Challenging combinatorial problems

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Francqui Lecture, KUL, May 2010

Outline

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- recall some challenging combinatorial problems from previous lectures

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- introduce a couple of new ones

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- introduce a couple of new ones
- difficulty: worth a variable number of bottles of wine (scale: from 1 to ∞).

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1. Combinatorial models in manufacturing

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   - Tool management
   - Robotic cells
2. Boolean functions and games
   - Dualization
   - Threshold functions and weighted majority games

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Setup minimization

Tool setup in automated manufacturing:

- limited size of the tool magazine (say, 10 to 120 tools)

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One-machine scheduling with tooling decisions:

Simultaneously

- sequence parts to be processed and
- allocate tools required to the machine

so as to minimize tool setup costs.

Setup minimization

Various models for setup minimization.

Common data

- M: number of tools;
- N : number of parts;
- A : $M \times N$ tool-part matrix:

  $$a_{ij} = 1 \text{ if part } j \text{ requires tool } i, 0 \text{ otherwise;}$$
- C : capacity of the tool magazine (= number of tool slots)

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Feasible batch

A batch of parts is feasible if it can be carried out without tool switches, i.e., if it requires at most C tools.

Example

Capacity: $C = 3$

Parts

Tools

	P_1	P_2	P_3	P_4	P_5
T_1	1	0	1	0	1
T_2	1	0	0	1	0
T_3	0	1	1	1	0
T_4	0	1	0	0	1

Example

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Batch selection model

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or...

Batch selection

Given a hypergraph, find a subset of C vertices that contains the largest possible number of hyperedges.

Pseudo-Boolean formulation

- define : $x_i = 1$ if tool i is selected, $x_i = 0$ otherwise

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- equivalently: part j can be processed if and only if $\prod_{i:a_{ij}=1} x_i = 1$

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So:

Nonlinear knapsack

Batch selection is equivalent to the *nonlinear (supermodular) knapsack problem*

$$\max \quad \sum_{j=1}^{N} \prod_{i:a_{ij}=1} x_i$$
$$\text{subject to } \sum_{i=1}^{M} x_i \leq C, \quad (x_1, ..., x_M) \in \{0,1\}^M$$

Complexity

Theorem

Batch selection is NP-hard.

Generalization of maximum clique.

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Many papers on this problem:

- integer programming
- heuristics
- special “graphical” case
- subproblem for *part grouping problem*: partition parts into small number of feasible batches (minimize setups).

Worst-case ratio

From a theoretical point of view, one may ask:

Worst-case ratio

What is the (theoretical) worst-case ratio of heuristics for the batch selection problem, where:

$$wcr = \frac{optimal\ value}{heuristic\ value}?$$

Analysis

Crama and van de Klundert (1999) proved:

Theorem

If there is a polynomial-time approximation algorithm with constant worst-case ratio for batch selection, then there is also a polynomial-time approximation scheme for this problem.

... meaning roughly that the optimal value can be approximated arbitrarily closely in polynomial time.

Conjectures

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There exists no polynomial-time approximation algorithm with constant worst-case ratio for batch selection, unless P = NP.

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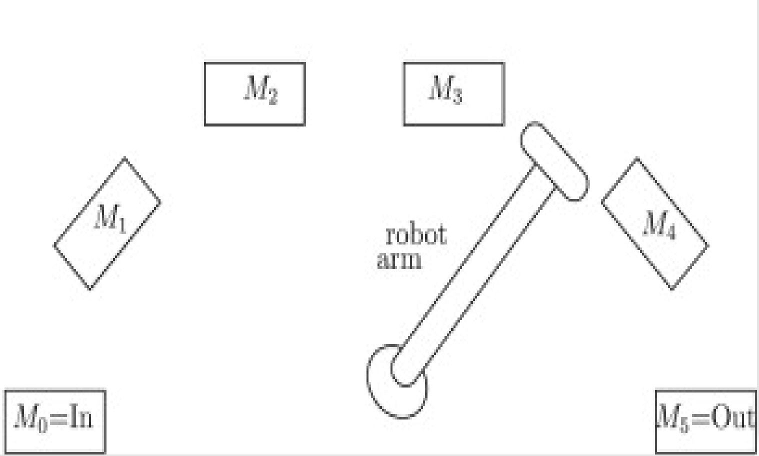
Circular robotic cell



