IFFT-EQUIVARIANT QUANTIZATIONS

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Abstract. The existence and uniqueness of quantizations that are equivariant with respect to conformal and projective Lie algebras of vector fields were recently obtained by Duval, Lecomte and Ovsienko. In order to do so, they computed spectra of some Casimir operators. We give an explicit formula for those spectra in the general framework of *IFFT*-algebras classified by Kobayashi and Nagano. We also define *tree-like* subsets of eigenspaces of those operators in which eigenvalues can be compared to show the existence of IFFT-equivariant quantizations. We apply our results to prove the existence and uniqueness of quantizations that are equivariant with respect to the infinitesimal action of the symplectic (resp. pseudo-orhogonal) group on the symplectic (resp. pseudo-orthogonal) Grassmann manifold.

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1. Introduction

The word "quantization" carries several different meanings, both in physics and mathematics. One approach — see for instance [12] — is to consider a quantization procedure as a linear bijection from the space of *symbols* $\mathrm{Pol}(T^*M)$ of smooth functions on the cotangent bundle of a manifold M that are polynomial along the fibres to the space $\mathcal{D}_{\frac{1}{2}}(M)$ of linear differential operators acting on half-densities. It is known that these spaces cannot be canonically identified. In other words, there does not exist a preferred quantization procedure.

The concept of *equivariant quantization* was introduced and developed in [10, 11] and [2]. These recent works take care of the symmetries of the classical situation to quantize.

If G is a group acting on the manifold M, a G-equivariant quantization is an isomorphism of representations of G between the spaces of symbols and of differential operators. Obviously, such an identification does not exist for all groups G acting on M : for instance those spaces are not equivalent as *Diff*(M)-modules. At the infinitesimal level, if G is a Lie group, its action gives rise to a Lie subalgebra $\mathfrak{g}$ of vector fields over M and one is led to build a $\mathfrak{g}$-equivariant linear bijection. Lecomte and Ovsienko examined the case of a projective structure on a manifold of dimension n, with $G = SL(n+1, R)$ and then, together with Duval, the case of the group $G = SO(p+1, q+1)$

on a manifold of dimension $p + q$. That latter group defines conformal transformations with respect to a pseudo-Riemannian metric.

In these works, the authors consider the more general modules $\mathcal{D}_{\lambda,\mu}$ of differential operators transforming λ-densities into μ-densities. These parameters give rise to the *shift* value $\delta = \mu - \lambda$ and to the special case $\delta = 0$, which can be specialized to the original problem. They obtain existence and uniqueness (up to normalization) results for a quantization procedure in both projective and conformal cases, provided the shift value does not belong to a *critical* set. Furthermore, they show that this set never contains zero.

In suitable charts, the subalgebras mentioned up to now are realized by polynomial vector fields and they share the property of being maximal proper subalgebras of the algebra of polynomial vector fields.

In [1], we investigated this maximality property and showed that the finite dimensional, graded and maximal proper subalgebras of the Lie algebra of polynomial vector fields over a Euclidean vector space correspond to the list of so called "Irreducible Filtered Lie algebras of Finite Type"(IFFT-algebras), classified by S. Kobayashi and T. Nagano in [7].

Our concern in this paper is to deal with the natural next question : *"Is it possible to build (unique) equivariant quantizations with respect to the IFFT-algebras ?"*

The original construction of the conformally equivariant quantization (see [2]) involves the computation of the spectrum of the Casimir operator of $so(p + 1, q + 1)$ acting on the space of symbols. The obstructions to the existence of a quantization show up as equalities among some eigenvalues of that operator. It was also shown in [2] how the relevant eigenvalues that should be compared are associated to *tree-like subsets* of eigenspaces.

Section 3 of the present article is devoted to this computation. We obtain, for a wide range of IFFT-algebras, a formula where the eigenvalues are expressed in terms of the dimension of the manifold and of the highest weights of some finite dimensional representations of the semisimple part of the linear isotropy algebra of $\mathfrak{g}$ (see [8]).

In Section 4, we propose a general definition for the above-mentioned tree-like subsets. A few elementary properties of these subspaces allow us to reformulate the existence theorem for equivariant quantizations in the framework of IFFT-algebras.

We later apply these results in Section 5. The Lie algebras of fundamental vector fields associated to the action of the symplectic (resp. pseudo-orthogonal) group on the Lagrangian (resp. pseudo-orthogonal) Grassmann manifold are indeed IFFT. We prove existence and uniqueness results for equivariant quantizations with respect to both of those algebras. Once more, these results hold outside of a critical set of values of the shift. We furthermore prove that this set never contains zero.

2. Basic definitions and notation

Here we recall the definitions of the fundamental objects involved in this work. For the most part, we will follow the notation of [2, 10] and we refer the reader to these papers for more detailed information. It will also be sufficient for our computations to fix our notation over vector spaces.

Throughout this section, V will be a d-dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Whenever E is a vector bundle over V, the space of *sections* of E, which we will write $\Gamma(E)$, is taken to be the space of C_∞ sections if $\mathbb{K} = \mathbb{R}$ or the space of holomorphic sections if $\mathbb{K} = \mathbb{C}$.

2.1. **Tensor densities and differential operators.** Let us denote by $\Delta^\lambda(V) \to V$ the line bundle of tensor densities of weight λ over V and by $\mathcal{F}_\lambda$ the space $\Gamma(\Delta^\lambda(V))$. There exists a natural representation L of the Lie algebra of vector fields $\mathrm{Vect}(V)$ on $\mathcal{F}_\lambda$. In local coordinates, the Lie derivative is given by

$$L_X\phi = X.\phi + \lambda\,\mathrm{tr}(\frac{\partial}{\partial x}X)\phi, \quad \forall X \in \mathrm{Vect}(M), \forall \phi \in \mathcal{F}_\lambda, \tag{1}$$

where $\frac{\partial}{\partial x}X$ denotes the Jacobian matrix of X.

Let now $\mathcal{D}_{\lambda,\mu}$ be the space of linear differential operators from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$. The representation $\mathcal{L}^{\lambda,\mu}$ of $\mathrm{Vect}(M)$ on $\mathcal{D}_{\lambda,\mu}$ is induced by L :

$$\mathcal{L}^{\lambda,\mu}D = L \circ D - D \circ L.$$

In order to keep the notations light, we will simply write $\mathcal{L}$ for $\mathcal{L}^{\lambda,\mu}$ unless that leads to confusion.

To the module $\mathcal{D}_{\lambda,\mu}$ is associated the *shift* value $\delta = \mu - \lambda$.

2.2. **Symbols.** The *symbol space of degree* k associated to $\mathcal{D}_{\lambda,\mu}$, which we denote by $\mathcal{S}^k_\delta$ is the space of contravariant symmetric tensor fields of degree k, with coefficients in δ-densities, that is

$$\mathcal{S}^k_\delta = \Gamma(S^kTV \otimes \Delta^\delta(V)).$$

We also consider the whole symbol space

$$\mathcal{S}_\delta = \bigoplus_{k\geq 0} \mathcal{S}^k_\delta.$$

As we continue, we will identify symbols with functions on T^*V that are polynomial along the fibre and we will denote by ξ their generic argument in the fibre of T^*V.

