EQUIVARIANT SYMBOL CALCULUS FOR DIFFERENTIAL OPERATORS ACTING ON FORMS

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Abstract. We prove the existence and uniqueness of a projectively equivariant symbol map (in the sense of Lecomte and Ovsienko) for the spaces $\mathcal{D}_p$ of differential operators transforming p-forms into functions, over $\mathbb{R}^n$. As an application, we classify the $\mathrm{Vect}(M)$-equivariant maps from $\mathcal{D}_p$ to $\mathcal{D}_q$ over a smooth manifold M, recovering and improving earlier results of N. Poncin. This provides the complete answer to a question raised by P. Lecomte about the extension of a certain intrinsic homotopy operator.

1. Introduction

Let $\mathcal{D}^k_\lambda(M)$ denote the space of differential operators of order at most k acting on λ-densities over a smooth manifold M. Let also $\mathcal{D}_\lambda(M)$ denote the filtered union of these spaces. Both $\mathcal{D}^k_\lambda(M)$ and $\mathcal{D}_\lambda(M)$ are in a natural way modules over the Lie algebra $\mathrm{Vect}(M)$ of vector fields over M.

In [6], for $M = \mathbb{R}^n$, P. Lecomte and V. Ovsienko considered the space $\mathcal{D}_\lambda(M)$ as a module over the subalgebra of $\mathrm{Vect}(M)$ made of infinitesimal projective transformations, which is isomorphic to $\mathfrak{sl}(n+1,\mathbb{R})$. They showed that this module is actually isomorphic to the module of symbols $\mathrm{Pol}(T^*M)$, which is the space of functions on T^*M that are polynomial along the fibres. Up to a natural normalization condition, the isomorphism Q_λ from $\mathrm{Pol}(T^*M)$ to $\mathcal{D}_\lambda(M)$ is unique and named the *projectively equivariant quantization map*. Its inverse σ_λ is the (projectively equivariant) *symbol map*. An explicit formula is given in [6] for both mappings in terms of a divergence operator.

Furthermore, when M is an arbitrary smooth manifold, the knowledge of the equivariant symbol map proves useful to classify the $\mathrm{Vect}(M)$-equivariant maps between $\mathcal{D}_\lambda(M)$ and $\mathcal{D}_\mu(M)$.

Several extensions of this work have recently been proposed. In [2], one considers the spaces $\mathcal{D}_{\lambda,\mu}(M)$ of differential operators mapping λ-densities into μ-densities as modules over the Lie algebra of infinitesimal conformal transformations, while in [1], the algebras under consideration correspond to the action of the symplectic (resp. pseudo-orthogonal) group on the Lagrangian (resp. pseudo-orthogonal) Grassmann manifolds. The results in both cases are the existence and uniqueness of an equivariant symbol map from $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^n)$ to the corresponding symbol space, provided that the value $\mu-\lambda$ does not belong to a countable critical set. Unfortunately, in these situations, a general formula for the symbol and quantization maps seems out of reach.

In this paper, we present a first example of projectively equivariant symbol calculus for differential operators acting on tensor fields. We consider the spaces $\mathcal{D}_p$ of differential operators mapping p-forms into functions over a smooth manifold, whose dimension is assumed to be greater than 1. We still denote by $\mathcal{D}_p^k$ the subspace of operators with order at most k. The corresponding symbol space $\mathcal{S}_p$ is made up of polynomial functions valued in contravariant antisymmetric tensor fields.

We show first that, for $M = \mathbb{R}^n$, there exists a unique (up to normalisation) projectively equivariant symbol map from $\mathcal{D}_p$ to $\mathcal{S}_p$. We obtain an explicit formula in terms of the divergence operator and classical invariants of the space of symbols, such as the Koszul differential. Next, we use this formula to classify the $\mathrm{Vect}(M)$-invariant maps from $\mathcal{D}_p^k$ to $\mathcal{D}_q^l$ over *any* manifold M, such that $n = \dim M > 1$. We recover in this way the results by Poncin [9] for $p \leqslant n-2$ and give the complete list of invariants for $p = n-1$ and $p = n$.

The paper is organized as follows. In Section 2, we recall basic definitions and notation. In Section 3, we collect some properties of the spaces of symbols and describe the projectively equivariant quantization map (see Theorem 4). In Section 4, we determine the invariant operators from $\mathcal{S}_p$ to $\mathcal{S}_q$. This is a first step towards the classification of $\mathrm{Vect}(M)$-invariant maps from $\mathcal{D}_p^k$ to $\mathcal{D}_q^l$ over an arbitrary manifold M, which is completed in Section 5.

2. Basic Definitions and Notation

Throughout this paper, we assume manifolds to be second-countable, smooth, Hausdorff, and connected. Let M be such a manifold.

2.1. **Differential operators acting on p-forms.** Let $\Omega^p(M)$ denote the space of p-forms over M. The action of $\mathrm{Vect}(M)$ on $\Omega^p(M)$ is standard:

$$(L_X\omega)_x = (X.\omega)_x - \rho(D_xX)\omega_x, \tag{1}$$

where $X.$ denotes the differentiation along the vector field X, D_xX is the Jacobian matrix of X, and ρ denotes the natural action of $\mathfrak{gl}(n,\mathbb{R})$ on the fibre $\wedge^p T_x^*M$.