Robotic cell flowshops

- m machines in line (or on a circle), without buffer space: $M_1, M_2, \ldots, M_m$
- loading station M_0 and unloading station M_{m+1}
- set of parts to be produced by the line
- a unique robot loads and unloads the parts

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Robotic cell scheduling

Determine

- a sequence of parts,
- a robot activity sequence,
- a production schedule (start/end times),
- so as to minimize cycle time (maximize throughput).

Assumptions

We concentrate here on:

- repetitive production of identical parts
- no intermediate buffers
- additive robot travel times.

1-Unit cycles

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(In particular, every activity is performed exactly once and the cycle can be repeated indefinitely.)

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(In particular, every activity is performed exactly once and the cycle can be repeated indefinitely.)

Crama and van de Klundert (1997) proved:

Theorem

For a robotic cell with m machines, a 1-unit cycle that minimizes the average cycle time can be computed in $O(m^3)$ time.

Optimality of 1-unit cycles

But... 1-unit cycles do not necessarily minimize the cycle time.

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Optimality of 1-unit cycles

But... 1-unit cycles do not necessarily minimize the cycle time.

k-Unit cycles

A *k-unit cycle* is a sequence of activities which unloads exactly k parts in the output buffer and which returns the cell to its initial state.

Note: For some k, some k-unit cycle is optimal.

Open question

Main open question:

Complexity

What is the complexity of computing an optimal (cyclic) robot move sequence?

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Definitions

Recall

Boolean functions

A Boolean function is a mapping $\varphi : \{0,1\}^n \rightarrow \{0,1\}$. Function φ is *positive* (monotone, isotone) if

$$X \leq Y \Rightarrow \varphi(X) \leq \varphi(Y).$$

Definitions

Recall

Boolean functions

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Set functions:

Boolean functions on $\{0,1\}^n$ can also be viewed as *set functions*, that is, functions defined on subsets of $\{1,2,\ldots,n\}$.

Example

x_1	x_2	x_3	S	φ
0	0	0	$\emptyset$	0
0	0	1	$\{3\}$	1
0	1	0	$\{2\}$	0
0	1	1	$\{2,3\}$	1
1	0	0	$\{1\}$	0
1	0	1	$\{1,3\}$	1
1	1	0	$\{1,2\}$	1
1	1	1	$\{1,2,3\}$	1

Simple games

- A positive Boolean function φ defines a *simple game* or *voting game*.
- Interpretation: φ describes the voting rule which is adopted by the players when a decision is to be made.
- If S is a subset of players, then $\varphi(S)$ is the outcome of the voting process when all players in S say "Yes".
- Positivity means:

$$S \subseteq T \Rightarrow \varphi(S) \leq \varphi(T).$$

Mimimal true points

- A positive Boolean function φ, or a simple game, is completely defined by the list of its *mimimal true points*, that is, minimal subsets of players $S_1, S_2, ..., S_m$ such that $\varphi(S_i) = 1$ for $i = 1, 2, ..., m$.

Example

x_1	x_2	x_3	S	φ	MTP
0	0	0	$\emptyset$	0	
0	0	1	$\{3\}$	1	S_1
0	1	0	$\{2\}$	0	
0	1	1	$\{2,3\}$	1	
1	0	0	$\{1\}$	0	
1	0	1	$\{1,3\}$	1	
1	1	0	$\{1,2\}$	1	S_2
1	1	1	$\{1,2,3\}$	1	

Maximal false points

- A positive Boolean function φ, or a simple game, is completely defined by the list of its *mimimal true points*, that is, minimal subsets of players $S_1, S_2, ..., S_m$ such that $\varphi(S_i) = 1$ for $i = 1, 2, ..., m$.
- Similarly, φ is completely defined by the list of its *maximal false points*, that is, maximal subsets of players $T_1, T_2, ..., T_k$ such that $\varphi(T_j) = 0$ for $j = 1, 2, ..., k$.

