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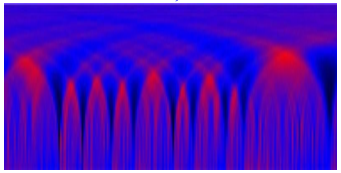

On generalized Hölder spaces

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Abstract. *The Hölder spaces* $C^\alpha(\mathbb{R}^d)$ *(* $\alpha > 0$ *) provide a natural way for measuring the smoothness of a function. These spaces appear in different areas such as approximation theory and multifractal analysis and lead to natural definitions of the notion of fractal function; for example a function belonging to* $C^\alpha(\mathbb{R}^d)$ *(* $\alpha \in (0,1)$ *) typically has a fractal graph. The purpose of this poster is to present a generalization of such spaces as well as some recent results about their characterizations .*

Notation $\Delta_h^1 f(x) = f(x+h) - f(x), \quad \Delta_h^{n+1}(x) = \Delta_h^n f(x+h) - \Delta_h^n f(x)$

Definition of Hölder spaces $C^\alpha(\mathbb{R}^d)$

Let $f \in L^\infty(\mathbb{R}^d)$ and $\alpha > 0$; we say that f belongs to $C^\alpha(\mathbb{R}^d)$ if there exists $C, R > 0$ such that

$$\sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \alpha \rfloor + 1} f\|_{L^\infty(\mathbb{R}^d)} \leq C 2^{-j\alpha}, \quad \forall j \in \mathbb{N}.$$

The Hölder exponent of f is $h_f = \sup\{\alpha : f \in C^\alpha(\mathbb{R}^d)\}$.

Definition of admissible sequences

A sequence $\sigma = (\sigma_j)_{j \in \mathbb{N}}$ of positive numbers is called *admissible* if there exists two positive constants d_0 and d_1 such that

$$d_0 \sigma_j \leq \sigma_{j+1} \leq d_1 \sigma_j, \qquad j \in \mathbb{N}.$$

Let

$$\underline{\sigma}_j := \inf_{k \geq 0} \frac{\sigma_{j+k}}{\sigma_k} \qquad \text{and} \qquad \overline{\sigma}_j := \sup_{k \geq 0} \frac{\sigma_{j+k}}{\sigma_k}, \qquad j \in \mathbb{N}.$$

The *lower and upper Boyd index* are respectively defined by

$$\underline{s}(\sigma) := \lim_{j \to +\infty} \frac{\log_2(\underline{\sigma}_j)}{j} \quad \text{and} \quad \overline{s}(\sigma) := \lim_{j \to +\infty} \frac{\log_2(\overline{\sigma}_j)}{j}.$$

Definition of generalized Hölder spaces $C^{\sigma,\alpha}(\mathbb{R}^d)$

Let $\alpha > 0$ and σ an admissible sequence. A function $f \in L^\infty(\mathbb{R}^d)$ belongs to the *generalized Hölder space* $C^{\sigma,\alpha}(\mathbb{R}^d)$ if there exists $C > 0$ such that

$$\sup_{x, |h| \leq 2^{-j}} |\Delta_h^{\lfloor \alpha \rfloor + 1} f(x)| \leq C \sigma_j \qquad \forall j \in \mathbb{N}_0.$$

Remark The notion of admissible sequence generalizes the notion of modulus of continuity. Indeed, moduli of continuity are exactly decreasing admissible sequences .

Link with generalized Besov spaces

If $\underline{s}(\sigma^{-1}) > 0$, it can be shown that generalized Hölder spaces $C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ are indeed generalized Besov spaces $B^{\sigma^{-1}}_{\infty,\infty}$ (see [4]).

Example Let $\sigma_j := (2^{-j})^{\frac{1}{2}} |\log|\log(2^{-j})||^{\frac{1}{2}}$ for $j \in \mathbb{N}_0$. A. Khintchine proved that the trajectories of a Brownian Motion belong almost surely to $C^{\sigma,\alpha}(\mathbb{R})$ ($0 < \alpha < 1$).