We denote by $\mathcal{D}_p^k$ the space of linear differential operators of order at most k from $\Omega^p(M)$ to $\Omega^0(M) = C^\infty(M)$. The action of $\mathrm{Vect}(M)$ on this space is given by the commutator: if $D \in \mathcal{D}_p^k$,

$$\mathcal{L}_X D = L_X \circ D - D \circ L_X. \tag{2}$$

The filtered union $\mathcal{D}_p = \cup_k \mathcal{D}_p^k$ thus inherits a $\mathrm{Vect}(M)$-module structure as well.

2.2. **Space of symbols.** For any D in $\mathcal{D}_p^k$, in local coordinates over a chart domain diffeomorphic to $\mathbb{R}^n$,

$$(D\omega)_x = \sum_{|\alpha|\leqslant k} \left\langle A_\alpha(x), \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n} \omega \right\rangle, \tag{3}$$

where $\langle , \rangle$ denotes the evaluation and $A_\alpha \in \Gamma(\wedge^p T\mathbb{R}^n)$ for each $\alpha \in \mathbb{N}^n$.

The principal symbol $\sigma(D)$ of D is the smooth section of the bundle

$$E_p^k = \wedge^p TM \otimes S^k TM \to M$$

defined, for all $x \in M$ and $\xi \in T_x^*M$ by

$$\sigma(D)_x(\xi) = \sum_{|\alpha|=k} (\xi_1)^{\alpha_1} \cdots (\xi_n)^{\alpha_n} A_\alpha(x). \tag{4}$$

We call the kth *order symbol space* and denote by $\mathcal{S}_p^k$ the space $\Gamma(E_p^k)$. The *total symbol space* $\mathcal{S}_p$ is then the graded sum of these. Both $\mathcal{S}_p^k$ and $\mathcal{S}_p$ are modules over the Lie algebra of vector fields. The Lie derivative is still given by (1), ρ now standing for the natural action of $\mathfrak{gl}(n,\mathbb{R})$ on $\wedge^p T_xM \otimes S^k T_xM$.

2.3. **Projectively equivariant quantization and symbol maps.** Let $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{Vect}(M)$. A $\mathfrak{g}$-*equivariant symbol map* is a $\mathfrak{g}$-module isomorphism $\sigma_{\mathfrak{g}} : \mathcal{D}_p(M) \to \mathcal{S}_p(M)$, such that for all $D \in \mathcal{D}_p^k(M)$,

$$\sigma_{\mathfrak{g}}(D) - \sigma(D) \in \bigoplus_{r=0}^{k-1} \mathcal{S}_p^r.$$

In analogy with [1, 2], the inverse map $Q_{\mathfrak{g}}$ will be called a $\mathfrak{g}$-*equivariant quantization map*.

Let us consider a basic example. We take $M = \mathbb{R}^n$ and consider the affine subalgebra Aff of $\mathrm{Vect}(\mathbb{R}^n)$, which is generated by constant and linear vector fields. The map

$$\sigma_{\mathrm{Aff}} : \mathcal{D}_p \to \mathcal{S}_p : \sum_{|\alpha|\leqslant k} \left\langle A_\alpha(x), \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n} \right\rangle \mapsto \sum_{|\alpha|\leqslant k} (\xi_1)^{\alpha_1} \cdots (\xi_n)^{\alpha_n} A_\alpha(x).$$

is a linear bijection. Moreover,

$$\sigma_{\mathrm{Aff}} \circ \mathcal{L}_X = L_X \circ \sigma_{\mathrm{Aff}}$$

for all $X \in \mathrm{Aff}$. In other words, the map σ_{Aff} is an affinely equivariant symbol map. The *projective Lie algebra* is the subalgebra of $\mathrm{Vect}(\mathbb{R}^n)$ generated by the vector fields

$$\frac{\partial}{\partial x^r}, \quad x^s \frac{\partial}{\partial x^r}, \quad \text{and} \quad x^r \mathcal{E} \quad (r, s \in \{1, \ldots, n\}), \tag{5}$$

where $\mathcal{E}$ denotes the Euler (or Liouville) vector field $\sum_{r=1}^n x^r \partial/\partial x^r$. We denote it by $\mathfrak{sl}_{n+1}$ to recall that it is isomorphic to $\mathfrak{sl}(n+1,\mathbb{R})$.

It is well-known that $\mathfrak{sl}_{n+1}$ is a maximal subalgebra of the Lie algebra $\mathrm{Vect}_*(\mathbb{R}^n)$ of vector fields with polynomial coefficients (see [6] for a proof).

A *projectively equivariant symbol (resp. quantization) map* is a $\mathfrak{sl}_{n+1}$-equivariant symbol (resp. quantization) map.

2.4. **Polynomial formalism.** In order to simplify the computations involving differential operators, it is convenient to represent them by polynomial functions belonging to the space of symbols, “à la Fourier”. We will still denote by $\mathcal{L}$ the Lie derivative of differential operators expressed in these symbolic terms. In other words, we equip $\mathcal{S}_p$ with a $\mathrm{Vect}(\mathbb{R}^n)$-module structure isomorphic to that of $\mathcal{D}_p$, the isomorphism then being σ_{Aff}. Seeking a $\mathfrak{g}$-equivariant quantization map then amounts to seeking a $\mathfrak{g}$-module isomorphism from $(\mathcal{S}_p, L)$ to $(\mathcal{S}_p, \mathcal{L})$. Moreover, as in [5], we consider the natural extension of σ_{Aff} to multidifferential operators. We therefore represent such operators by polynomial functions of several variables.

NOTATION. As we continue, we represent by η the derivatives acting on the coefficients of an element of $\mathcal{S}_p$ and by ζ the derivatives acting on a vector field. We also write the symbols as polynomial functions of the indeterminate ξ. Besides, following the notation of [2], we denote by Div the operator $(\eta\partial_\xi) = \sum_i \eta_i \partial_{\xi_i}$.

