COHERENT STATES,
WAVELETS AND APPLICATIONS

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Deficient splines wavelets

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1 Introduction

Splines are intensively used in different domains (numerical analysis, approximation theory,...) It is also well known that wavelets which are spline functions have been constructed some years ago; let us quote the works of Chui and Wang, Mallat, Meyer ([2], [6],[7]).

In some problems of numerical analysis (see for example [3], [8]), deficient splines are prefered to classical splines. Then a natural question arises in the context of wavelets: is it possible to construct bases wavelets which are splines of that kind? This problem is treated in a very general aspect in the papers [4], [5].

Here we present a constructive and direct way to obtain a mutiresolution analysis generated by deficient splines which are piecewise polynomials of degree 5, regularity 3, and have a compact support. Then, by a natural procedure, we obtain wavelets which are also functions of that type.

2 The results

Let us denote V_0 the following set of quintic splines

$$V_0 := \{f \in L_2(\mathbb{R}) : \ f|_{[k,k+1]} = P_k^{(5)}, k \in \mathbb{Z} \text{ and } f \in C_3(\mathbb{R})\}.$$

Looking for $f \in V_0$ with support $[0,3]$ (smaller interval does not give anything), we are lead to a homogenous linear system of 18 unknowns and 16 equations; this make us think that two scaling functions will be needed to generate V_0.

Proposition 2.1 *A function f with support $[0,3]$ belongs to V_0 if and only if*

$$f(x) = \begin{cases} nx^4 + ax^5 & \text{if } x \in [0,1] \\ bx^5 + cx^4 + dx^3 + ex^2 + fx + g & \text{if } x \in [1,2] \\ h(3-x)^4 + j(3-x)^5 & \text{if } x \in [2,3] \\ 0 & \text{if } x < 0 \text{ or } x > 3 \end{cases}$$

with

$$n = -\tfrac{4}{5}c - \tfrac{3}{10}d \qquad a = \tfrac{7}{15}c + \tfrac{8}{45}d \qquad b = -\tfrac{19}{75}c - \tfrac{19}{450}d$$

$$e = -\tfrac{-18}{5}c - \tfrac{13}{5}d \qquad f = \tfrac{18}{5}c + \tfrac{21}{10}d \qquad g = -\tfrac{27}{25}c - \tfrac{29}{50}d$$

$$h = -c - \tfrac{1}{3}d \qquad j = \tfrac{46}{75}c + \tfrac{91}{450}d$$

Theorem 2.2 *The following functions φ_a and φ_s*

$$\varphi_a(x) = \begin{cases} x^4 - \frac{11}{15}x^5 & \text{if } x \in [0,1] \\ -\frac{9}{8}(x-\frac{3}{2}) + 3(x-\frac{3}{2})^3 - \frac{38}{15}(x-\frac{3}{2})^5 & \text{if } x \in [1,2] \\ -(3-x)^4 + \frac{11}{15}(3-x)^5 & \text{if } x \in [2,3] \\ 0 & \text{if } x < 0 \text{ or } x > 3 \end{cases}$$

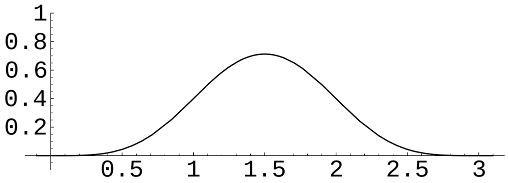
$$\varphi_s(x) = \begin{cases} x^4 - \frac{3}{5}x^5 & \text{if } x \in [0,1] \\ \frac{57}{80} - \frac{3}{2}(x-\frac{3}{2})^2 + (x-\frac{3}{2})^4 & \text{if } x \in [1,2] \\ (3-x)^4 - \frac{3}{5}(3-x)^5 & \text{if } x \in [2,3] \\ 0 & \text{if } x < 0 \text{ or } x > 3 \end{cases}$$

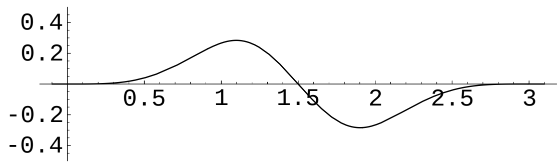
are respectively antisymmetric and symmetric with respect to $\frac{3}{2}$ *and the family*

$$\{\varphi_a(.-k),\ k \in \mathbb{Z}\} \cup \{\varphi_s(.-k),\ k \in \mathbb{Z}\}$$

constitutes a Riesz basis of V_0*.*

Here is a picture of φ_s, φ_a.