The Lie derivative of symbols is also natural. It is an extension of (1). We recall that the natural action of $gl(d,\mathbb{K})$ on $\Delta^\delta(\mathbb{K}^d)$ is given by

$$\rho(A)\phi = -\delta\mathrm{tr}(A)\phi, \quad \forall A \in gl(d,\mathbb{K}), \forall\phi \in \Delta^\delta(\mathbb{K}^d).$$

Then in local coordinates, the Lie derivative of $P \in \mathcal{S}^k_\delta$ in the direction of a vector field X writes

$$L_XP = X.P - \rho(\frac{\partial}{\partial x}X)P, \tag{2}$$

where ρ is the natural action of $gl(d,\mathbb{K})$ on the typical fibre $S^k\mathbb{K}^d\otimes\Delta^\delta(\mathbb{K}^d)$ of the space of symbols.

The link between differential operators and symbols is the following : the space $\mathcal{D}_{\lambda,\mu}$ is the filtered union $\bigcup_{k\in\mathbb{N}}\mathcal{D}^k_{\lambda,\mu}$ of the submodules of differential operators of order at most k. In local coordinates, any $D\in\mathcal{D}^k_{\lambda,\mu}$ may be written

$$f\in\mathcal{F}_\lambda\mapsto\sum_{|\alpha|\leq k}c_\alpha d^\alpha f\in\mathcal{F}_\mu,$$

where α is a multi-index, d^α stands for $(\frac{\partial}{\partial x^1})^{\alpha_1}\cdots(\frac{\partial}{\partial x^d})^{\alpha_d}$ and $c_\alpha\in\mathcal{F}_\delta$. The *principal symbol* of D is then

$$\sigma(D)=\sum_{|\alpha|=k}c_\alpha\xi^\alpha \tag{3}$$

It is well-known that $\sigma:\mathcal{D}^k_{\lambda,\mu}\to\mathcal{S}^k_\delta$ intertwines the actions of $\mathrm{Vect}(V)$ on these spaces :

$$\sigma\circ\mathcal{L}=L\circ\sigma.$$

Moreover, its kernel is by definition $\mathcal{D}^{k-1}_{\lambda,\mu}$. The module $(\mathcal{S}_\delta,L)$ is then the graded module associated to $(\mathcal{D}_{\lambda,\mu},\mathcal{L})$.

2.3. **Equivariant quantizations and symbol maps.** Let $\mathfrak{g}$ be a subalgebra of $\mathrm{Vect}(V)$. A $\mathfrak{g}$-*equivariant symbol map* is a $\mathfrak{g}$-module isomorphism

$$\sigma_\mathfrak{g}:\mathcal{D}_{\lambda,\mu}\to\mathcal{S}_\delta$$

that induces the identity on the associated graded module. Explicitly, this latter requirement means that

$$D\in\mathcal{D}^k_{\lambda,\mu}\implies\sigma_\mathfrak{g}(D)-\sigma(D)\in\bigoplus_{l<k}\mathcal{S}^l_\delta.$$

The inverse map of such an application is named $\mathfrak{g}$-*equivariant quantization*.

Let us quote a first example of equivariant symbol map that will be useful as we continue. Since V is a vector space, it makes sense to consider constant and linear vector fields. These vector fields generate the *affine* subalgebra Aff of $\mathrm{Vect}(V)$. Now, it is well-known that the *total symbol map*, which is also known as the *standard ordering*,

$$\sigma_{\mathrm{Aff}}:\mathcal{D}_{\lambda,\mu}\to\mathcal{S}_\delta:\sum_{|\alpha|\leq k}c_\alpha d^\alpha\mapsto\sum_{|\alpha|\leq k}c_\alpha\xi^\alpha$$

is an isomorphism of Aff-representations.

Remark : We can endow $\mathcal{S}_\delta$ with the module structure that turns σ_{Aff} into a module isomorphism. This is done by considering the representation

$$\sigma_{\mathrm{Aff}}\circ\mathcal{L}^{\lambda,\mu}\circ(\sigma_{\mathrm{Aff}})^{-1},$$

which we still denote $\mathcal{L}^{\lambda,\mu}$ or simply $\mathcal{L}$. The comparison of spaces of differential operators and tensor fields as modules over a given subalgebra of vector fields becomes the comparison of the modules $(\mathcal{S}_\delta,L)$ and $(\mathcal{S}_\delta,\mathcal{L})$, provided one keeps in mind that two parameters, namely λ and μ, are attached to the second one.

2.4. **Equivariance algebras.** In [2] and [10], the authors considered the problem of equivariant quantization with respect to the subalgebras of vector fields generated by infinitesimal projective (or conformal) transformations, over a manifold endowed with a flat projective (or conformal) structure. Both algebras are realized over suitable charts as subalgebras of polynomial vector fields. They are graded by the degree of polynomials and of finite dimension. They are moreover maximal in the set of proper subalgebras of the algebra of polynomial vector fields. In this sense, they represent a maximal set of equivariance conditions that one can impose to a quantization procedure. In [1], we determined all the graded, finite-dimensional maximal proper subalgebras of polynomial vector fields over a vector space V (real or complex). We proved that these subalgebras are the *Irreducible Filtered Finite-dimensional Transitive* Lie algebras, listed by Kobayashi and Nagano in [7]. The most important properties of these algebras are the following :

- They are simple.
- Their grading contains exactly three terms :

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

- $\mathfrak{g}_0$ is reductive : one has

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathbb{K}\mathcal{E},$$

  where $\mathfrak{h}_0$ is the semisimple part of $\mathfrak{g}_0$ and where the *Euler* element $\mathcal{E}$ spans a one-dimensional center.
- $\mathfrak{g}_p$ is the eigenspace of eigenvalue p of $ad(\mathcal{E})$.

It is worth noticing that in [7], the authors listed simple matrix algebras together with their gradings. But in [8], they described a standard procedure to view these algebras as subalgebras of polynomial vector fields over the vector space $V = \mathfrak{g}_{-1}$. Namely, if we denote by X^h the vector field over $\mathfrak{g}_{-1}$ which corresponds to $h \in \mathfrak{g}$,

$$\left\{ \begin{array}{lcl} X^h_x & = & -h \quad \forall h \in \mathfrak{g}_{-1} \\ X^h_x & = & -[h,x] \quad \forall h \in \mathfrak{g}_0 \\ X^h_x & = & -\frac{1}{2}[[h,x],x] \quad \forall h \in \mathfrak{g}_1 \end{array} \right. \tag{4}$$

In [1], we proved that the subalgebra of vector fields obtained in this way is a maximal proper subalgebra, provided it meets the additional requirement :

- When the base field is $\mathbb{R}$, the representation $\mathfrak{g}_{-1}$ of $\mathfrak{g}_0$ has no complex structure.

