Multiqubit symmetric states with high geometric entanglement

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Geometric measure of entanglement of an N-qubit pure state

Definition : $E_G(|\psi\rangle) = 1 - \max_{|\Phi\rangle = |\phi_1,\phi_2,\phi_3,\ldots\rangle} |\langle\psi|\Phi\rangle|^2$

where the maximum is taken over all separable states $|\Phi\rangle$. [1]

The explicit value is only known in a few cases because the optimization procedure is very complicated in the general case.

Upper bound : $E_G(|\psi\rangle) \leqslant 1 - \frac{1}{2^{N-1}}$ [2]

[1] T.-C. Wei and P.M. Golbart, PRA 68, 042307 (2003)

[2] E. Jung et al., PRA 77, 062317 (2008)

Majorana representation of an N-qubit symmetric state

Any symmetric state can be written in the form

2-qubit : $|\psi_S\rangle = \mathcal{N}(|\phi_1,\phi_2\rangle + |\phi_2,\phi_1\rangle)$

3-qubit : $|\psi_S\rangle = \mathcal{N}(|\phi_1,\phi_2,\phi_3\rangle + |\phi_1,\phi_3,\phi_2\rangle + |\phi_2,\phi_1,\phi_3\rangle + \cdots)$

N-qubit : $|\psi_S\rangle = \mathcal{N}\sum_\sigma |\phi_{\sigma(1)},\cdots,\phi_{\sigma(N)}\rangle$

where the sum runs over all permutations ($N!$).

Thus, any N-qubit symmetric state is fully dtermined by N single qubit states

$$|\phi_i\rangle = \cos(\theta_i/2)|0\rangle + e^{i\varphi_i}\sin(\theta_i/2)|1\rangle$$

and can be represented by N points on the Bloch sphere (Majorana points). Any symmetric separable state is of the form $|\Phi\rangle = |\phi,\ldots,\phi\rangle$ and is represented by N identical points.

Geometric measure of entanglement of a pure symmetric state

Theorem : $E_G(|\psi_S\rangle) = 1 - \max_{|\Phi\rangle = |\phi,\phi,\phi,\ldots\rangle} |\langle\psi_S|\Phi\rangle|^2$

where the maximum is taken over all symmetric separable states (huge simplification !). [3]

Upper bound (this work) : $E_G(|\psi_S\rangle) < 1 - \frac{1}{N+1}$ [4]

↘ average |overlap|^2

[3] R. Hübener et al., PRA 80, 032324 (2009)

[4] J. Martin et al., PRA 81, 062347 (2010)

Some examples [1]

GHZ states : $|\text{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0,\ldots,0\rangle + |1,\ldots,1\rangle)$

$E_G(|\text{GHZ}_N\rangle) = 1/2$

Dicke states : $|D_N(k)\rangle = \frac{1}{\sqrt{C_N^k}}\sum_\sigma |\underbrace{0,\ldots,0}_{N-k},\underbrace{1,\ldots,1}_{k}\rangle$

$$E_G(|D_N(k)\rangle) = 1 - C_N^k\left(\frac{k}{N}\right)^k\left(\frac{N-k}{N}\right)^{N-k}$$

Balanced Dicke states ($k = [N/2]$) :

$$E_G(|D_N([N/2])\rangle) = 1 - \sqrt{\frac{2}{\pi N}} + \mathcal{O}(N^{-3/2})$$

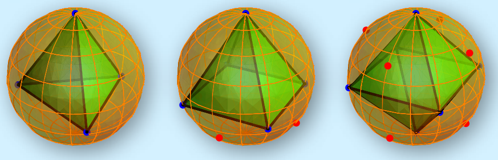
Highest geometric entanglement configurations

N=2 : $E_G|_{\text{max}} = 1/2$ for $|\psi_S\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$

N=3 : $E_G|_{\text{max}} = 5/9$ for $|\psi_S\rangle = \frac{1}{\sqrt{3}}(|0,0,1\rangle + |0,1,0\rangle + |1,0,0\rangle)$ → $|D_3(1)\rangle \equiv |W\rangle$

N=4 : $E_G|_{\text{max}} = 2/3$

N=5 : $E_G|_{\text{max}} = 0.7011\ldots$ N=6 : $E_G|_{\text{max}} = 7/9$

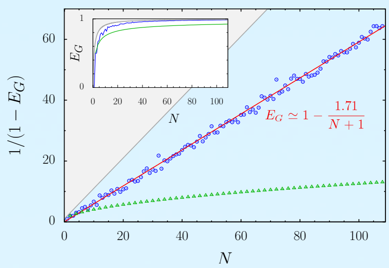


$N = 4$ $N = 5$ $N = 6$

Majorana points of the maximally entangled symmetric states for $N = 4 - 6$ (polyhedron vertices). Red points correspond to the closest separable states.

Coulomb configurations

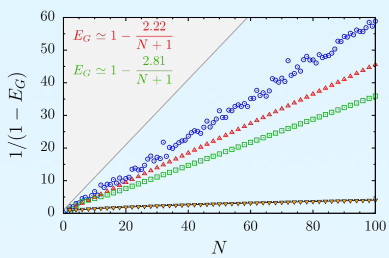
The highest geometric entanglement is obtained with states having points largely spread on the Bloch sphere, similar to how N equal electrical charges tend to place themselves as far as possible from each other when they are constrained to the surface of a conducting sphere (Thomson problem). Though similar, the Thomson problem remains distinct from the quest of maximal entanglement as it corresponds to finding charge positions $\mathbf{r}_i$ minimizing the electrostatic energy $E = \sum_{i,j}\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$. We nevertheless can expect high E_G :

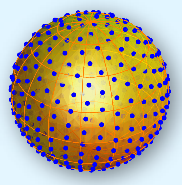


Geometric measure of entanglement of symmetric states for the Coulomb arrangement of the Majorana points for N up to 110 (blue circles). Green triangles = balanced Dicke states. Grey shaded area = domain ruled out by the upper bound. For some N (notably 4 and 6), it corresponds to the highest possible geometric entanglement.

More regular behavior with respect to N is obtained when considering equally weighed superpositions of Dicke states with pseudo-random quadratic phases :

$$|\psi_\gamma(N)\rangle = \sum_{k=0}^{N}\frac{e^{i\gamma k^2}}{\sqrt{N+1}}|D_N(k)\rangle \quad (1)$$





Left : Geometric measure of entanglement as a function of N for (blue) Coulomb arrangement, (red) states (1) with $\gamma = 2/3$, (green) states (1) with $\gamma = 1$, and (orange) equally weighted superpositions with linear phases $e^{i\gamma k}$ instead → much less entanglement. Right : Majorana representation of state (1) for $\gamma = 2/3$ and $N = 400$.