Partition of the circle in cells of equal area and shape

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Abstract

In the present paper, a new method for partitioning a circle in cells is presented. Cells have equal areas and shapes, so the method is termed the Isocell method. Its most interesting property is that cell centres are uniformly distributed inside the circle. Among possible applications of the Isocell method, the calculation of view factors by ray-tracing (image rendering, radiative heat transfer modelling) is presented.

Keywords: circle partition, cells, ray-tracing, view factor, rendering, projection

1 Introduction

Partition of common geometrical shapes and surfaces has always been a major concern in many different fields such as geography, cartography, computer graphics or scientific visualisation. Partition is often linked with tessellation, subdivision or tiling methods. Even though the purposes of these methods are different, the principle remains the same: decompose a complex surface in basic elements, usually triangles or quadrangles.

In the field of Digital Terrain Models, Bjørke et al. [?] develop a global grid model based on quadrilateral cells of almost constant area. They apply their model to the case of a space corpse with an ellipsoid shape. Chukkapalli et al. [?] present a scheme to generate unstructured grids on the sphere. Their method is based on a spiral going from one pole to the other. Stuhne and Peltier [?] study the shallow water equations on the sphere using an icosahedral grid-point discretisation. Leopardi [?] presents the partitions of the unit sphere into regions of equal area and small diameter as well as several applications (optimal packing of caps, Voronoi tessellation ...). Cox [?] studies the problem of deformable bubbles confined in an equilateral triangle and a circle (Fig. 1). The shapes are filled with hexagonal cells and their method is based on Voronoi diagrams.

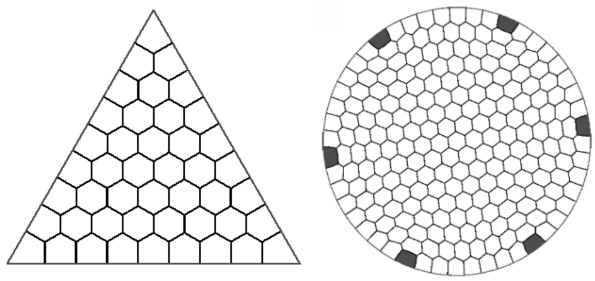

Figure 1: Hexagonal cells in an equilateral triangle and a circle [?].

The starting point of this work was the need for an efficient shooting method for ray-tracing, in order to compute view factors more accurately. Deriving from the famous Nusselt's analogy [?], ray directions may be set by uniformly sampling points within a circle (each point defines the 3D direction of a ray, see section 4). The most used technique is to distribute points in a pure random way (Monte Carlo based methods, [?]) but a large number of rays need to be cast in order to obtain a good precision.

An interesting alternative is to uniformly divide the circle in cells and shoot one ray within each cell. The most common partition of the circle as well as the

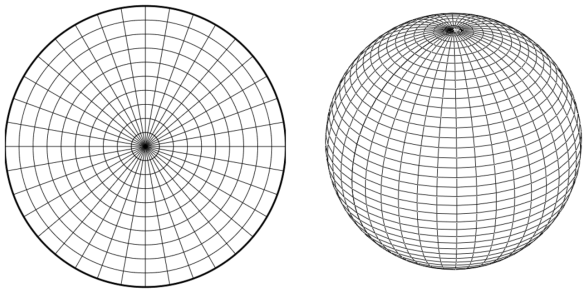

Figure 2: Partition of the circle and the sphere along meridians and parallels.

sphere (Fig. 2) is the well-known division in meridians and parallels (as the earth globe). The drawbacks of this method are the non uniform cell shape and aspect ratio and the poor distribution of vertices inside the circle (concentration near the centre) or on the sphere (concentration at the earth poles). In order to overcome this issue, a new partition method was investigated.

2 Circle partition

Following the previous discussion, a proper partition of the circle should lead to cells of almost constant area and shape. First, the circle of unit radius is divided into n equally spaced rings having a radial height $\triangle r$ equal to $1/n$. The smallest ring is divided in N_1 cells (Fig. 3).