In the present paper, we will compare the modules $(\mathcal{S}_\delta, L)$ and $(\mathcal{S}_\delta, \mathcal{L})$ over the base space $V = \mathfrak{g}_{-1}$.

3. Casimir operators

In [2], the computation of the Casimir operator of the space of symbols was based on the knowledge of explicit formulas for the action of generators of the conformal algebra . From now on, we will consider an IFFT-algebra $\mathfrak{g}$ realized as a maximal proper subalgebra of vector fields over $\mathfrak{g}_{-1}$. We will derive a general formula for the spectrum of the Casimir operator of the space of symbols, based on the analysis of finite-dimensional representations of $\mathfrak{h}_0$. We will denote by B the Killing form of $\mathfrak{g}$, set $d = \dim(\mathfrak{g}_{-1})$ and denote by B_0 the Killing form of $\mathfrak{h}_0$.

3.1. Choice of a basis.

Let us first describe suitable bases of $\mathfrak{g}$ in order to simplify the computation of the Casimir operators.

Proposition 1. *Let (e_i) $(i = 1, \ldots, d)$ denote a basis of $\mathfrak{g}_{-1}$ and (h_j) $(j = 1, \ldots, \dim(\mathfrak{h}_0))$ a basis of $\mathfrak{h}_0$. There exist unique bases (ϵ^i) and (h_j^*) of $\mathfrak{g}_1$ and $\mathfrak{h}_0$ respectively such that the bases $(e_i, \mathcal{E}, h_j, \epsilon^i)$ and $(\epsilon^i, \frac{1}{2d}\mathcal{E}, h_j^*, e_i)$ of $\mathfrak{g}$ are dual to each other with respect to B.*

Moreover, one has

$$\sum_i [e_i, \epsilon^i] = -\frac{1}{2}\mathcal{E}. \tag{5}$$

Proof. The existence and uniqueness of the basis (ϵ^i) in $\mathfrak{g}_1$ such that $B(e_i, \epsilon^j) = \delta_{i,j}$ follows from the relations (proved in [7])

$$B(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1, \mathfrak{g}_0) = B(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) = B(\mathfrak{g}_1, \mathfrak{g}_1) = 0.$$

But $\mathfrak{h}_0$ and $\mathbb{K}\mathcal{E}$ are orthogonal to each other too. It is sufficient to note that $\mathfrak{h}_0$ is equal to its derived ideal and that

$$B(\mathcal{E}, [x_0, y_0]) = B([\mathcal{E}, x_0], y_0) = 0, \quad \forall x_0,\, y_0 \in \mathfrak{h}_0.$$

This ensures the existence and uniqueness of the basis (h_j^*) in $\mathfrak{h}_0$.

Finally, for every $x_0 \in \mathfrak{g}_0$, we have

$$\begin{aligned} B(x_0, \mathcal{E}) &= \operatorname{tr}(ad(x_0)_{|\mathfrak{g}_1}) - \operatorname{tr}(ad(x_0)_{|\mathfrak{g}_{-1}}) \\ &= \sum_i B(e_i, [x_0, \epsilon^i]) - \sum_i B(\epsilon^i, [x_0, e_i]) = -2B(x_0, \sum_i [e_i, \epsilon^i]) \end{aligned}$$

The second relation shows that $B(\mathcal{E}, \mathcal{E}) = 2d$, while the third one proves (5). □

3.2. The cocycle γ.

Since the Lie derivatives $\mathcal{L}_X^{\lambda,\mu}$ and L_X coincide for every X in the affine algebra, the obstuctions to build a $\mathfrak{g}$-equivariant quantization come from $\mathfrak{g}_1$. They are best seen in the difference of the Casimir operators on differential operators and symbols. As we continue, we will denote C_δ the Casimir operator of $(\mathcal{S}_\delta, L)$ and by $\mathcal{C}_{\lambda,\mu}$ the Casimir operator of $(\mathcal{S}_\delta, \mathcal{L}^{\lambda,\mu})$. The following maps will also play an important role :

$$\gamma : \mathfrak{g} \to gl(\mathcal{S}_\delta) : X \mapsto \mathcal{L}_X - L_X,$$

and

$$N_{\mathcal{C}} : \mathcal{S}_\delta \to \mathcal{S}_\delta : P \mapsto 2\sum_i \gamma(\epsilon^i) \circ L_{e_i} P.$$

Let us analyse their most important properties.

Proposition 2. *The map γ has the following properties*

- *It is a Chevalley-Eilenberg cocycle with values in the representation $(gl(\mathcal{S}_\delta), L')$ of $\mathfrak{g}$, where*

$$L'_X : gl(\mathcal{S}_\delta) \to gl(\mathcal{S}_\delta) : T \mapsto \mathcal{L}_X \circ T - T \circ L_X.$$

- *Its restriction to $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ vanishes.*
- *For every X in $\mathfrak{g}_1$ and every $k \in \mathbb{N}$, $\gamma(X) : \mathcal{S}^k_\delta \to \mathcal{S}^{k-1}_\delta$ is a differential operator of order zero with constant coefficients.*

Proof. The first statement is a direct consequence of the Jacobi identity for $\mathcal{L}^{\lambda\mu}$. For the second one, we recall that $\mathcal{L}^{\lambda,\mu}_X$ and L_X coincide for every X in the affine algebra, while the third one is the result of a straightforward computation. □

The next proposition shows the link between the Casimir operators.

Proposition 3. *The Casimir operators $\mathcal{C}_{\lambda,\mu}$ and C_δ are related by the formula*

$$\mathcal{C}_{\lambda,\mu} = C_\delta + N_{\mathcal{C}}.$$

Proof. Using the notation of Proposition 1, the Casimir operator $\mathcal{C}_{\lambda,\mu}$ can be rewritten as follows:

$$\begin{aligned}
\mathcal{C}_{\lambda,\mu} &= \sum_i (\mathcal{L}_{e_i} \circ \mathcal{L}_{\epsilon^i} + \mathcal{L}_{\epsilon^i} \circ \mathcal{L}_{e_i}) + \frac{1}{2d}(\mathcal{L}_{\mathcal{E}})^2 + \sum_j \mathcal{L}_{h_j} \circ \mathcal{L}_{h_j^*} \\
&= 2\sum_i \mathcal{L}_{\epsilon^i} \circ \mathcal{L}_{e_i} + \mathcal{L}_{\sum_i [e_i,\epsilon^i]} + \frac{1}{2d}(\mathcal{L}_{\mathcal{E}})^2 + \sum_j \mathcal{L}_{h_j} \circ \mathcal{L}_{h_j^*}.
\end{aligned}$$

The conclusion is then a direct consequence of the vanishing of γ on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$. □

Let us end this section by stating some properties of $N_{\mathcal{C}}$.