A result *à la* Lion-Peetre

Let $1 < m \in \mathbb{N}$, $\alpha > 0$ with $1 \leq \alpha \leq m$, $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ an admissible sequence and f a bounded continuous function on $\mathbb{R}$ such that $\sup_{|h| \leq 2^{-j}} \|\Delta_h^m f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0.$

If

$$\sum_{j=1}^{+\infty} 2^{j(m-\alpha)} \sigma_j \to$$

then $\forall \lambda \in]0,1[$, there exists two functions

$$F_1^\lambda \in C^{m-(\alpha-1)}(\mathbb{R}), \quad F_2^\lambda \in C^{m-\alpha}(\mathbb{R})$$

with $f = F_1^\lambda + F_2^\lambda$ and such that for $K_\lambda := \lfloor 2 \log_2(1/\lambda) \rfloor + 1$ we've got

$$\sup_{|h| \leq 2^{-l}} \|\Delta_h^m F_2^\lambda\|_{L^\infty} \leq C_1 2^{-l(m-\alpha)} \sum_{j=K_\lambda+1}^{+\infty} 2^{j(m-\alpha)} \sigma_j$$

and

$$\sup_{|h| \leq 2^{-l}} \|\Delta_h^m F_1^\lambda\|_{L^\infty} \leq C_2 2^{-l(m-\alpha+1)} \lambda^{-2} \sum_{j=1}^{K_\lambda} 2^{j(m-\alpha)} \sigma_j$$

(where C_1 and C_2 are two constant independent of λ).

Corollary (link with classical regularity)

Let $K \in \mathbb{N}_0$ and $\sigma = (\sigma_j)_{j \in \mathbb{N}}$ an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$ and $\sum_{j=1}^{+\infty} 2^{jK} \sigma_j < \infty.$

If $f \in C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ then f is K-times continuously differentiable.

A characterization by the convolution

Let $\sigma = (\sigma_j)_{j \in \mathbb{N}}$ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$. Then

$$C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = \left\{ f \in L^\infty(\mathbb{R}^d) : \exists \Phi \in C_c^\infty(\mathbb{R}^d) \quad \sup_{j \in \mathbb{N}} \left(\sigma_j^{-1} \sup_{\delta \leq 2^{-j}} \|f \star \Phi_\delta - f\|_{L^\infty} \right) < \infty \right\}.$$

(where $\Phi_\delta = \delta^{-d} \Phi(x/\delta)$).

A characterization by polynomials

Let $\sigma = (\sigma_j)_{j \in \mathbb{N}}$ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$. Let $M \in \mathbb{N}$ such that $M > \overline{s}(\sigma)$, then

$$C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = \left\{ f \in L^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \left(\sup_{j \in \mathbb{N}} \left(\sigma_j^{-1} \inf_{P \in \mathbb{P}_{M-1}} \|f - P\|_{L^\infty(B(x,2^{-j}))} \right) \right) < \infty \right\}.$$

A characterization by wavelet coefficients (see [3])

Let $N \in \mathbb{N}_0$ and $\sigma = (\sigma_j)_{j \in \mathbb{N}}$ be a decreasing admissible sequence such that

$$\sum_{j=0}^{J} 2^{Nj} \sigma_j \leq C 2^{NJ} \sigma_J,$$
$$\sum_{j=J}^{+\infty} 2^{(N-1)j} \sigma_j \leq C 2^{(N-1)J} \sigma_J$$

For all $J \in \mathbb{N}$. Let be a multiresolution analysis of regularity $r > N$. Then the following are equivalent:

1. $f \in C^{\sigma,N-1}(\mathbb{R}^d)$;
2. $\exists C > 0 : \begin{cases} \sup_{k \in \mathbb{Z}^d} |C_k| \leq C \\ \sup_{k \in \mathbb{Z}^d} |c_{j,k}^i| \leq C\sigma_j, \quad \forall j \geq 0, \forall i \in \{1, \ldots, 2^d - 1\}, \end{cases}$

where C_k and $c_{j,k}^i$ are the classical wavelet coefficients associated with the multiresolution analysis (they correspond respectively to the father wavelet and the mother wavelet).

References.

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