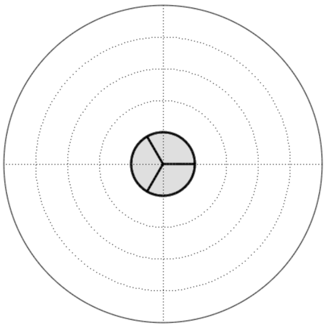

Figure 3: Division of the first ring ($N_1 = 3$).

Each of the N_1 first cells has an area equal to

$$A_1 = \frac{\pi \Delta r^2}{N_1} = \frac{\pi}{N_1 n^2} \quad (1)$$

The second ring (Fig. 4) has to be divided in such a way that cell area is as close as possible to area A_1. Dividing the ring area by the area of the first ring cells A_1, the number of rings N_2 is given by

$$N_2 = \frac{\pi 4 \Delta r^2 - \pi \Delta r^2}{A_1} = 3N_1 \quad (2)$$

A first interesting property is that an integer number $(3N_1)$ of cells of area A_1 fit in the second ring.

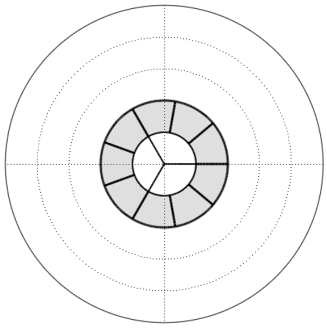

Figure 4: Division of the second ring ($N_1 = 3$).

Following the same scheme, the third ring contains a number of cells equal to

$$N_3 = \frac{\pi 9 \Delta r^2 - \pi 4 \Delta r^2}{A_1} = 5N_1 \quad (3)$$

This division leads to n rings each composed of $(2i-1)N_1$ cells of area A_1 as shown in Fig. 5. The number of cells in the n rings is equal to the suite of the first n odd numbers multiplied by the number of divisions of the first ring N_1.

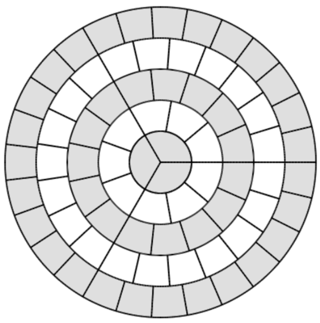

Figure 5: Final division of the circle ($N_1 = 3$).

Finally the circle is divided into a number of cells equal to

$$N_{tot} = N_1 \sum_{i=1}^{n} (2i-1) = N_1 n^2 \qquad (4)$$

Fig. 6 shows the partitions obtained with different numbers of initial divisions (N_1) and different total numbers of cells.

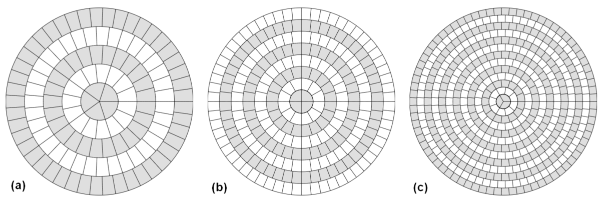

Figure 6: Circle partition with $N_1 = 5$ and 125 cells (a), $N_1 = 4$ and 256 cells (b) and $N_1 = 3$ and 507 cells (c).

Provided a required number of cells N_{target}, the real number of rings n^* is given by

$$n^* = \sqrt{\frac{N_{target}}{N_1}} \qquad (5)$$

and the actual number of rings n is the integer value strictly greater than n^*. Then the total number of cells is given by Eq. 4.

3 Cell properties

By construction, each cell has a unique area A_1 and a quadrangle-shape with two circular arcs, except the N_1 inner cells that degenerate into curved triangles.

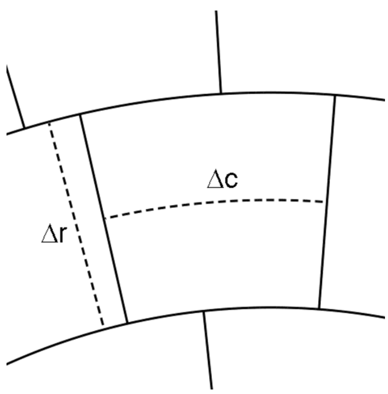

Figure 7: Cell height and width.