Proposition 4. *For every $k \in \mathbb{N}$, the map $N_{\mathcal{C}} : \mathcal{S}^k_\delta \to \mathcal{S}^{k-1}_\delta$ is a differential operator of order one with constant coefficients. Moreover, for every $X \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, we have*

$$L_X \circ N_{\mathcal{C}} = N_{\mathcal{C}} \circ L_X.$$

Proof. The first statement is a corollary of Proposition 2 while the second one is a consequence of Proposition 3. □

3.3. **Spectrum of $\mathcal{C}_\delta$.** We will now compute the spectrum of the Casimir operator on the space of symbols. We first recall that the Lie derivative of a symbol P in the direction of a vector field X writes

$$L_X P = X \cdot P - \rho(\frac{\partial}{\partial x}X)P \tag{6}$$

where ρ is the natural representation of $gl(\mathfrak{g}_{-1})$ on the fibre of the space of symbols and $\frac{\partial}{\partial x}X$ is the Jacobian matrix of X.

Note that, in view of formula (4), the map

$$-\frac{\partial}{\partial x} : \mathfrak{g}_0 \to gl(\mathfrak{g}_{-1}) : X \mapsto -\frac{\partial}{\partial x}X$$

is just the (matrix realization of the) adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$. As we continue, we will denote by ρ^k the natural extension of the adjoint representation of $\mathfrak{g}_0$ on the fibre $S^k\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$ of $\mathcal{S}^k_\delta$.

It is also noteworthy that we have $ad(\mathfrak{h}_0) \subset sl(\mathfrak{g}_{-1})$, since $\mathfrak{h}_0$ is semisimple. Therefore, as a representation of $\mathfrak{h}_0$, $S^k\mathfrak{g}_{-1}\otimes\Delta^\delta(\mathfrak{g}_{-1})$ is isomorphic to $S^k\mathfrak{g}_{-1}$. Now we can come to the first result.

Proposition 5. *For every $P \in \mathcal{S}^k_\delta$, one has*

$$\mathcal{C}_\delta P = \frac{1}{2d}(d\delta - k)(d(\delta-1)-k)P + \sum_{j=1}^{\dim(\mathfrak{h}_0)} \rho^k(h_j)\rho^k(h_j^*)P. \tag{7}$$

Proof. As in Proposition 3, we write

$$\mathcal{C}_\delta = \sum_i L_{\epsilon^i} \circ L_{e_i} - \frac{1}{2}L_\mathcal{E} + \frac{1}{2d}(L_\mathcal{E})^2 + \sum_j L_{h_j} \circ L_{h_j^*}.$$

The operator $\mathcal{C}_\delta$ commutes with the action of $\mathfrak{g}_{-1}$. It is therefore a differential operator with constant coefficients. Hence we only need to sum the constant terms in the right-hand side of the last formula. In view of formula (6) of L, it is clear that the Lie derivatives with respect to a quadratic vector field do not contribute to such terms. Furthermore we have

$$L_\mathcal{E} P = \sum_i x^i \frac{\partial}{\partial x^i}P + (d\delta - k)P \tag{8}$$

and

$$L_{h_j}P = \sum_i (h_j)^i \frac{\partial}{\partial x^i}P + \rho^k(h_j)P.$$

Hence the result. □

In order to state the main theorem, we introduce a few more notations. From now on to the end of this Section, for each vector space (resp. Lie algebra) E, we will denote by $E^\mathbb{C}$ the complexified vector space (resp. Lie algebra) $E \otimes_\mathbb{R} \mathbb{C}$. We will set

$$\widetilde{E} = \begin{cases} E \text{ if the base field } \mathbb{K} \text{ is } \mathbb{C}, \\ E^\mathbb{C} \text{ if } \mathbb{K} = \mathbb{R}. \end{cases}$$

Furthermore, we fix a Cartan subalgebra $\mathfrak{C}$ in $\widetilde{\mathfrak{h}_0}$, a root system Λ, a simple root system Λ_S. Finally, let us denote by ρ_S half the sum of the positive roots and by $(\cdot,\cdot)$ the scalar product induced by the extension of B_0 to $\widetilde{\mathfrak{h}_0}$ on the real vector space spanned by the roots.

If E is an irreducible module over $\mathfrak{h}_0$, then either $\tilde{E}$ is irreducible as a complex representation of $\widetilde{\mathfrak{h}_0}$ and we denote by μ_E its highest weight or E admits a complex structure as a module over $\mathfrak{h}_0$. In this latter case, we set μ_E to be the highest weight of E as a complex representation of $\widetilde{\mathfrak{h}_0}$. Recall that the latter case never occurs when E is taken to be $\mathfrak{g}_{-1}$.

Finally, as a representation of $\mathfrak{h}_0$, $S^k\mathfrak{g}_{-1}\otimes\Delta^\delta(\mathfrak{g}_{-1})$ is decomposed as a sum of irreducible representations, say

$$S^k\mathfrak{g}_{-1}\otimes\Delta^\delta(\mathfrak{g}_{-1})=\oplus_{p=1}^{n_k}I_{k,p},$$

and for each irreducible representation $I_{k,p}$ we denote by $E_{k,p}$ the corresponding space of sections, that is

$$E_{k,p}=\Gamma(I_{k,p}).$$

We are now in position to present the main result.

Theorem 6. *Let $\mathfrak{g}$ be an IFFT-algebra such that $\widetilde{\mathfrak{h}_0}$ is simple. Then the Casimir operator $\mathcal{C}_\delta$ is diagonalizable.*

Indeed, for every $k\in\mathbb{N}$, the restriction of $\mathcal{C}_\delta$ to $E_{k,p}$ is equal to

$$\frac{1}{2d}(d\delta-k)(d(\delta-1)-k) \\ +\frac{\dim(\mathfrak{h}_0)}{2(\mu_{\mathfrak{g}_{-1}},\mu_{\mathfrak{g}_{-1}}+2\rho_S)d+\dim(\mathfrak{h}_0)}(\mu_{I_{k,p}},\mu_{I_{k,p}}+2\rho_S) \quad (9)$$

times the identity of $E_{k,p}$.

Proof. Let us assume first that $\mathbb{K}=\mathbb{C}$ and consider an irreducible submodule $I_{k,p}$. Using Proposition 5, we only have to compute the operator

$$\sum_{j=1}^{\dim(\mathfrak{h}_0)}\rho^k(h_j)\rho^k(h_j^*)$$

on $I_{k,p}$. Under the assumption that $\mathfrak{h}_0$ be simple, there exists $l\in\mathbb{C}\setminus\{0\}$ such that

$$B_{|\mathfrak{h}_0}=lB_0. \quad (10)$$

Then we consider the bases (f_j) and (f_j^*) of $\mathfrak{h}_0$ defined by $f_j=h_j$ and $f_j^*=lh_j^*$. These bases are dual with respect to B_0 and so we have

$$\sum_{j=1}^{\dim(\mathfrak{h}_0)}\rho^k(h_j)\rho^k(h_j^*)=\frac{1}{l}C_{\mathfrak{h}_0,I_{k,p}},$$

where $C_{\mathfrak{h}_0,I_{k,p}}$ is the Casimir operator of the representation $I_{k,p}$ of $\mathfrak{h}_0$. Moreover, it is well-known that

$$C_{\mathfrak{h}_0,I_{k,p}} = (\mu_{I_{k,p}}, \mu_{I_{k,p}} + 2\rho_S) \tag{11}$$

times the identity (see for instance [6, p. 122]).