The formulas for the Lie derivatives can then be stated in a concise way.

Lemma 1. *The Lie derivatives on the space of symbols $\mathcal{S}^k_p$ are given by the expressions*

$$L_X u = \langle X, \eta\rangle u - \rho(X\otimes\zeta)u,$$

and

$$\mathcal{L}_X u = L_X u + \sum_{r=1}^{k} t_r(X)u, \tag{6}$$

where

$$t_r(X) : \mathcal{S}^k_p \to \mathcal{S}^{k-r}_p : \Lambda\otimes P \mapsto -\Lambda\otimes X\frac{(\zeta\partial_\xi)^{r+1}}{(r+1)!}P - X\wedge i_\zeta\Lambda\otimes\frac{(\zeta\partial_\xi)^r}{r!}P. \tag{7}$$

3. Projectively Equivariant Quantization Map

Let us first review some properties of the space of symbols.

3.1. Classical operators on the space of symbols. The typical fibre W^k_p of E^k_p is isomorphic to $\wedge^p\mathbb{R}^n\otimes S^k\mathbb{R}^n$. The space $\oplus_{k,p}W^k_p$ is thus the tensor product of the exterior and symmetric algebras over $\mathbb{R}^n$. Now, any $X\in\mathbb{R}^n$ can be identified with $X\otimes 1$ or $1\otimes X$. We define two operators of multiplication by X:

$$\pi_a(X) : W^k_p \to W^k_{p+1} : \Lambda\otimes P \mapsto X\wedge\Lambda\otimes P$$

and

$$\pi_s(X) : W^k_p \to W^{k+1}_p : \Lambda\otimes P \mapsto \Lambda\otimes XP.$$

We will also make use of the corresponding inner products by an element $\alpha\in\mathbb{R}^{n*}$:

$$i_\alpha : W^k_p \to W^k_{p-1} : \Lambda\otimes P \mapsto i_\alpha(\Lambda)\otimes P$$

and

$$\alpha\partial : W^k_p \to W^{k-1}_p : \Lambda\otimes P \mapsto \Lambda\otimes i_\alpha(P) = \Lambda\otimes\sum_{j=1}^n \alpha_j\partial_{\xi_j}P.$$

Now, if (v_i) is a basis of $\mathbb{R}^n$ and if (β^i) is the dual basis of $\mathbb{R}^{n*}$, the Koszul differential can be defined as

$$\delta : W^k_p \to W^{k-1}_{p+1} : u \mapsto \sum_{j=1}^n \pi_a(v_j)\circ(\beta^j\partial)(u), \tag{8}$$

while the associated differential δ^* is defined by

$$\delta^* : W^k_p \to W^{k+1}_{p-1} : u \mapsto \sum_{j=1}^n i_{\beta^j}\circ\pi_s(v_j)(u). \tag{9}$$

The properties of these operators were discussed, for instance, in [3]. Let us quote that they commute with the natural action of $\mathfrak{gl}(n,\mathbb{R})$ on W^k_p, and that the following relations hold on W^k_p:

$$\delta^2 = 0, \quad (\delta^*)^2 = 0, \quad \delta\circ\delta^* + \delta^*\circ\delta = (k+p)\mathrm{id}.$$

It follows that if $k+p\neq 0$, the mappings $\delta\circ\delta^*/(k+p)$ and $\delta^*\circ\delta/(k+p)$ are projectors onto $\mathfrak{gl}(n,\mathbb{R})$-submodules of W^k_p. We denote by A^k_p and B^k_p their respective images. At last, we know from the representation theory of $\mathfrak{sl}(n,\mathbb{R})$ that the decomposition

$$W^k_p = A^k_p \oplus B^k_p$$

is the decomposition of W_p^k into irreducible $\mathfrak{sl}(n,\mathbb{R})$-submodules. It is worth noticing that, in terms of Young diagrams, A_p^k is represented by a line of $k+1$ boxes on top of $p-1$ lines with one single box, while B_p^k is isomorphic A_{p+1}^{k-1}.

Finally, the operators defined in this section can be canonically extended to the spaces $\mathcal{S}_p^k$. The operators δ and δ^* obtained in this way commute with the action of $\mathrm{Vect}(\mathbb{R}^n)$. We write $\mathcal{A}_p^k = \operatorname{im}\delta \cap \mathcal{S}_p^k$ and $\mathcal{B}_p^k = \operatorname{im}\delta^* \cap \mathcal{S}_p^k$.

3.2. **Existence and uniqueness of the equivariant quantization.** In order to build the equivariant quantization, we use a method due to Duval, Lecomte, and Ovsienko. It appeared in [2], in a conformal equivariance setting. In [1], it proved to be relevant within a more general framework.

This method relies on the comparison of eigenvalues and eigenvectors of two Casimir operators. We denote by C the Casimir operator associated to the representation $(\mathcal{S}_p, L)$ of $\mathfrak{sl}_{n+1}$ and by $\mathcal{C}$ the Casimir operator associated to the representation $(\mathcal{S}_p, \mathcal{L})$ of the same algebra.

Let us recall briefly the main stages of the process.

Assume that an $\mathfrak{sl}_{n+1}$-equivariant quantization $\mathcal{Q}$ is defined. Then

$$\mathcal{Q}\circ C = \mathcal{C}\circ\mathcal{Q}. \tag{10}$$

In particular, if P is a homogeneous eigenvector of C with eigenvalue α, then $\mathcal{Q}(P)$ is an eigenvector of $\mathcal{C}$ with the same eigenvalue. The principal symbol of $\mathcal{Q}(P)$ is P.