The result is obtained using Fourier techniques for the computations.

To get the Riesz condition, we first show that each of the families $\{\varphi_a(.-k),\ k \in \mathbb{Z}\}$ and $\{\varphi_s(.-k),\ k \in \mathbb{Z}\}$ satisfies this condition. Then, using the fact that the $L^2(\mathbb{R})$-norm of functions of the linear hull of the union of the families can be written as the $L^2([0,1])$-norm of functions which belongs to a linear space of finite dimension, and the fact that, separately, the families satisfies the Riesz condition, we get the result.

To obtain that these families generate V_0, we just solve the equations obtained when we write down the problem.

Now, we construct multiresolution analysis.

For every $j \in \mathbb{Z}$ we define

$$V_j = \{f \in L^2(\mathbb{R}) \ :\ f(2^{-j}.) \in V_0\}.$$

Proposition 2.3 *We have* $V_j \subset V_{j+1}$ *for every* j *and*

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

Moreover, the functions φ_a, φ_s *satisfy the following scaling relation*

$$\begin{pmatrix} \widehat{\varphi_s}(2\xi) \\ \\ \widehat{\varphi_a}(2\xi) \end{pmatrix} = M_0(\xi) \begin{pmatrix} \widehat{\varphi_s}(\xi) \\ \\ \widehat{\varphi_a}(\xi) \end{pmatrix}$$

where $M_0(\xi)$ *is the matrix (called filter matrix)*

$$M_0(\xi) = \frac{e^{-3i\xi/2}}{64} \begin{pmatrix} 51\cos(\frac{\xi}{2}) + 13\cos(\frac{3\xi}{2}) & -9i(\sin(\frac{\xi}{2}) + \sin(\frac{3\xi}{2})) \\ \\ i(11\sin(\frac{3\xi}{2}) + 21\sin(\frac{\xi}{2})) & -7\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2})) \end{pmatrix}.$$

And finally, using standard procedure, we get wavelets. We denote W_0 the orthogonal complement of V_0 in V_1.

First we define the matrix $W(\xi)$ as follows

$$W(\xi) = \begin{pmatrix} \omega_s(\xi) & \omega_m(\xi) \\ \omega_m(\xi) & \omega_a(\xi) \end{pmatrix}$$

with

$$\begin{aligned}
\omega_a(\xi) &= \sum_{l=-\infty}^{+\infty} |\widehat{\varphi_a}(\xi+2l\pi)|^2 = \frac{23247-21362\cos\xi-385\cos(2\xi)}{311850} \\
\omega_s(\xi) &= \sum_{l=-\infty}^{+\infty} |\widehat{\varphi_s}(\xi+2l\pi)|^2 = \frac{14445+7678\cos\xi+53\cos(2\xi)}{34650} \\
\omega_m(\xi) &= \sum_{l=-\infty}^{+\infty} \widehat{\varphi_s}(\xi+2l\pi)\overline{\widehat{\varphi_a}(\xi+2l\pi)} = -\frac{i}{51975}\sin\xi\ (6910+193\cos\xi).
\end{aligned}$$

Proposition 2.4 *A function f belongs to W_0 if and only if there exists $p, q \in L^2_{loc}$, $2\pi-$ periodic such that*

$$\widehat{f}(2\xi) = p(\xi)\widehat{\varphi_s}(\xi) + q(\xi)\widehat{\varphi_a}(\xi)$$

and

$$\overline{M_0(\xi)}W(\xi)\begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} + \overline{M_0(\xi+\pi)}W(\xi+\pi)\begin{pmatrix} p(\xi+\pi) \\ q(\xi+\pi) \end{pmatrix} = 0 \ \ a.e.$$

We explicitely solved this matrix equation and find

Proposition 2.5 *there exist symmetric and antisymmetric solutions with support in* $[0, 5]$.