In order to compute l, we recall that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are dual representations of $\mathfrak{h}_0$. Then, for all $x, y \in \mathfrak{h}_0$, we have

$$\begin{aligned} B_{|\mathfrak{h}_0}(x,y) &= 2\mathrm{tr}(ad(x)_{|\mathfrak{g}_{-1}} ad(y)_{|\mathfrak{g}_{-1}}) + \mathrm{tr}(ad(x)_{|\mathfrak{h}_0} ad(y)_{|\mathfrak{h}_0}) \\ &= 2B_{\rho^1}(x,y) + B_0(x,y), \end{aligned}$$

where B_{ρ^1} is the bilinear form associated to the representation $\mathfrak{g}_{-1}$ of $\mathfrak{h}_0$.

The latter formula also writes

$$B_{\rho^1} = \frac{l-1}{2} B_0.$$

Note that $l \neq 1$ because B_{ρ^1} is non-singular since ρ^1 is not zero (see [5, p. 143]). Then we look at the Casimir operator $C_{\mathfrak{h}_0,\mathfrak{g}_{-1}}$ of the representation $\mathfrak{g}_{-1}$. We have, as above

$$C_{\mathfrak{h}_0,\mathfrak{g}_{-1}} = (\mu_{\mathfrak{g}_{-1}}, \mu_{\mathfrak{g}_{-1}} + 2\rho_S)$$

times the identity of $\mathfrak{g}_{-1}$. But the bases (f_j) and $(\frac{2}{l-1} f_j^*)$ are dual with respect to B_{ρ^1} and then

$$\begin{aligned} \mathrm{tr}(C_{h_0,\mathfrak{g}_{-1}}) &= d(\mu_{\mathfrak{g}_{-1}}, \mu_{\mathfrak{g}_{-1}} + 2\rho_S) \\ &= \mathrm{tr}(\textstyle\sum_j \rho^1(f_j)\rho^1(f_j^*)) \\ &= \tfrac{l-1}{2} \textstyle\sum_j B_{\rho^1}(f_j, \tfrac{2}{l-1} f_j^*) = \tfrac{l-1}{2} \dim(\mathfrak{h}_0). \end{aligned}$$

Hence the result over the field of complex numbers.

Let us now handle the case $\mathbb{K} = \mathbb{R}$. We first remark that Formula (10) still holds — with $l \in \mathbb{R}$, this time — since $\mathfrak{h}_0$ has no complex structure. Now, let us adapt Formula (11). If $I_{k,p}^{\mathbb{C}}$ is a simple representation of $\mathfrak{h}_0^{\mathbb{C}}$, its Casimir operator is the $\mathbb{C}$-linear extension of the Casimir operator of $I_{k,p}$, since the Killing form of $\mathfrak{h}_0^{\mathbb{C}}$ is just the extension of B_0. The Casimir operator of $I_{k,p}^{\mathbb{C}}$ is then the real multiple of the identity given by Formula (11). If $I_{k,p}^{\mathbb{C}}$ is reducible, then $I_{k,p}$ admits a complex structure and becomes a simple complex representation of $\mathfrak{h}_0^{\mathbb{C}}$. We then conclude using the same arguments. □

The eigenvalue formula (9) is easily shown to coincide, when $\mathfrak{g}$ is taken to be $sl(n+1,\mathbb{R})$, with the formula given in [11, Prop. 2].

4. Building equivariant quantizations

Throughout this Section, we assume that $\widetilde{\mathfrak{h}_0}$ is simple, in order to apply Theorem 6. We will denote by $\alpha_{k,p}$ the eigenvalue of C_δ on $E_{k,p}$.

4.1. **The tree-like subspace associated to γ.** We identify tensors in $S^k\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$ with symbols with constant coefficients. Since for every $X \in \mathfrak{g}_1$, $\gamma(X)$ has constant coefficients, we can consider that $\gamma(X)$ is defined on $S^k\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$.

Lemma 7. *Let $k \in \mathbb{N}$ and F be a submodule of $S^k\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$ over $\mathfrak{h}_0$. Then $\gamma(\mathfrak{g}_1)(F)$ is a submodule of $S^{k-1}\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$ over $\mathfrak{h}_0$.*

Proof. It is easy to see that the cocycle relation for γ implies

$$L_Y \circ \gamma(X)P = \gamma(X) \circ L_Y P + \gamma([Y, X])P \in \gamma(\mathfrak{g}_1)F$$

for every $Y \in \mathfrak{g}_0$, $X \in \mathfrak{g}_1$ and $P \in F$. We conclude by noticing that on the subspace of symbols with constant coefficients, L reduces to ρ^k. $\square$

We define the *tree-like subspace associated to* γ, starting at an irreducible submodule $I_{k,p}$:

$$\mathcal{T}_\gamma(I_{k,p}) = \bigoplus_{l\in\mathbb{N}} \mathcal{T}^l_\gamma(I_{k,p}),$$

where $\mathcal{T}^0_\gamma(I_{k,p}) = I_{k,p}$ and $\mathcal{T}^{l+1}_\gamma(I_{k,p}) = \gamma(\mathfrak{g}_1)(\mathcal{T}^l_\gamma(I_{k,p}))$ for all $l \in \mathbb{N}$. The spaces $\mathcal{T}^l_\gamma(E_{k,p})$ are defined in the same way.

Recall that the module structure defined by $\mathcal{L}$ is related to two parameters λ and μ and that their difference δ is called *shift*. As one would expect, the possibility of building equivariant quantization depends on the values of λ and μ. We will say that a couple of parameters (λ, μ) is *critical* if there exist k, p such that the eigenvalue $\alpha_{k,p}$ belongs to the spectrum of the restriction of C_δ to $\bigoplus_{l\geq 1} \mathcal{T}^l_\gamma(E_{k,p})$. In the same way, we will say that a shift value δ is critical if there exists a value of λ such that (λ, μ) is critical in the previous sense.

The following straightforward lemmas show the link between the existence of a $\mathfrak{g}$-equivariant quantization and the last definition.

Lemma 8. *Let $I_{k,p}$ be an irreducible submodule of $S^k\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$ and let $\mathfrak{g}^*_{-1} \otimes I_{k,p}$ denote the subspace of $E_{k,p}$ made up of sections with linear coefficients. Then*

$$N_{\mathcal{C}}(\mathfrak{g}^*_{-1} \otimes I_{k,p}) = \gamma(\mathfrak{g}_1)(I_{k,p}).$$

Proof. In the basis (e_i) of $\mathfrak{g}_{-1}$ chosen in Proposition 1, L_{e_i} takes the local form $\frac{\partial}{\partial x^i}$. It then follows that

$$N_{\mathcal{C}}(\sum_{l,m} a^l_m x^m u_l) = 2 \sum_{l,m} a^l_m \gamma(\epsilon^m) u_l,$$

for all $a^l_m \in \mathbb{K}$ and $u_l \in I_{k,p}$. $\square$

In a similar fashion, we have

Lemma 9. *For all $u \in E_{k,p}$, $N_{\mathcal{C}}(u) \in \gamma(\mathfrak{g}_1)(E_{k,p})$.*

Theorem 10. *If (λ, μ) is not critical, then there exists a $\mathfrak{g}$-equivariant quantization.*

Proof. The proof machinery goes as in [2]. We give it for the sake of completeness.

