :

$$\nabla(\mathcal{F}) : Vect(M, \mathcal{F}) \times Vect(M, \mathcal{F}) \to Vect(M, \mathcal{F})$$

Adapted connection :

$$\nabla_{\mathcal{F}} : Vect_{\mathcal{F}}(M) \times Vect_{\mathcal{F}}(M) \to Vect_{\mathcal{F}}(M)$$
$$: Vect_{\mathcal{F}}(M) \times T\mathcal{F} \to T\mathcal{F}$$

Foliated connection :

$$\nabla(\mathcal{F}) : Vect(M, \mathcal{F}) \times Vect(M, \mathcal{F}) \to Vect(M, \mathcal{F})$$

Projection : $(\pi_{\nabla}\nabla_{\mathcal{F}})_{[X]}[Y] := [\nabla_{\mathcal{F},X} Y]$

Adapted differential operator : ($\to \mathcal{D}_{\mathcal{F}}(M)$)
$D|_U = \sum_{|\alpha|\leq k} D_\alpha \, \partial_{x^1}^{\alpha^1} \ldots \partial_{x^p}^{\alpha^p} \partial_{y^1}^{\alpha^{p+1}} \ldots \partial_{y^q}^{\alpha^{p+q}}$,
with $D_\alpha, \alpha^1 = \ldots = \alpha^p = 0$ foliated functions.

Adapted differential operator : $(\rightarrow \mathcal{D}_{\mathcal{F}}(M))$
$D|_U = \sum_{|\alpha|\leq k} D_\alpha \, \partial_{x^1}^{\alpha^1} \ldots \partial_{x^p}^{\alpha^p} \partial_{y^1}^{\alpha^{p+1}} \ldots \partial_{y^q}^{\alpha^{p+q}}$,
with $D_\alpha, \alpha^1 = \ldots = \alpha^p = 0$ foliated functions.

Foliated differential operator : $(\rightarrow \mathcal{D}(M, \mathcal{F}))$

$D|_U = \sum_{|\alpha|\leq k} D_\alpha \, \partial_{y^1}^{\alpha^1} \ldots \partial_{y^q}^{\alpha^q}$,
with D_α foliated functions.

Adapted differential operator : ($\to \mathcal{D}_{\mathcal{F}}(M)$)
$D|_U = \sum_{|\alpha|\leq k} D_\alpha \,\partial_{x^1}^{\alpha^1} \ldots \partial_{x^p}^{\alpha^p} \partial_{y^1}^{\alpha^{p+1}} \ldots \partial_{y^q}^{\alpha^{p+q}}$,
with $D_\alpha, \alpha^1 = \ldots = \alpha^p = 0$ foliated functions.

Foliated differential operator : ($\to \mathcal{D}(M, \mathcal{F})$)

$D|_U = \sum_{|\alpha|\leq k} D_\alpha \,\partial_{y^1}^{\alpha^1} \ldots \partial_{y^q}^{\alpha^q}$,
with D_α foliated functions.

Projection π_D :
$\sum_{|\alpha|\leq k} D_\alpha \,\partial_{x^1}^{\alpha^1} \ldots \partial_{x^p}^{\alpha^p} \partial_{y^1}^{\alpha^{p+1}} \ldots \partial_{y^q}^{\alpha^{p+q}} \mapsto$
$\sum_{|\alpha|\leq k, \alpha^1 = \ldots = \alpha^p = 0} D_\alpha \,\partial_{y^1}^{\alpha^1} \ldots \partial_{y^q}^{\alpha^q}$.

Adapted symbols : graded space $\mathcal{S}_{\mathcal{F}}(M)$ associated to $\mathcal{D}_{\mathcal{F}}(M)$

Adapted symbols : graded space $\mathcal{S}_{\mathcal{F}}(M)$ associated to $\mathcal{D}_{\mathcal{F}}(M)$

Foliated symbols : graded space $\mathcal{S}(M, \mathcal{F})$ associated to $\mathcal{D}(M, \mathcal{F})$

Adapted symbols : graded space $\mathcal{S}_{\mathcal{F}}(M)$ associated to $\mathcal{D}_{\mathcal{F}}(M)$

Foliated symbols : graded space $\mathcal{S}(M, \mathcal{F})$ associated to $\mathcal{D}(M, \mathcal{F})$

Projection π_S : $\pi_S[D] = [\pi_D D]$

Adapted symbols : graded space $\mathcal{S}_{\mathcal{F}}(M)$ associated to $\mathcal{D}_{\mathcal{F}}(M)$

Foliated symbols : graded space $\mathcal{S}(M,\mathcal{F})$ associated to $\mathcal{D}(M,\mathcal{F})$

Projection π_S : $\pi_S[D] = [\pi_D D]$

Conclusion : $\pi_D(Q_{ad}(\nabla_{ad})(S_{ad})) = Q_{fol}(\pi_\nabla \nabla_{ad})(\pi_S S_{ad})$