Conversely, assume that

(1) The Casimir operator C is diagonalizable.
(2) To any homogeneous eigenvector P of C with eigenvalue α corresponds a unique eigenvector of $\mathcal{C}$, which we denote $\mathcal{Q}(P)$, such that $\mathcal{Q}(P)$ has eigenvalue α and P is the principal symbol of $\mathcal{Q}(P)$.

Then the map $\mathcal{Q}$, linearly extended to $\mathcal{S}_p$, is the unique $\mathfrak{sl}_{n+1}$-equivariant quantization of $\mathcal{S}_p$.

The spectrum of C is described by Proposition 2. The difference $\mathcal{C}-C$ is computed in Proposition 3. Finally, the unique $\mathfrak{sl}_{n+1}$-equivariant quantization is described in Theorem 4.

Proposition 2. *The restriction of C to $\mathcal{S}_p^k$ equals*

$$\frac{k+n+1}{n+1}\delta\circ\delta^* + \frac{k+n}{n+1}\delta^*\circ\delta.$$

In particular, C is diagonalizable and its spectrum is

$$\left\{\overbrace{\frac{(k+n+1)(k+p)}{n+1}}^{\alpha_p^k} : k\in\mathbb{N}\right\} \cup \left\{\overbrace{\frac{(k+n)(k+p)}{n+1}}^{\beta_p^k} : k\in\mathbb{N}\right\}.$$

Proof. We fix two bases of $\mathfrak{sl}_{n+1}$. The first one is made up of the vector fields

$$e_i = \frac{\partial}{\partial x^i},\quad \mathcal{E},\quad h_A = -\sum_{k,l} A_l^k x^l \frac{\partial}{\partial x^k},\quad \epsilon^i = -\frac{1}{2(n+1)} x^i \mathcal{E},$$

where $i \in \{1, \ldots, n\}$ and A runs over a basis of $\mathfrak{sl}(n, \mathbb{R})$. The second one is its dual with respect to the Killing form of $\mathfrak{sl}_{n+1}$. We write

$$\epsilon^i, \frac{1}{2n}\mathcal{E}, \frac{n}{n+1}h_{A^*}, e_i,$$

where A^* is dual to A with respect to the Killing form of $\mathfrak{sl}(n, \mathbb{R})$.

Then,

$$C = 2\sum_{i=1}^{n} L_{\epsilon^i} \circ L_{e_i} + L_{\sum_i [e_i, \epsilon^i]} + \frac{1}{2n}(L_{\mathcal{E}})^2 + \frac{n}{n+1}\sum L_{h_A} \circ L_{h_{A^*}}. \tag{11}$$

Since C commutes with the constant vector fields, it has constant coefficients. Hence, we just collect terms with such coefficients in the last right-hand side. From $L_X = X. - \rho(DX)$, where ρ denotes the natural representation of $\mathfrak{gl}(n, \mathbb{R})$, we get

$$C|_{\mathcal{S}^k_p} = \left(\frac{k+p}{2} + \frac{(k+p)^2}{2n}\right)\mathrm{id} + \frac{n}{n+1}\sum_A \rho(A) \circ \rho(A^*).$$

The second factor of the last term is the Casimir operator of $\mathfrak{sl}(n, \mathbb{R})$ acting on W^k_p. It is a multiple of the identity on each irreducible component A^k_p and B^k_p. The computation of the eigenvalues is classical (see, for instance, [4, p.122]).

Since

$$\frac{1}{k+p}\delta \circ \delta^* \quad \text{and} \quad \frac{1}{k+p}\delta^* \circ \delta$$

are the projectors on A^k_p and B^k_p, the conclusion follows. □

Proposition 3. *Let $N_{\mathcal{C}}$ denote the difference $\mathcal{C} - C$. Then $N_{\mathcal{C}}$ equals*

$$\frac{1}{n+1}(\delta \circ \mathrm{Div} \circ \delta^* + \delta^* \circ \mathrm{Div} \circ \delta).$$

Proof. We use the bases defined in the proof of Proposition 2. As a consequence of (6) and (11),

$$N_{\mathcal{C}} = \mathcal{C} - C = 2\sum_{i=1}^{n} t_1(\epsilon^i) \circ L_{e_i}.$$

Using (7), we get

$$N_{\mathcal{C}}|_{\mathcal{S}^k_p} = \frac{k+p}{n+1}\mathrm{Div} - \frac{1}{n+1}i_\eta \circ \delta$$

But, if $k > 0$,

$$N_{\mathcal{C}}|_{\mathcal{S}^k_p} = N_{\mathcal{C}} \circ \frac{\delta \circ \delta^* + \delta^* \circ \delta}{k+p}.$$

Since $[\mathrm{Div}, \delta] = 0$ and $[\mathrm{Div}, \delta^*] = i_\eta$, the conclusion follows. If $k = 0$, $N_{\mathcal{C}}$ vanishes. □

Theorem 4. *The map $\mathcal{Q}$, defined by its restrictions*

$$\mathcal{Q}|_{\mathcal{S}^k_p} = \mathrm{id}_{\mathcal{S}^k_p} + \sum_{l=1}^{k} \mathcal{Q}_{l,p},$$

with $\mathcal{Q}_{l,p} = \mathcal{Q}'_{l,p} + \mathcal{Q}''_{l,p}$, where

$$\mathcal{Q}'_{l,p} = \left(\frac{1}{n+1}\right)^l \left(\prod_{1\leqslant j\leqslant l} \frac{1}{\alpha^k_p - \alpha^{k-j}_p}\right) (\delta \circ \mathrm{Div} \circ \delta^*)^l$$

and

$$\mathcal{Q}''_{l,p} = \left(\frac{1}{n+1}\right)^l \left(\prod_{1\leqslant j\leqslant l} \frac{1}{\beta_p^k - \beta_p^{k-j}}\right) (\delta^* \circ \mathrm{Div} \circ \delta)^l,$$

is the unique $\mathfrak{sl}_{n+1}$*-equivariant quantization map of* $\mathcal{S}_p$.