Let $P \in E_{k,p}$. We first prove that there exists a unique $\hat{P} \in \mathcal{T}_\gamma(E_{k,p})$ such that P is the principal symbol of $\hat{P}$ and that $\hat{P}$ is an eigenvector of $\mathcal{C}_{\lambda\mu}$ associated to the eigenvalue $\alpha_{k,p}$. For all $R \in \mathcal{S}_\delta$, write R_l the projection of R onto $\mathcal{S}^l_\delta$. With these notations, the equation $\mathcal{C}_{\lambda\mu}\hat{P} = \alpha_{k,p}\hat{P}$ can be rewritten :

$$\begin{cases} C_\delta P = \alpha_{k,p} P \\ (C_\delta - \alpha_{k,p}\mathrm{id})\hat{P}_l = N_{\mathcal{C}}\hat{P}_{l+1} \end{cases} \tag{12}$$

where the last equation must be satisfied for all $l < k$. The existence and the properties of the correspondence $P \mapsto \hat{P}$ follow from the observation that the latter system is triangular and admits a unique solution. Indeed, the right hand side of the equations involving $N_{\mathcal{C}}$ always belongs to $\bigoplus_{l\geq 1} \mathcal{T}^l_\gamma(E_{k,p})$ and the restriction of $C_\delta - \alpha_{k,p}\mathrm{id}$ to this space is non-singular.

Now, let $\mathcal{Q}$ denote the linear extension of this correspondence. It remains to prove that it is equivariant with respect to $\mathfrak{g}$. It suffices to check that

$$\mathcal{L}_X \circ \mathcal{Q}(P) = \mathcal{Q} \circ L_X(P),$$

for all $X \in \mathfrak{g}$, all $k \in \mathbb{N}$ and all eigenvectors $P \in \mathcal{S}^k_\delta$ of C_δ associated to any eigenvalue $\alpha_{k,p}$. But both sides of this condition are eigenvectors of $\mathcal{C}_{\lambda\mu}$ associated to the same eigenvalue $\alpha_{k,p}$. Moreover, they have the same principal symbol: $L_X(P)$. Since $E_{k,p}$ and $\mathcal{T}_\gamma(E_{k,p})$ are respectively closed under L_X and $\mathcal{L}_X$, both sides belong to the latter tree. The first part of the proof ensures that they coincide. □

5. Examples

We will now apply the method described in the previous section to two particular algebras. The treatment will be done in a concurrent way. Throughout this Section, $\mathfrak{g}$ denotes one of the algebras $\mathfrak{O}^{(n)}$ and $\mathfrak{S}^{(n)}$ defined below.

We will significantly refine Theorem 10 by proving that 0 is never a critical shift value and obtaining the uniqueness of the quantization.

5.1. **Orthogonal and Symplectic algebras.** From now on, we assume that n is an integer greater than 2. It is well-known that the Lie subalgebras $so(n, n, \mathbb{K})$ and $sp(2n, \mathbb{K})$ of the general linear algebra $gl(2n, \mathbb{K})$ can be realized as 3-graded algebras. These are described in [7, pp. 893-894].

For the constructions below to be self-contained, we only need to recall that $so(n, n, \mathbb{K})$ is written

$$\mathfrak{O}^{(n)} = \mathfrak{O}^{(n)}_{-1} \oplus \mathfrak{O}^{(n)}_0 \oplus \mathfrak{O}^{(n)}_1,$$

where $\mathfrak{O}^{(n)}_{-1} = \wedge^2\mathbb{K}^n$, $\mathfrak{O}^{(n)}_1 = \wedge^2\mathbb{K}^{n*}$ and $\mathfrak{O}^{(n)}_0 = gl(n, \mathbb{K})$. For all $A \in \mathfrak{O}^{(n)}_0$ and $h \in \mathfrak{O}^{(n)}_{-1} \oplus \mathfrak{O}^{(n)}_1$,

$$[A, h] = \rho(A)h,$$

where ρ is the natural representation of $\mathfrak{O}_0^{(n)}$ on $\mathfrak{O}_{-1}^{(n)} \oplus \mathfrak{O}_1^{(n)}$. The Euler element is $-\frac{1}{2}\mathrm{id} \in gl(n, \mathbb{K})$. We will refer to $\mathfrak{O}^{(n)}$ as the *orthogonal* algebra.

Similarly, $sp(2n, \mathbb{K})$ is written

$$\mathfrak{S}^{(n)} = \mathfrak{S}_{-1}^{(n)} \oplus \mathfrak{S}_0^{(n)} \oplus \mathfrak{S}_1^{(n)},$$

where $\mathfrak{S}_{-1}^{(n)} = S^2\mathbb{K}^n$, $\mathfrak{S}_1^{(n)} = S^2\mathbb{K}^{n*}$ and $\mathfrak{S}_0^{(n)} = gl(n, \mathbb{K})$. The same statements about the bracket and Euler element hold. We will refer to this algebra as the *symplectic* algebra.

5.2. **Casimir operator eigenvalues.** In the examples under consideration, the subalgebra $\mathfrak{h}_0$ is isomorphic to $sl(n, \mathbb{K})$ and $\widetilde{\mathfrak{h}_0} = sl(n, \mathbb{C})$ is obviously simple. The data introduced to state Theorem 6 are classical. Let us denote by $d(n, \mathbb{K})$ the matrix subalgebra of diagonal matrices of $gl(n, \mathbb{K})$ and $D_j \in d(n, \mathbb{K}), (j = 1, \dots, n-1)$, as the diagonal matrix

$$\mathrm{diag}(0, \dots, 0, \overset{(j)}{1}, 0, \dots, 0, -1).$$

These diagonal matrices generate the Cartan subalgebra $sl(n, \mathbb{K}) \cap d(n, \mathbb{K})$ of $sl(n, \mathbb{K})$. In its dual space, we define δ_j by $\delta_j(D_i) = \delta_{ij}$ for all $i, j \in \{1, \dots, n-1\}$. As it is common, we set $\delta_n = -\sum_{i=1}^{n-1} \delta_i$ as well. Then, a simple root system of $sl(n, \mathbb{K})$ is given by the $\delta_i - \delta_{i+1}$, $(i = 1, \dots, n-1)$ and $\rho_S = \sum_i (n-i)\delta_i$. The Killing form B_0 of $sl(n, \mathbb{K})$ is given by $B_0(A, B) = 2n\,\mathrm{tr}(AB)$ for all $A, B \in sl(n, \mathbb{K})$. The induced scalar product satisfies

$$(\delta_i, \delta_j) = \frac{1}{2n^2}(n\delta_{ij} - 1) \text{ and } (\delta_i, 2\rho_S) = \frac{n - 2i + 1}{2n}, \tag{13}$$

for all $i = 1, \dots, n$.