For the i^{th} ring, the aspect ratio λ_i of a cell, defined as the ratio between the radial height Δr and the circular width Δc at the middle of the cell (Fig. 7), is given by

$$\lambda_i = \frac{\Delta r}{\Delta c} = \frac{N_1}{\pi} \qquad (6)$$

The aspect ratio of the cells is thus a constant depending on the number of division of the first ring N_1. The most compact shapes correspond to an aspect ratio as close as possible to one. Thus the best number N_1 of initial divisions is 3 (the real optimum being π).

The perimeter ρ_i of the cells of the i^{th} ring is given by

$$\rho_i = 2\Delta r + \Delta c_{int} + \Delta c_{ext} = 2\Delta r \left(1 + \frac{\pi}{N_1}\right) = \rho \qquad (7)$$

The division of the circle leads to cells of constant area A_1 and constant perimeter ρ. Thus, the shape coefficient β, defined as the ratio between the square of the cell perimeter and the cell area, is also constant and equal to

$$\beta = \frac{\rho^2}{A_1} = \frac{4\left(N_1 + \pi\right)^2}{\pi N_1} \qquad (8)$$

The plot of coefficient β against the number of initial divisions N_1 is given in Fig. 8. Values of the shape coefficient for the circle (4π) and for the square (16) are added in the figure. In order to obtain the most compact cells, coefficient β must be as small as possible, i.e. achieving the smallest perimeter for a given area.

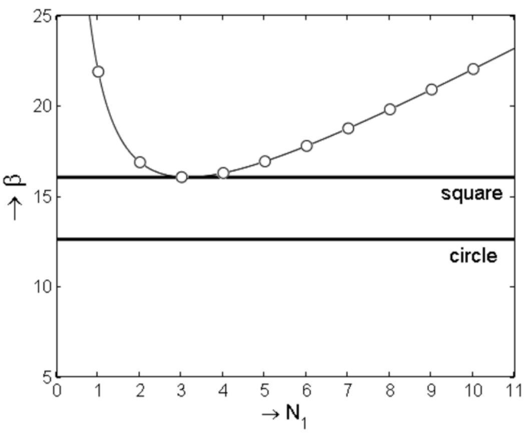

Figure 8: Shape coefficient for different values of the initial division N_1.

The first derivative of coefficient β

$$\frac{d\beta}{dN_1} = \frac{4}{\pi}\frac{N_1^2 - \pi^2}{N_1^2} \quad (9)$$

is equal to zero when N_1 is equal to π while its second derivative,

$$\frac{d^2\beta}{dN_1^2} = \frac{8\pi}{N_1^3} \quad (10)$$

is always positive. Thus the optimal value of N_1 is π for which the shape coefficient β is equal to 16. The integer closest to π is 3, for which cells are almost as good as squares ($\beta = 16.0085$).

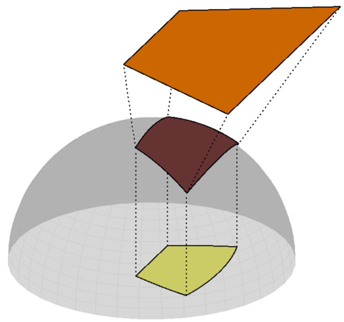

Figure 10: Orthographic projection.

4 Ray-tracing

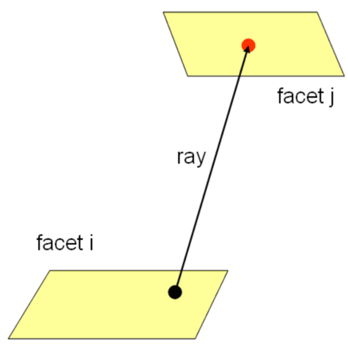

Figure 9: A ray emitted by facet i and reaching facet j.

In heat transfer problems, radiative exchange simulation requires the computation of the view factors between facets of 3D finite element models [?]. For strictly diffuse radiation, the definition of the view factor $F_{i\to j}$ between facet i and facet j (Fig. 9) is the amount of diffuse energy emitted by facet i that reaches facet j divided by the total amount of energy emitted by facet i. Nusselt [?] demonstrated that the point wise view factor from a point P on a facet i to a facet j is equal to the area of its orthographic projection divided by π. The orthographic projection is composed of a projection on the unit sphere centred on point P and an orthogonal projection onto the plane of facet i (Fig. 10). This is known as Nusselt's analogy.