Proof. Let $P \in \mathcal{S}_p^k$ be an eigenvector of C. The computation of $\mathcal{Q}(P)$ is similar according to whether $P \in \mathcal{A}_p^k$ or $P \in \mathcal{B}_p^k$ (cf. Subsection 3.1).
Assume that $P \in \mathcal{A}_p^k$. Then $C(P) = \alpha_p^k P$.

The linear system

$$\mathcal{C}(\mathcal{Q}(P)) = \alpha_p^k \mathcal{Q}(P), \tag{12}$$

which defines $\mathcal{Q}(P)$, is a triangular one. Indeed, $C(\mathcal{S}_p^k) \subset \mathcal{S}_p^k$ and $N_{\mathcal{C}}(\mathcal{S}_p^k) \subset \mathcal{S}_p^{k-1}$. Noticing that $N_{\mathcal{C}}$ stabilizes $\operatorname{im}\delta$, we write $\mathcal{Q}(P) = P + \sum_{r=1}^k P_{k-r}$, with $P_{k-r} \in \mathcal{S}_p^{k-r}$, and we observe that (12) amounts to

$$P_{k-r} = \frac{1}{(n+1)(\alpha_p^k - \alpha_p^{k-r})} \delta \circ \mathrm{Div} \circ \delta^*(P_{k-r+1}).$$

Hence the result. □

4. Invariant Differential Operators from $\mathcal{S}_p^k$ to $\mathcal{S}_q^l$

In this section, we will characterize the differential operators $\mathcal{T}$ from $\mathcal{S}_p^k$ to $\mathcal{S}_q^l$ that intertwine the action L of $\mathfrak{sl}_{n+1}$ on these spaces. The arguments are essentialy those used in [1, Lemmas 10, 12]. We refer the reader to this paper for more detailed computations.

Lemma 5. *If* $\mathcal{T} : \mathcal{S}_p^k \to \mathcal{S}_q^l$ *commutes with the action of constant vector fields, then it has constant coefficients. If in addition, it commutes with the action of the Euler field, it is homogeneous of order* $r = (k+p) - (l+q)$.

Theorem 6. *The space of* $\mathfrak{sl}_{n+1}$*-invariant differential operators from* $\mathcal{S}_p^k$ *to* $\mathcal{S}_q^l$ *is trivial if* $(l,q) \notin \{(k+1,p-1),(k,p),(k-1,p+1)\}$. *The nontrivial spaces are generated by* δ^*, id *and* δ^* *and* δ, *respectively.*

Proof. The assertion is trivial if $k+p = 0$. Assume that $k+p > 0$ and let $\mathcal{T}$ denote an invariant mapping from $\mathcal{S}_p^k$ to $\mathcal{S}_q^l$. We may compose $\mathcal{T}$ on both sides with the projectors on $\operatorname{im}\delta$ and $\operatorname{im}\delta^*$. We obtain in this way (at most) four invariant operators. Let us show for instance how one can characterize an invariant operator from $\mathcal{A}_p^k$ to $\mathcal{B}_q^l$.

Since $\mathcal{T}$ has constant coefficients and is homogeneous with order $r = (k+p) - (l+q)$, its restriction to the subspace of $\mathcal{A}_p^k$ of sections with degree r polynomial coefficients has values in the subspace of constant sections of $\mathcal{B}_q^l$. As modules over the Lie algebra of linear and divergence free vector fields, these subspaces identify with $S^r\mathbb{R}^{n*} \otimes A_p^k$ and B_q^l respectively, endowed with the standard representation of $\mathfrak{sl}(n,\mathbb{R})$.

If $\mathcal{T}$ is not trivial, since B_q^l is irreducible, the decomposition of $S^r\mathbb{R}^n \otimes A_p^k$ into $\mathfrak{sl}(n,\mathbb{R})$-irreducible submodules contains a factor isomorphic to B_q^l. We can compute the Young diagrams associated to these submodules using Littlewood-Richardson rule and compare these to the diagram associated to B_q^l. We then observe that the diagram associated to B^l_q is obtained by deleting r boxes in the diagram associated to A^k_p. This amounts to the information $l \leqslant k+1$ and $q+1 \leqslant p$.

Since $\mathcal{T}$ commutes with the Casimir operator C, we also have $\alpha^k_p = \beta^l_q$, and thus

$$0 = (n+1)(\alpha^k_p - \beta^l_q) = (k+p)(k+n+1) - (l+q)(l+n) \geqslant (l+q)(k-l+1) \geqslant 0.$$

This implies successively $l+q > 0$, $l = k+1$, $p = q+1$, and finally $r = 0$. Therefore $\mathcal{T}$ defines an $\mathfrak{sl}(n,\mathbb{R})$-invariant map from A^k_p to B^{k+1}_{p-1}. The space of such invariants is by Schur's lemma at most one dimensional, and is thus generated by the map δ^*. □

5. Invariant Operators from $\mathcal{D}^k_p$ to $\mathcal{D}_q$

In this section, we let M denote an arbitrary manifold. We show how the projectively equivariant quantization map allows us to classify the $\mathrm{Vect}(M)$-invariant operators T from $\mathcal{D}^k_p(M)$ to $\mathcal{D}_q(M)$, i.e. those T such that

$$T(\mathcal{L}_X D) = \mathcal{L}_X T(D) \tag{13}$$

for all $X \in \mathrm{Vect}(M)$ and $D \in \mathcal{D}^k_p(M)$. We denote by $\mathcal{I}^k_{p,q}$ the space of these invariant operators.