Now, let $\mathbb{K} = \mathbb{C}$. The decomposition of $S^k\mathfrak{g}_{-1}$ into irreducible submodules over $\mathfrak{h}_0$ is given in [4]. These submodules may be generated by the action of real matrices on their (real) highest weight. Therefore, these results can be used when $\mathbb{K} = \mathbb{R}$ as well. As it is well-known, irreducible submodules can be conveniently indexed by Ferrers diagrams, which in turn can be denoted by elements of $\mathbb{N}^n$. We will respectively denote by $(5, 5, 2, 2)$ and $(6, 4, 2, 2)$ the Ferrers diagrams given in Figure 1. They admit $5(\delta_1 + \delta_2) + 2(\delta_3 + \delta_4)$ (resp. $6\delta_1 + 4\delta_2 + 2(\delta_3 + \delta_4)$) as highest weight.

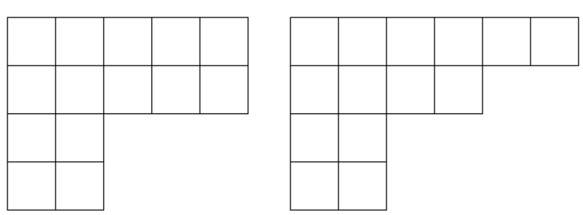

FIGURE 1. Ferrers diagrams of irreducible submodules of $S^k\mathfrak{O}_{-1}^{(n)}$ and $S^k\mathfrak{S}_{-1}^{(n)}$ $(n \geq 4)$.

Theorem 5.2.11 in [4] states that $S^k\mathfrak{O}^{(n)}_{-1} \otimes \Delta^\delta(\mathfrak{O}^{(n)}_{-1}) \cong S^k\mathfrak{O}^{(n)}_{-1}$ splits as a sum of one copy of each irreducible submodule of weight $\sum_{i=1}^n \mu_i\delta_i$ such that

(1) $\mu_1 \geq \cdots \geq \mu_n \geq 0$, ($\mu$ is dominant nonnegative),
(2) $\sum_i \mu_i = 2k$,
(3) $\mu_{2i-1} = \mu_{2i}, \quad \forall i \in \{1, \ldots, \lfloor n/2 \rfloor\}$,
(4) $\mu_n = 0$ if n is odd.

Theorem 5.2.9 ibidem states that $S^k\mathfrak{S}^{(n)}_{-1} \otimes \Delta^\delta(\mathfrak{S}^{(n)}_{-1}) \cong S^k\mathfrak{S}^{(n)}_{-1}$ splits as a sum of one copy of each irreducible submodule of weight $\sum_{i=1}^n \mu_i\delta_i$ such that

(1) $\mu_1 \geq \cdots \geq \mu_n \geq 0$, ($\mu$ is dominant nonnegative),
(2) $\sum_i \mu_i = 2k$,
(3) $\mu_i \in 2\mathbb{N}, \quad \forall i \in \{1, \ldots, n-1\}$.

Let us compute explicitly the value of Expression (9). For all submodules R of $\mathfrak{O}^{(n)}$ or $\mathfrak{S}^{(n)}$ with highest weight μ_R described by a Ferrers diagram $(k_1, \ldots, k_n)$, Formula (13) shows that

$$(\mu_R, \mu_R + 2\rho_S) = \frac{1}{2n^2} \sum_{i,j=1}^{n} (k_i k_j (n\delta_{ij} - 1) + 2k_i(n-j)(n\delta_{ij} - 1)). \quad (14)$$

In the *orthogonal* case, $d = \frac{n(n-1)}{2}$ and the highest weight of $\mathfrak{g}_{-1} = \mathfrak{O}^{(n)}_{-1}$ is $\delta_1 + \delta_2$. Let R now denote an irreducible submodule of $S^k\mathfrak{O}^{(n)}_{-1}$ associated to a Ferrers diagram $\vec{k} = (k_1, \ldots, k_n)$. A direct computation using (9) and (14) shows that the eigenvalue of C_δ associated to $R \otimes \Delta^\delta(\mathfrak{O}^{(n)}_{-1})$ equals

$$\alpha_o(\vec{k}) = \frac{n(n-1)}{4}\delta^2 - (k + \frac{n(n-1)}{4})\delta + \frac{n}{n-1}k + \frac{\sum_{i=1}^n k_i(k_i - 2i)}{4(n-1)}. \quad (15)$$

In the *symplectic* case, $d = \frac{n(n+1)}{2}$ and the highest weight of $\mathfrak{g}_{-1} = \mathfrak{S}^{(n)}_{-1}$ is $2\delta_1$. Let R now denote an irreducible submodule of $S^k\mathfrak{S}^{(n)}_{-1}$ associated to a Ferrers diagram $(k_1, \ldots, k_n)$. Then the eigenvalue of C_δ associated to $R \otimes \Delta^\delta(\mathfrak{O}^{(n)}_{-1})$ equals

$$\alpha_s(\vec{k}) = \frac{n(n+1)}{4}\delta^2 - (k + \frac{n(n+1)}{4})\delta + k + \frac{\sum_{i=1}^n k_i(k_i - 2i)}{4(n+1)}. \quad (16)$$

5.3. **Another tree.** In both symplectic and orthogonal cases, it is easy to check that the difference of two eigenvalues corresponding to different degrees k cannot be identically zero. Indeed, such a difference is a linear expression in δ with rational coefficients. Thus there exist infinitely many values of the shift for which a quantization exists.

We will now develop two important refinements. First, we will determine a set CV that contains the critical shift values and we will show that this set does not contain zero in both symplectic and orthogonal cases. Then, given any value of the shift outside CV, we will prove that the only equivariant quantization is the one we have built.

In order to prove that 0 is not critical, it is unfortunately not sufficient to check all the eigenvalues by a straight inspection. For instance, the eigenvalues associated to diagrams $(6,2,2,2)$ and $(6,4)$ are equal when $n=5$ in the symplectic case. But it is clear from Equation (12) that only some of those equalities can actually prevent the quantization from existing.

Let $I \subset S^k\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$ be an irreducible submodule over $\mathfrak{h}_0$. We define a bigger tree than $\mathcal{T}_\gamma(I)$ as follows. Let $\widetilde{\mathcal{T}}^1(I)$ be the sum of all irreducible submodules J_p in $S^{k-1}\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$ that are isomorphic to an irreducible submodule of $\mathfrak{g}^*_{-1} \otimes I$. Define $\widetilde{\mathcal{T}}^2(I) = \bigoplus_{(p)} \widetilde{\mathcal{T}}^1(J_p)$ and continue recursively. We write

$$\widetilde{\mathcal{T}}(I) = I \oplus \bigoplus_{k\geq 1} \widetilde{\mathcal{T}}^k(I).$$

Consider now the natural representation of $\mathfrak{h}_0$ on $\mathfrak{g}^*_{-1} \otimes I$. It is isomorphic to the representation defined by the Lie derivative in the direction of (linear) vector fields of $\mathfrak{h}_0$ on the space of sections valued in I with linear coefficients. Lemma 8 and Proposition 4 then show that for all λ, $\mathcal{T}_\gamma(I)$ is indeed a subset of $\widetilde{\mathcal{T}}(I)$.