Several methods were derived from Nusselt's analogy. Ray-tracing based methods are the most general and efficient ones. Malley [?] developed a method for which the view factor is computed by uniformly sampling the unit disc (i.e. the orthogonal projection of the unit sphere). Each point on the unit disc defines the direction of a ray in the space (Fig. 11).

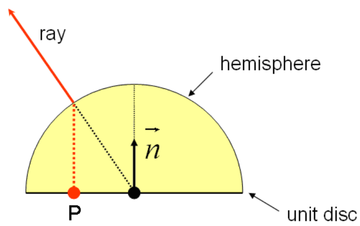

Figure 11: A point P on the unit disc defines the ray direction.

Ray-tracing consists in shooting from each facet a set of rays in the half-space surrounding it. The view factor $F_{i\to j}$ is equal to

$$F_{i\to j} = \frac{M_j}{M_i} \quad (11)$$

where M_j is the number of rays emitted by facet i that reach facet j and M_i is the total number of rays emitted by facet i.

The most used method to distribute the rays within the unit disc is the Monte Carlo method [?], i.e. the ray positions are set in a purely random way (Fig. 12a). This method always converges but requires a great number of rays to achieve a good precision. Vueghs et al. [?] propose an improved method called the Stratified Hemisphere and based on a prior division of the circle in cells of equal area (similar to the meridians and parallels of the earth globe). A ray is shot in a random position inside each cell (Fig. 12b). This method proved to be more efficient than the purely random method since the rays are more smoothly distributed within the circle. Its main drawback is that the cells have very different shapes, needle-shaped in the centre of the circle and flat near the perimeter.

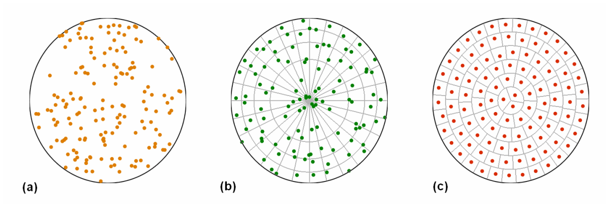

Figure 12: Ray distribution methods; Monte Carlo (a), Stratified Hemisphere (b) and Isocell (c).

The Isocell method does not exhibit such behaviour as shown in Fig. 12c. The cell shape is constant over the circle and so is the precision. In order to avoid aliasing, the first cell of each ring is shifted from 0 with a random angle so that the distribution exhibits no symmetry.

Basically, the distribution methods (Monte Carlo, Stratified Hemisphere and Isocell) are similar to an integration method over a circular domain, the integration points being the ray positions within the circle (we will denote them "cell centres" for the sake of simplicity). Several tests were performed to compare these three distribution methods. The first one consists in moving a simple shape (a circle for instance) of a known area A_0 inside the unit disc and measuring its approximated area A calculated as the sum of the areas of cells for which the centre lies inside the moving shape. In the example in Fig. 13a, there are four centres lying inside the circular shape so its approximated area is the sum of the area of the four coloured cells (Fig. 13b).

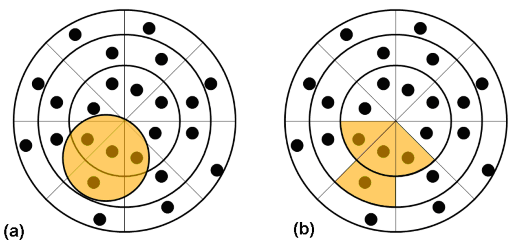

Figure 13: Cell centres lying inside the moving circle (a) and approximated area (b).

For a given position of the moving shape, the relative error is given by

$$\varepsilon = \frac{|A - A_0|}{A_0} \qquad (12)$$

The moving shape is moved thousands of times and the relative error is plot each time in function of the moving shape radial position. Representative results are given in Figs. 14 and 15.