The classification method presented here allows us to recover quite easily the results of [9] in the case $p \leqslant n-2$ ($n = \dim M$) and to complete the classification of invariant maps in the cases $p = n-1$ and $p = n$.

The following result is proved in [9].

Proposition 7. *If $T \in \mathcal{I}^k_{p,q}$, then T is a local operator.*

We can thus consider the restrictions of invariant operators to relatively compact chart domains over the manifold. We can furthermore consider chart domains that are diffeomorphic to $\mathbb{R}^n$, and it will be sufficient to compute the spaces $\mathcal{I}^k_{p,q}$ over the manifold $M = \mathbb{R}^n$. Now, we use the projectively equivariant quantization map as a tool in the following way. To any $T \in \mathcal{I}^k_{p,q}$, we associate the operator

$$\mathcal{T} : \bigoplus_{r=0}^{k} \mathcal{S}^r_p \to \mathcal{S}_q : u \mapsto (\mathcal{Q}^{-1} \circ T \circ \mathcal{Q})(u).$$

We also denote by $\mathcal{T}_{r,p}$ the restriction of $\mathcal{T}$ to $\mathcal{S}^r_p$. In this framework, equation (13) means that $\mathcal{T}$ has to commute with the operator $\mathcal{Q}^{-1} \circ \mathcal{L}_X \circ \mathcal{Q}$, for all $X \in \mathrm{Vect}(\mathbb{R}^n)$.

Restricted to $\mathcal{S}^r_p$, the latter operator equals

$$L_X + \sum_{i=1}^{r} \gamma_{i,p}(X),$$

where each $\gamma_{i,p}(X) : \mathcal{S}^r_p \to \mathcal{S}^{r-i}_p$ is a differential operator with respect to X and vanishes for all X in $\mathfrak{sl}_{n+1}$. Equation (13) then first implies

$$L_X \circ \mathcal{T}_{r,p} = \mathcal{T}_{r,p} \circ L_X,$$

for any $X \in \mathfrak{sl}_{n+1}$, and Theorem 6 ensures that no more than three nontrivial invariant types exist:

- If $q = p-1$, the map $\mathcal{T}_{r,p} : \mathcal{S}^r_p \to \mathcal{S}^{r+1}_{p-1}$ writes $a_{r,p}\delta^*$ for some constant $a_{r,p}$.
- If $q = p$, the map $\mathcal{T}_{r,p} : \mathcal{S}^r_p \to \mathcal{S}^r_p$ writes $b_{r,p}\delta \circ \delta^* + c_{r,p}\delta^* \circ \delta$ for some constants $b_{r,p}$ and $c_{r,p}$.
- If $q = p+1$, the map $\mathcal{T}_{r,p} : \mathcal{S}^r_p \to \mathcal{S}^{r-1}_{p+1}$ writes $d_{r,p}\delta$ for some constant $d_{r,p}$.

The commutation relation (13) then forces $\mathcal{T}$ to fulfil

$$\gamma_{i,q}(X) \circ \mathcal{T}_{r,p} = \mathcal{T}_{r-i,p} \circ \gamma_{i,p}(X) \tag{14}$$

for all $r \leqslant k$, $i \leqslant r$, and $X \in \mathrm{Vect}(\mathbb{R}^n)$.

An important remark helps simplifying the computations further at this stage. Notice that it is sufficient to require that condition (13), and therefore (14), be satisfied for all X with polynomial coefficients. Indeed, the Lie derivatives are differential operators with respect to X. But then, since $\mathfrak{sl}_{n+1}$ is a maximal subalgebra of $\mathrm{Vect}_*(\mathbb{R}^n)$ (cf. Subsection 2.3), it suffices to verify that (14) holds for a given polynomial vector field X out of $\mathfrak{sl}_{n+1}$ in order to ensure that T be a $\mathrm{Vect}(\mathbb{R}^n)$-invariant.

NOTATION. From now on, we therefore fix $X = (x^1)^2\partial/\partial x^2$.

Lemma 8. *The following relations hold on $\mathcal{S}^r_p$:*

- *For all i, r, and p: $\delta^* \circ \gamma_{i,p}(X) = \gamma_{i,p-1}(X) \circ \delta^*$.*
- *For all $r \geqslant 1$ and $1 \leqslant p \leqslant n-1$, $\delta^* \circ \gamma_{1,p}(X) \neq 0$.*
- *The operator $\gamma_{1,p}$ vanishes iff $r = 0$ or $p = n$ or $(r,p) = (1,0)$.*
- *For all $r \geqslant 3$ and $p \in \{0, \ldots, n-2\}$, $\delta \circ \gamma_{1,p}(X) \circ \delta \neq 0$.*
- *For all $r \geqslant 2$, $\delta^* \circ \gamma_{2,n}(X) \neq 0$.*