Example

x_1	x_2	x_3	S	φ	MTP	MFP
0	0	0	$\emptyset$	0		
0	0	1	$\{3\}$	1	S_1	
0	1	0	$\{2\}$	0		T_1
0	1	1	$\{2,3\}$	1		
1	0	0	$\{1\}$	0		T_2
1	0	1	$\{1,3\}$	1		
1	1	0	$\{1,2\}$	1	S_2	
1	1	1	$\{1,2,3\}$	1		

Dualization

A fundamental algorithmic problem:

Dualization

- Input: the list of minimal true points of a positive Boolean function.
- Output: the list of maximal false points of the function.

(or conversely)

Problem investigated in Boolean theory, game theory, integer programming, electrical engineering, artificial intelligence, reliability, combinatorics, etc.

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Dualization

Dualization amounts to generating

- all minimal transversals of a hypergraph (V, E), $E = (E_1, \ldots, E_m)$, $E_i \subseteq V$ (in particular: all maximal stable sets of a graph);
- all minimal solutions of a set covering problem:

$$\sum_{j \in E_i} x_j \geq 1 \ (i = 1, ..., m);$$

$$x_j \in \{0, 1\} \ (j = 1, ..., n).$$

Complexity

Note: the output is uniquely defined, but its size can be exponentially large in the size of the input.

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(Lawler, Lenstra, Rinnooy Kan 1980; Johnson, Papadimitriou, Yannakakis 1988; Bioch, Ibaraki 1995; etc.)

Can positive Boolean functions be dualized in *total polynomial time*, that is, in time polynomial in the combined size of the input and of the output?

Equivalent problem

Dualization is “polynomially equivalent” to the problem:

Test Dual

- Input: the MTPs of a Boolean function φ, and a list L of points.
- Question: is L the list of MFPs of φ ?

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- Dualization can be solved in total polynomial time if and only if Test Dual can be solved in polynomial time.
- Test Dual does not require exponential outputs.

Quasi-polynomial algorithm

Fredman and Khachiyan have shown

Fredman and Khachiyan (1996)

Dualization can be solved in time $O(m^{\log m})$, where m is the combined size of the input and of the output of the problem.

Quasi-polynomial algorithm

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Fredman and Khachiyan (1996)

Dualization can be solved in time $O(m^{\log m})$, where m is the combined size of the input and of the output of the problem.

- Several generalizations of this result have been obtained by Boros, Elbassioni, Gurvich, Khachiyan, Makino, etc.
- But the central questions remain open:

Open problems

Complexity of Dualization

Can dualization be solved in total polynomial time?

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Complexity of Test Dual

Can Test Dual be solved in polynomial time? Is it NP-hard (unlikely)?

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Weighted majority games

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A simple game φ is a *weighted majority game* if

- each player i carries a voting weight w_i

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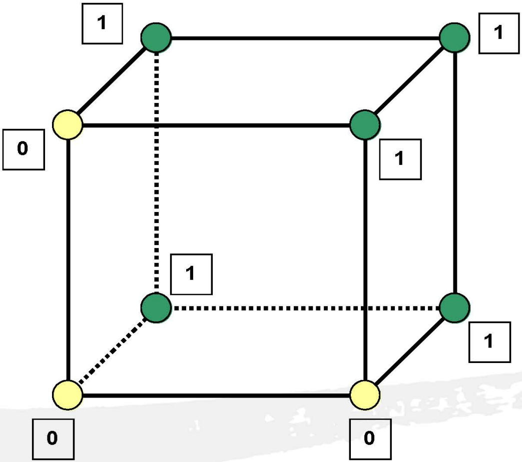
In Boolean theory, a weighted majority game is called a *threshold function*.