It is customary to order the Ferrers diagrams as follows:

$$\vec{k} \leq \vec{l} \equiv (k_i \leq l_i, \forall i \leq n)$$

and of course

$$\vec{k} < \vec{l} \equiv (k \leq l \text{ and } k \neq l).$$

Then we can describe $\widetilde{\mathcal{T}}(I)$ in the examples under consideration.

Lemma 11. *Let $K \subset S^k\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$ be an irreducible submodule over $\mathfrak{h}_0$ whose type is given by the Ferrers diagram $\vec{k}$. If an irreducible submodule $L \subset S^l\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$ with type $\vec{l}$, $(l < k)$ is in $\widetilde{\mathcal{T}}(K)$ then $\vec{l} < \vec{k}$.*

Proof. It suffices to determine the diagrams occurring in the decomposition of $\mathfrak{g}^*_{-1} \otimes K$ into irreducible components using Littlewood-Richardson rule (see for instance [3, pp. 455-456]).

Let us detail the proof in the orthogonal case, for which $\mathfrak{g}^*_{-1}$ is represented by a column of height $n-2$ and width 1. The irreducible components of $K \otimes \mathfrak{g}^*_{-1}$ are then associated to diagrams made up by adding one box to $n-2$ rows of the diagram associated to K.

Then, one needs to know which of these new diagrams represent irreducible components isomorphic to one occurring in the decomposition of $S^{k-1}\mathfrak{g}_{-1}$. But the latter admit diagrams with $2(k-1)$ boxes while the former have $2k+n-2$. In order to describe isomorphic $sl(n,\mathbb{K})$ submodules, they should differ by a column of height n and width 1 on the left. The diagram with $2k-n+2$ boxes may thus only be isomorphic to a diagram smaller than the original diagram of K. The conclusion follows by induction. □

Theorem 12. *All critical shift values belong to the set*

$$CV = \{\frac{n}{n+1} + \frac{\sum_{i=1}^n (k_i - l_i)(k_i - l_i + 2i)}{4(n-1)(k-l)} : \vec{k} > \vec{l}\}$$

in the orthogonal case and

$$CV = \{1 + \frac{\sum_{i=1}^{n}(k_i - l_i)(k_i - l_i + 2i)}{4(n+1)(k-l)} : \vec{k} > \vec{l}\}$$

in the symplectic case, where $\vec{k}$ and $\vec{l}$ describe all the admissible Ferrers diagrams. In particular, they are greater than 0 *and there exists a* $\mathfrak{g}$*-equivariant quantization into operators that preserve the weight of their arguments.*

Proof. Assume that two eigenvalues associated to K and L taken as above are equal. For instance, in the orthogonal case, we have, using Equation (15)

$$\alpha_o(\vec{k}) - \alpha_o(\vec{l}) = 0 \Leftrightarrow \delta = \frac{n}{n-1} + \frac{\sum_{i=1}^{n}(k_i - l_i)(k_i + l_i - 2i)}{4(n-1)(k-l)},$$

hence the description of the set CV. The right-hand side of the last equation is not less than

$$\frac{n}{n-1} + \frac{\sum_i (k_i^2 - l_i^2)}{4(n-1)(k-l)} - 2n\frac{\sum_i (k_i - l_i)}{4(n-1)(k-l)},$$

which is greater than 0. Indeed, the last term sums up to the first and there exists an index i such that $k_i > l_i$.

The proof goes the same way in the symplectic case. Hence the result. □

Let us now turn to the uniqueness problem. Here we restrict ourselves to the real case in order to apply the results of [9].

Lemma 13. *Assume that δ is not in the set CV of Theorem 12 and let $k, l \in \mathbb{N}$ such that $l < k$. Then there exists no (non-trivial)* $\mathfrak{g}$*-equivariant map from $(\mathcal{S}^k_\delta, L)$ to $(\mathcal{S}^l_\delta, L)$.*

Proof. Assume that T is such a map. As proved in [9, Lemma 7.1], the equivariance of T with respect to $\mathfrak{g}_{-1}$ (i.e. every constant vector field) and $\mathcal{E}$ implies that it is a differential operator with constant coefficients. We can thus write

$$T = \sum_{r=0}^{R} T_r,$$

with T_r an homogeneous differential operator of order r.

In view of (8), $[L_{\mathcal{E}}, T] = 0$ leads furthermore to

$$\sum_{r=0}^{R}(k - l - r)T_r = 0$$

and, therefore, $T = T_{k-l}$.

Let $I_{k,p}$ be an irreducible submodule of $S^k\mathfrak{g}_{-1} \otimes \Delta^\delta(\mathfrak{g}_{-1})$ over $\mathfrak{h}_0$, described by $\vec{\imath}$. We know that T is entirely defined by its values on the sections in $\Gamma(I_{k,p})$ with polynomial coefficients of degree $k - l$. We recall that the Lie derivative in the direction of vector fields in $\mathfrak{h}_0$ on those has no effect on the "density part" and corresponds to the natural representation of $\mathfrak{h}_0$ on

$$S^{k-l}\mathfrak{g}^*_{-1} \otimes I_{k,p}. \tag{17}$$

The image of such sections through the application of T is made of sections with constant coefficients. This image corresponds to a submodule F of $S^l(\mathfrak{g}_{-1}) \otimes \Delta^\delta(\mathfrak{g}_{-1})$ over $\mathfrak{h}_0$. The irreducible components of F necessarily appear in the decomposition of (17) and thus the one of $\otimes^{k-l}\mathfrak{g}^*_{-1} \otimes I_{k,p}$.

Our last argument goes as in the proof of Lemma 11. Let $\vec{f}$ describe a submodule of F isomorphic to a submodule L of $\otimes^{k-l}\mathfrak{g}^*_{-1} \otimes I_{k,p}$. Let $\vec{l}$ be the diagram describing L. On the one hand, in application of the Littlewood-Richardson rule, in the symplectic (resp. orthogonal) case $\vec{l}$ is obtained by adding $2(k-l)(n-1)$ (resp. $(k-l)(n-2)$) boxes to $\vec{\imath}$, with no more than $2(k-l)$ (resp. $(k-l)$) boxes in a single row. On the other hand, since $\vec{f}$ contains exactly $2l$ boxes, it is obtained by removing $2(k-l)$ (resp. $(k-l)$) columns on the left of $\vec{l}$.

Therefore, $\vec{f} < \vec{\imath}$. But the invariance of T ensures that the values of C_δ on F and $I_{k,p}$ coincide, which contradicts the hypothesis on the shift. $\square$

Corollary 14. *If the shift is not in the set CV of theorem 12 then the $\mathfrak{g}$-equivariant quantization is unique.*

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