Proof. The proof is straightforward. Let us perform it for the first claim. Note that the maps $\gamma_{i,p}$ are polynomial functions of the operators L_X , $t_1(X)$ and $\mathcal{Q}_{j,p}$ $(1 \leqslant j \leqslant i)$. For instance, we have

$$\gamma_{1,p}(X) = t_1(X) + [L_X, \mathcal{Q}_{1,p}].$$

In order to conclude, we notice that $[\delta^*, L_X] = [\delta^*, t_1(X)] = 0$ and that $\delta^* \circ \mathcal{Q}_{i,p} = \mathcal{Q}_{i,p-1} \circ \delta^*$. □

Corollary 9. *For all p, $\mathcal{D}^0_p$ is isomorphic to $\mathcal{S}^0_p$ as a $\mathrm{Vect}(M)$-module, while $\mathcal{D}^1_p$ is isomorphic to $\mathcal{S}^1_p \oplus \mathcal{S}^0_p$, if and only if $p = 0$ or $p = n$.*

Proof. Over any chart domain U of M, Theorem 4 allows us to define a unique isomorphism $\mathcal{Q}_U$ of $\mathfrak{sl}_{n+1}$-modules from the space of symbols to the space of differential operators over the chart. By Lemma 8, the operator $\gamma_{1,p}(X)$ vanishes and the maps $\mathcal{Q}_U$ actually commute with the action of all vector fields over the chart. The uniqueness of the maps $\mathcal{Q}_U$ as $\mathrm{Vect}(U)$-module isomorphisms then implies that they glue toghether to define a global isomorphism. □

Now, before turning to the classification results, let us recall the expression of some classical invariant maps and introduce some new ones.

5.1. **Some invariant maps.** The dual d^* of the de Rham differential is defined by

$$d^* : \mathcal{D}^k_p \to \mathcal{D}^{k+1}_{p-1} : D \mapsto D \circ d.$$

It is well-known that it commutes with the action of $\mathrm{Vect}(M)$. The space $\mathcal{D}_0$ is the direct sum of the space of differential operators vanishing on the constants and of $\mathcal{D}^0_0$. We denote by $\mathcal{I}_0$ the projection onto the second summand:

$$\mathcal{I}_0 : \mathcal{D}_0 \to \mathcal{D}^0_0 : D \mapsto D(1),$$

which is of course invariant.

The *conjugation operator* was for instance presented in [8, 5.5.3] in the framework of differential operators acting on densities. Let U be a chart domain of M. Densities with weight 0 over U are nothing but functions, while 1-densities identify with n-forms. Here, the conjugation operator $\mathcal{C} : \mathcal{D}_n^k \to \mathcal{D}_n^k$ is locally defined by the condition

$$\int_U D(f)g = \int_U f\mathcal{C}(D)(g),$$

where f and g are arbitrary compactly supported n-forms over U and $D \in \mathcal{D}_n^k$ has also compact support in U. It turns out that $\mathcal{C}$ is then a globally defined invariant.

Corollary 9 allows us to define a new invariant map, acting on $\mathcal{D}_p^k$ when $k = 1$ and $p \geqslant 1$ or $(k,p) = (2, n-1)$:

$$K : D_p^k \to \mathcal{D}_{p+1}^{k-1} : D \mapsto \mathcal{Q} \circ \delta \circ \sigma(D).$$

Finally, we recall the existence of an invariant map $K' : \mathcal{D}_0^2 \to D_1^1$, which is defined by its local expression over a chart domain of M:

$$\mathcal{Q}^{-1} \circ K' \circ \mathcal{Q}|_{\mathcal{S}_0^2} = \frac{1}{2}\delta, \quad \mathcal{Q}^{-1} \circ K' \circ \mathcal{Q}|_{\mathcal{S}_0^1} = \delta, \quad K'|_{\mathcal{D}_0^0} = 0.$$

P. Lecomte presented this operator in a more general setting as a homotopy operator for d^* and proved its global existence and invariance (see [7, p. 188]).

5.2. **Classification.** We now state the classification of all invariant maps from $\mathcal{D}_p^k$ to $\mathcal{D}_q$. Theorem 10 takes into account the case $q = p-1$. Theorem 11 and Theorem 12 give the results for $q = p$. Only a few values of k and p yield invariants when $q = p+1$. They are given, as well as the corresponding invariants themselves, by Theorem 13.

Theorem 10. *If $p < n$ or $k = 0$, then $\mathcal{I}_{p,p-1}^k$ is generated by d^*. If $p = n$ and $k \neq 0$, then $\dim(\mathcal{I}_{p,p-1}^k) = 2$, an additional generator being given by $d^* \circ \mathcal{C}$.*

Theorem 11. *Let $1 \leqslant p \leqslant n-1$. If $k \neq 1$ and $(k,p) \neq (2, n-1)$, then $\mathcal{I}_{p,p}^k$ is generated by the identity map. If $k = 1$ or $(k,p) = (2, n-1)$, then another generator is given by $d^* \circ K$.*

Theorem 12. *The space $\mathcal{I}_{0,0}^0$ (resp. $\mathcal{I}_{n,n}^0$) is generated by* id*. If $k \neq 0$, $\mathcal{I}_{0,0}^k$ (resp. $I_{n,n}^k$) is generated by* id *and $\mathcal{I}_0$ (resp. $\mathcal{C}$).*

Theorem 13. *If $k \neq 1$ and $(k,p) \notin \{(2,0),(2,n-1)\}$, then $\mathcal{I}_{p,p+1}^k = \{0\}$. If $k = 1$ or $(k,p) = (2,n-1)$, the invariants are multiples of K. If $(k,p) = (2,0)$, they are multiples of K'.*

Proof of Theorem 10. Let $T \in \mathcal{I}_{p,p-1}^k$. We follow the method outlined and use the notation introduced at the beginning of this Section. We have $\mathcal{T}_{r,p} = a_{r,p}\delta^*$, for all $r \in \{0,\ldots,k\}$. The conclusion follows if $k = 0$.