Example



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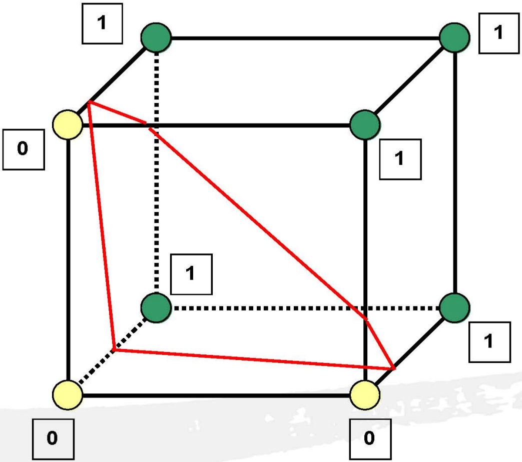

Applications

Numerous applications:

- game theory (weighted voting)
- electrical engineering (gates)
- optimization (knapsack)
- neural networks (perceptrons)
- databases (concurrent access)
- etc.

Threshold recognition

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Peled and Simeone (1985)

Threshold recognition can be solved in polynomial time.

Generic approach

Observations:

- every threshold function φ is *regular*: up to a permutation of its variables, $\varphi(\ldots 0 \ldots 1 \ldots) \leq \varphi(\ldots 1 \ldots 0 \ldots)$

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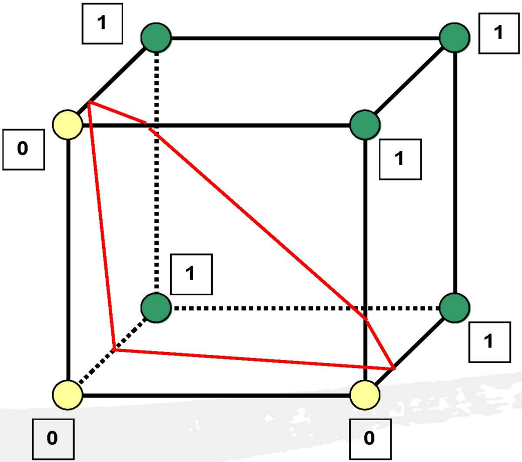
Observations:

- every threshold function φ is *regular*: up to a permutation of its variables, $\varphi(\ldots 0 \ldots 1 \ldots) \leq \varphi(\ldots 1 \ldots 0 \ldots)$
- regular functions can be recognized and dualized in polynomial time (Peled-Simeone 1985, Crama 1989, etc.)
- recall: dualizing amounts to generating all maximal 0s from all minimal 1s.

Example



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- solve the linear programming problem (in $w_1, \ldots, w_n$):

$$w_1 x_1 + w_2 x_2 + \ldots + w_n x_n \leq q \quad \text{if } \varphi(x_1, \ldots, x_n) = 0$$

$$w_1 x_1 + w_2 x_2 + \ldots + w_n x_n > q \quad \text{if } \varphi(x_1, \ldots, x_n) = 1$$

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Complexity

- $O(n^2 m)$ to test regularity and to dualize
- complexity of linear programming for separation

Open problem

Purely combinatorial recognition procedure

Can we avoid to solve an LP in order to recognize threshold functions?

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Purely combinatorial recognition procedure

Can we avoid to solve an LP in order to recognize threshold functions?

- Some failed attempts in that direction in the 60's (Chow, Winder, Dertouzos, etc.)
- Question could be taken up again in the light of advances in complexity theory and optimization.
- Recent attempts by Smaus (incomplete?)

Open problem

Purely combinatorial recognition procedure

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Conclusion

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Thank you for your presence and for your attention!!