On generalized Hölder spaces

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Abstract. The Hölder spaces \( C^\alpha(\mathbb{R}^d) \ (\alpha > 0) \) provide a natural way for measuring the smoothness of a function. These spaces appear in different areas such as approximation theory and multifractal analysis and lead to natural definitions of the notion of fractal function; for example a function belonging to \( C^{\alpha}(\mathbb{R}^d) \ (\alpha \in (0,1)) \) typically has a fractal graph. The purpose of this poster is to present a generalization of such spaces as well as some recent results about their characterizations.

Notation \( \Delta^k f(x) = f(x + h) - f(x) - \Delta^{k+1} f(x) \)

Definition of Hölder spaces \( C^\alpha(\mathbb{R}^d) \)

Let \( f \in L^p(\mathbb{R}^d) \) and \( \alpha > 0 \); we say that \( f \) belongs to \( C^\alpha(\mathbb{R}^d) \) if there exists \( C > 0 \) such that

\[
\sup_{|h| < |\epsilon|} \frac{|\Delta^k f(x)|}{|h|^\alpha} \leq C, \quad \forall x \in \mathbb{R}^d.
\]

The Hölder exponent of \( f \) is \( \alpha_f = \sup\{\alpha : f \in C^\alpha(\mathbb{R}^d)\} \).

Definition of admissible sequences

A sequence \( \sigma = (\sigma_j)_{j \in \mathbb{N}} \) of positive numbers is called admissible if there exist two positive constants \( d_0 \) and \( d_1 \) such that

\[
d_0 \leq \sigma_j \leq d_1 \sigma_{j+1}, \quad \forall j \in \mathbb{N}.
\]

Let \( \sigma_j = \inf_{k \geq j} \sigma_{k+1} \) and \( \sigma_j = \sup_{k \geq j} \sigma_{k+1} \), \( \forall j \in \mathbb{N} \).

The lower and upper Boyd index are respectively defined by

\[
s(\sigma) = \lim_{j \to +\infty} \log_{\sigma_{j+1}}(\sigma_j) \quad \text{and} \quad \sigma(\sigma) = \lim_{j \to +\infty} \log_{\sigma_{j+1}}(\sigma_j).
\]

Definition of generalized Hölder spaces \( C^{\sigma,\alpha}(\mathbb{R}^d) \)

Let \( \alpha > 0 \) and \( \sigma \) an admissible sequence. A function \( f \in L^p(\mathbb{R}^d) \) belongs to the generalized Hölder space \( C^{\sigma,\alpha}(\mathbb{R}^d) \) if there exists \( C > 0 \) such that

\[
\sup_{|h| < |\epsilon|} \frac{|\Delta^{k+1} f(x)|}{|h|^\alpha} \leq C \sigma_j, \quad \forall x \in \mathbb{R}^d, \quad \forall j \in \mathbb{N}.
\]

Remark The notion of admissible sequence generalizes the notion of modulus of continuity. Indeed, moduli of continuity are exactly decreasing admissible sequences.

Link with generalized Besov spaces

If \( \sigma(\sigma) = 0 \), it can be shown that generalized Hölder spaces are indeed generalized Besov spaces \( \mathcal{B}^{\sigma,\alpha}_{\infty,\infty} \) (see [4]).

Example Let \( \sigma_j = (2^j)^{|\log_2(2^j)|} \) for \( j \in \mathbb{N} \), A. Khintchine proved that the trajectories of a Brownian Motion belong almost surely to \( C^{\sigma,\alpha}(\mathbb{R}) \) (\( 0 < \alpha < 1 \)).

A result à la Lion-Peetre

Let \( 1 < m < n \) and \( \alpha > 0 \); with \( 1 < \alpha < m \), \( \sigma = (\sigma_j)_{j \in \mathbb{N}^n} \) an admissible sequence and \( f \) a bounded continuous function on \( \mathbb{R} \) such that

\[
\sup_{|h| < |\epsilon|} \frac{|\Delta^k f(x)|}{|h|^\alpha} \leq C \sigma_j, \quad \forall x \in \mathbb{R}, \quad \forall j \in \mathbb{N}.
\]

We've got

\[
\sup_{|h| < |\epsilon|} \frac{|\Delta^k f(x)|}{|h|^\alpha} \leq C \sigma_j, \quad \forall x \in \mathbb{R}, \quad \forall j \in \mathbb{N}.
\]

A characterization by polynomials

Let \( N \in \mathbb{N} \) and \( \sigma = (\sigma_j)_{j \in \mathbb{N}^n} \) be a decreasing admissible sequence such that

\[
\sum_{j=1}^{\infty} \frac{2^{m(\sigma_j)}}{\sigma_j} \sum_{j=1}^{\infty} 2^{j(\sigma_j)} \sigma_j < \infty
\]

A characterization by polynomials (see [3])

\[
C^{\sigma,\alpha}(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : \sup_{j \in \mathbb{N}} \left( \sup_{|x| < |\epsilon|} \frac{|\Delta^k f(x)|}{|x|^\alpha} \right) \leq C \sigma_j \right\}
\]

A characterization by wavelet coefficients (see [3])

\[
C^{\sigma,\alpha}(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : \sup_{j \in \mathbb{N}} \left( \sup_{|x| < |\epsilon|} \frac{|\Delta^k f(x)|}{|x|^\alpha} \right) \leq C \sigma_j \right\}
\]

Examples Let \( \sigma = (\sigma_j)_{j \in \mathbb{N}} \) be an admissible sequence such that \( g(\sigma) = 0 \). Then

\[
C^{\sigma,\alpha}(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : \sup_{j \in \mathbb{N}} \left( \sup_{|x| < |\epsilon|} \frac{|\Delta^k f(x)|}{|x|^\alpha} \right) \leq C \sigma_j \right\}
\]

A characterization by wavelet coefficients (see [3])

\[
C^{\sigma,\alpha}(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : \sup_{j \in \mathbb{N}} \left( \sup_{|x| < |\epsilon|} \frac{|\Delta^k f(x)|}{|x|^\alpha} \right) \leq C \sigma_j \right\}
\]

A characteristic by the convolution

Let \( \sigma = (\sigma_j)_{j \in \mathbb{N}} \) be an admissible sequence such that \( g(\sigma) > 0 \). Then

\[
C^{\sigma,\alpha}(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : \sup_{j \in \math{\bf \text{References.}}}
\]