Otherwise, the first of the commutation relations (14) then yields

$$a_{r,p}\gamma_{1,p-1}(X) \circ \delta^*|_{\mathcal{S}_p^r} = a_{r-1,p}\delta^* \circ \gamma_{1,p}(X)|_{\mathcal{S}_p^r}.$$

In view of Lemma 8, we get

$$(a_{r,p} - a_{r-1,p})\delta^* \circ \gamma_{1,p}(X)|_{\mathcal{S}_p^r} = 0.$$

If $p \neq n$, $\delta^* \circ \gamma_{1,p}(X)|_{\mathcal{S}_p^r} \neq 0$ and $\dim(\mathcal{I}_{p,p-1}^k) \leqslant 1$. Since d^* is a suitable invariant, the conclusion follows.

If $p = n$, the second relation of (14) yields

$$(a_{r,p} - a_{r-2,p})\delta^* \circ \gamma_{2,n}(X)|_{\mathcal{S}^r_n} = 0,$$

for all $r \geqslant 2$. Lemma 8 then ensures that $\dim(\mathcal{I}^k_{n,n-1}) = 2$. Hence, the result. □

Proof of Theorem 11. Let $T \in \mathcal{I}^k_{p,p}$. Then $d^* \circ T \in \mathcal{I}^k_{p,p-1}$ is an invariant operator classified by Theorem 10 and thus a multiple of d^*. If we add a suitable multiple of id to T, we may assume that

$$d^* \circ T = 0.$$

Writing $\mathcal{T}_{r,p} = b_{r,p}\delta \circ \delta^* + c_{r,p}\delta^* \circ \delta$, we get $(r+p)b_{r,p}\delta^*|_{\mathcal{S}^r_p} = 0$ and thus $b_{r,p} = 0$, since $p > 0$. The first relation of (14) yields

$$c_{r,p}\gamma_{1,p}(X) \circ \delta^* \circ \delta|_{\mathcal{S}^r_p} = c_{r-1,p}\delta^* \circ \delta \circ \gamma_{1,p}(X)|_{\mathcal{S}^r_p}$$

and then

$$\delta^* \circ (c_{r,p}\gamma_{1,p+1}(X) \circ \delta - c_{r-1,p}\delta \circ \gamma_{1,p}(X))|_{\mathcal{S}^r_p} = 0, \tag{15}$$

for all $r \in \{1, \ldots, k\}$. If $k = 1$, this equation is trivial and $\dim(\mathcal{I}^1_{p,p}) \leqslant 2$. If $k > 1$, we evaluate the left-hand side on $\operatorname{im}\delta$ and obtain

$$c_{r-1,p}\delta^* \circ (\delta \circ \gamma_{1,p}(X) \circ \delta)|_{\mathcal{S}^{r+1}_{p-1}} = 0.$$

Since $\operatorname{im}\delta \cap \ker\delta^* = \{0\}$, we conclude that $\mathcal{T} = \mathcal{T}_{k,p} = c_{k,p}\delta^* \circ \delta$. Another look at (15) shows that $c_{k,p} = 0$ if $p \neq n-1$. But, if $p = n-1$ and $k > 2$, the second relation of (14) forces $c_{k,p} = 0$. Hence the result. □

Proof of Theorem 12. We proceed as in the proof of the last two theorems. We know that $\mathcal{T}_{r,p} = e_{r,p}\mathrm{id}$, for $p \in \{0, n\}$. The first (resp. second) relation of (14) yields the result when $p = 0$ (resp. $p = n$). □

Proof of Theorem 13. Let $T \in \mathcal{I}^k_{p,p+1}$. We have $T_{r,p} = d_{r,p}\delta$ and thus $\mathcal{T}_{r,p} = 0$. The result for $k = 1$ is then clear.

From Theorem 11, we know that $d^* \circ T$ is a multiple of the identity, provided that $1 \leqslant p \leqslant n-2$. This is also true if $k \geqslant 3$ and $p = n-1$. But the restriction of $d^* \circ T$ to operators of order 0 vanishes. Therefore,

$$d^* \circ T = 0.$$

We deduce that $d_{r,p}\delta^* \circ \delta|_{\mathcal{S}^r_p} = 0$ and thus $T = 0$.

Now, if $p = n-1$ and $k = 2$, we get similarly that $d^* \circ T$ is a multiple of $d^* \circ K$.

If $p = 0$ and $k \geqslant 2$, $d^* \circ T$ is a combination of the identity and $\mathcal{I}_0$. It vanishes on operators of order 0. Therefore, $d^* \circ T = a(\mathrm{id} - \mathcal{I}_0)$ for some $a \in \mathbb{R}$. This equality implies $d_{r,0}\delta^* \circ \delta = a\,\mathrm{id}$ on $\mathcal{S}^r_0$, for all $r \geqslant 1$, which in turn proves the result for $k = 2$. If $k \geqslant 3$, the first relation of (14) forces once more T to vanish. Hence, the result. □

Acknowledgements

We would like to thank P. Lecomte for valuable suggestions about this work. The first two authors thank the Belgian National Fund for Scientific Research for their Postdoctoral Researcher position and Fellowship, respectively. The last author thanks the Luxembourgian Research Department for Grant MEN/C.U.L./99/007.

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