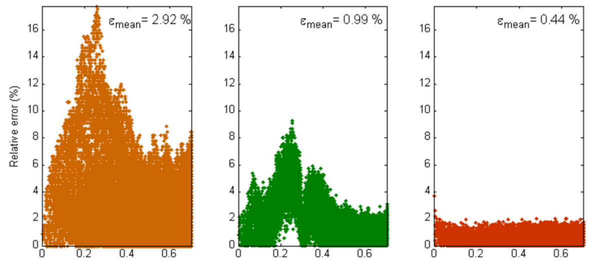

Figure 14: Relative error for a moving circle of radius 0.3 and 5000 rays; Monte Carlo distribution (left), Stratified Hemisphere (centre) and Isocell (right).

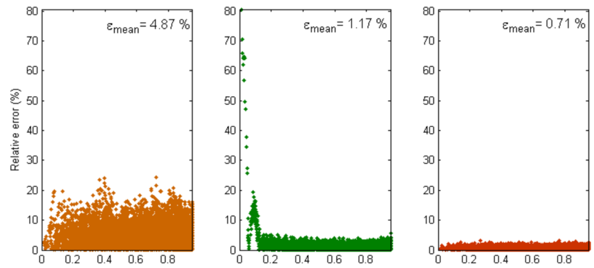

Figure 15: Relative error for a moving circle of radius 0.05 and 100000 rays; Monte Carlo distribution (left), Stratified Hemisphere (centre) and Isocell (right).

The integration error of the Monte Carlo method is very high compared to the other two methods. That is why it converges slowly and requires a high number of rays. The method of the Stratified Hemisphere presents a better behaviour except for some radial position of the moving shape: about 0.3 for a circle of radius 0.3 and about 0 for a circle of radius 0.05. This is due to the fact that the cell centres are badly distributed in the middle of the unit disc (Fig. 12b). In fact, the error is made on the cells that are partially covered by the moving shape, i.e. cells located on the circumference in the case of a moving circle. So, for the Stratified Hemisphere, the error increases when the border of the moving shape is located near the centre of the unit disc.

Unlike the stratified hemisphere method, the Isocell method performs equally for any radial position of the moving shape. Compared to Monte Carlo based methods, the Isocell method achieves a precision about 10 times greater for a given number of rays. Put it another way, it is possible to shoot 10 times less rays with the Isocell method to achieve the same precision. This is very useful since ray-tracing methods are known to be highly time consuming.

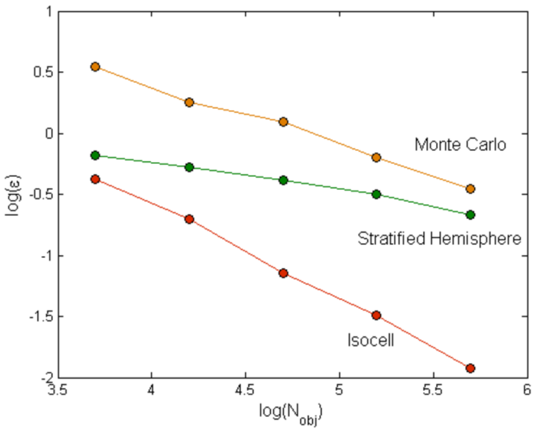

Figure 16: Convergence of distribution methods.

Convergence tests were also carried out. Here, the shape is located at a given position and the number of rays is increased from 5000 to 500000 (Fig. 16). The Monte Carlo method converges slower than the Isocell method, the rates being respectively equal to 0.5 and 0.75. Again, depending on the shape position, the method of the Stratified Hemisphere may converge even slower than the pure random method (0.25 in the case of Fig. 16).

5 Conclusion

The Isocell method is a new method for partitioning the circle in cells of equal area and shape. Its natural simplicity makes it appealing for various applications. It outclasses the usual partition in cells along radiuses and inner circles (meridians and parallels). The only weakness of the Isocell method is that the cells are not conforming such as for a usual mesh. Nevertheless, the Isocell method should apply very well for most calculation methods based on a circular domain. The computation of the view factors with ray-tracing, presented in this paper, is an application for which the Isocell method brings a great improvement on other methods in terms of performance and precision.

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