Projectively equivariant quantizations over the superespace $\mathbb{R}^{p|q}$

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•Geometric quantization Q_G : $Q_G = Q|_{\mathcal{A}}$.

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•$\exists Q : L_X Q(S) = Q(L_X S)\ \forall X \in \mathfrak{sl}(m+1, \mathbb{R})$.

- Casimir operator method : $(\mathcal{S}(\mathbb{R}^m), L)$ and $(\mathcal{D}(\mathbb{R}^m), \mathcal{L})$ are representations of $\mathfrak{sl}(m+1, \mathbb{R})$.

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• C and $\mathcal{C}$: second-order Casimir operators of $\mathfrak{sl}(m+1, \mathbb{R})$ on $\mathcal{S}(M)$ and $\mathcal{D}(M)$.

•Necessary and sufficient condition to have the existence of an $\mathfrak{sl}(m+1, \mathbb{R})$-equivariant quantization : if $C(S) = \alpha S$, then $\exists!$ $Q(S)$ s.t. $\mathcal{C}(Q(S)) = \alpha Q(S)$, $\sigma(Q(S)) = S$ and s.t. a technical property is satisfied.

Projectively equivariant quantization over $\mathbb{R}^{p|q}$

- $f \in C^{\infty p|q} : f(x^1, \ldots, x^p) = \sum_{I \subseteq \{1,\ldots,q\}} f_I(x^1, \ldots, x^p)\theta^I.$

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- λ-density over $\mathbb{R}^{p|q}$: smooth function $f \in C^{\infty p|q}$ s.t.

$$L_X^\lambda f = X(f) + \lambda \operatorname{div}(X) f,$$

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- Space of tensor fields of type $(V, \rho) = C^{\infty p|q} \otimes V$.

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$$D(f)=\sum_{|\alpha|\leqslant k} f_\alpha\left(\frac{\partial}{\partial x^1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x^p}\right)^{\alpha_p}\left(\frac{\partial}{\partial\theta^1}\right)^{\alpha_{p+1}}\cdots\left(\frac{\partial}{\partial\theta^p}\right)^{\alpha_{p+q}}f.$$

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- $(\mathcal{L}_X D)(f) = L^\mu_X \circ Df - (-1)^{\tilde{X}\tilde{D}} D \circ L^\lambda_X f$.

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- Quantization on $\mathbb{R}^{p|q}$: linear bijection $Q : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$ s.t. $\sigma_k(Q(S)) = S$ for all $S \in \mathcal{S}_\delta^k$.

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• If X is a linear super vector field over Ω, then $\pi(X) \in \mathrm{Vect}(\mathbb{R}^{p|q}) : \pi(X)(f) = i^{-1} \circ X \circ i(f)$, where $f \in C^{\infty p|q}$.

- $\pi \circ h_{p+1,q}(\mathrm{Id}) = 0$, thus $\pi \circ h_{p+1,q}$ induces a homomorphism from $\mathfrak{pgl}(p+1|q)$ to $\mathrm{Vect}(\mathbb{R}^{p|q})$.

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- $\mathfrak{pgl}(p+1|q) \longleftrightarrow \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\mathfrak{g}_{-1} = \mathbb{R}^{p|q}$, $\mathfrak{g}_0 = \mathfrak{gl}(p|q)$ and $\mathfrak{g}_1 = (\mathbb{R}^{p|q})^*$.

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- Projectively equivariant quantization : quantization Q s.t. $\mathcal{L}_{X^h} \circ Q = Q \circ L_{X^h}$ for every $h \in \mathfrak{pgl}(p+1|q)$.

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• Application γ :

$$\gamma : \mathfrak{g} \to \mathfrak{gl}(\mathcal{S}_\delta, \mathcal{S}_\delta) : h \mapsto \gamma(h) = \mathcal{L}_{X^h} - L_{X^h}.$$

• γ vanishes on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, $\gamma(h) : \mathcal{S}^k_\delta \to \mathcal{S}^{k-1}_\delta$.

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• Casimir operator of (V, β) :

$$\sum_{i=1}^{n} (-1)^{\tilde{u}_i} \beta(u_i)\beta(u_i') = \sum_{i=1}^{n} \beta(u_i')\beta(u_i).$$

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- The Casimir operator C of $\mathfrak{pgl}(p+1|q) \cong \mathfrak{sl}(p+1|q)$ on $(\mathcal{S}^k_\delta, L)$ is equal to $\alpha(k,\delta)\mathrm{Id}$, where

$$\alpha(k,\delta) = \frac{p-q}{2}\delta^2 - \frac{2k+p-q}{2}\delta + \frac{k(k+(p-q))}{(p-q+1)}.$$

- The Casimir operator $\mathcal{C}$ of $\mathfrak{pgl}(p+1|q) \cong \mathfrak{sl}(p+1|q)$ on $(\mathcal{S}^k_\delta, \mathcal{L})$ is equal to $C+N$, where N is defined in this way :

$$N : \mathcal{S}^k_\delta \to \mathcal{S}^{k-1}_\delta : S \mapsto 2\sum_i \gamma(\epsilon^i) L_{X^{e_i}} S,$$

where $\epsilon^r = \frac{(-1)^{\tilde{r}}}{2(p-q+1)}\varepsilon^r$.

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• If $S \in \mathcal{S}^k_\delta$, $Q(L_{X^h}S) = \mathcal{L}_{X^h}(Q(S))$ because they are eigenvectors of $\mathcal{C}$ of eigenvalue $\alpha(k,\delta)$ and because their term of degree k is exactly $L_{X^h}S$.

•Divergence operator :

$$\mathrm{div} : \mathcal{S}^k_\delta \to \mathcal{S}^{k-1}_\delta : S \mapsto \sum_{j=1}^{p+q} (-1)^{\tilde{y^j}} i(\varepsilon^j)\partial_{y^j} S.$$

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Theorem

If δ is not critical, then the map $Q : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$ defined by

$$Q(S)(f) = \sum_{r=0}^{k} C_{k,r} Q_{\mathrm{Aff}}(\mathrm{div}^r S)(f), \quad \textit{for all } S \in \mathcal{S}^k_\delta$$

is the unique $\mathfrak{sl}(p+1|q)$-equivariant quantization if

$$C_{k,r} = \frac{\prod_{j=1}^{r}((p-q+1)\lambda + k - j)}{r! \prod_{j=1}^{r}(p-q+2k-j-(p-q+1)\delta)} \; \textit{for all } r \geqslant 1.$$

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• Killing form of $\mathfrak{psl}(p+1|p+1)$ vanishes, but K defined by

$$K([A],[B]) = \mathrm{str} AB$$

is a nondegenerate invariant supersymmetric even form.

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- The $\mathfrak{psl}(p+1,p+1)$-equivariant quantizations are $\mathfrak{pgl}(p+1,p+1)$-equivariant (equivariance with respect to the Euler vector field).

• If $q = p + 1$, Q does not depend on δ and λ.