We use the notations

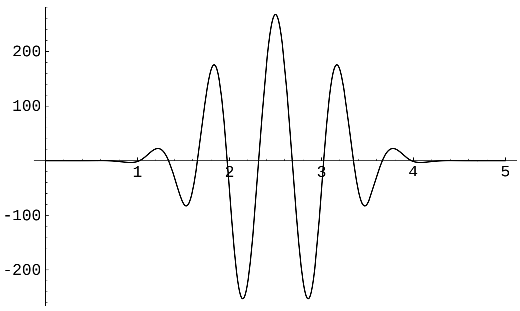
$$\psi_s(2\xi) = p_s(\xi)\widehat{\varphi_s}(\xi) + q_s(\xi)\widehat{\varphi_a}(\xi)$$

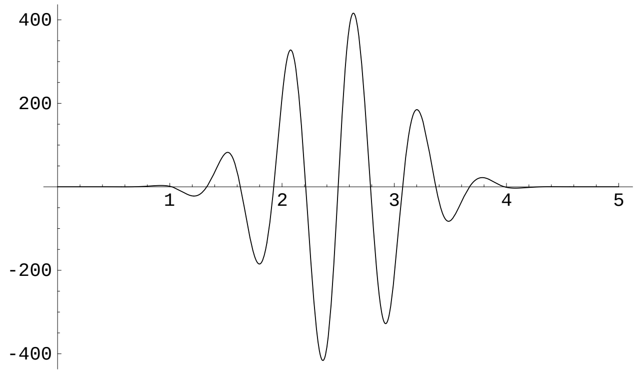
$$\psi_a(2\xi) = p_a(\xi)\widehat{\varphi_s}(\xi) + q_a(\xi)\widehat{\varphi_a}(\xi).$$

We have (with natural definition of M_1)

$$\begin{pmatrix} \widehat{\psi_s}(2\xi) \\ \widehat{\psi_a}(2\xi) \end{pmatrix} = M_1(\xi)\begin{pmatrix} \widehat{\varphi_s}(\xi) \\ \widehat{\varphi_a}(\xi) \end{pmatrix}.$$

Here are ψ_s, ψ_a (up to a multiplicative constant)





Proposition 2.6 *The family*

$$\{\psi_a(.-k) : k \in \mathbb{Z}\} \cup \{\psi_s(.-k) : k \in \mathbb{Z}\}$$

form a Riesz basis of W_0.

To obtain this result, we proceed as follows, using essentially techniques of Goodman and Lee. Define

$$W_\psi(\xi) = \begin{pmatrix} \omega_{\psi_s}(\xi) & \omega_{\psi_s,\psi_a}(\xi) \\ \overline{\omega_{\psi_s,\psi_a}(\xi)} & \omega_{\psi_a}(\xi) \end{pmatrix}$$

where

$$\begin{aligned}
\omega_{\psi_a}(\xi) &= \sum_{l=-\infty}^{+\infty} |\widehat{\psi_a}(\xi+2l\pi)|^2, \\
\omega_{\psi_s}(\xi) &= \sum_{l=-\infty}^{+\infty} |\widehat{\psi_s}(\xi+2l\pi)|^2, \\
\omega_{\psi_s,\psi_a}(\xi) &= \sum_{l=-\infty}^{+\infty} \widehat{\psi_s}(\xi+2l\pi)\overline{\widehat{\psi_a}(\xi+2l\pi)}.
\end{aligned}$$

The Riesz condition is satisfied if and only if there are $A, B > 0$ such that

$$A \leq \lambda_1(\xi),\ \lambda_2(\xi) \leq B$$

where $\lambda_i(\xi)$ are the eigenvalues of $W_\psi(\xi)$.

The functions form a basis if and only if for every ξ, the matrix

$$\begin{pmatrix} \overline{M_0(\xi)} & \overline{M_0(\xi+\pi)} \\ \overline{M_1(\xi)} & \overline{M_1(\xi+\pi)} \end{pmatrix} \begin{pmatrix} \overline{W(\xi)} & 0 \\ 0 & \overline{W(\xi+\pi)} \end{pmatrix}$$

is not singular.

The properties above are easily obtained using the three relations

$$\begin{aligned}
W(2\xi) &= M_0(\xi)W(\xi)M_0^*(\xi) + M_0(\xi+\pi)W(\xi+\pi)M_0^*(\xi+\pi) \\
0 &= M_1(\xi)W(\xi)M_0^*(\xi) + M_1(\xi+\pi)W(\xi+\pi)M_0^*(\xi+\pi) \\
W_\psi(2\xi) &= M_1(\xi)W(\xi){M_1}^*(\xi) + M_1(\xi+\pi)W(\xi+\pi){M_1}^*(\xi+\pi)
\end{aligned}$$

which are consequences of the previous constructions and properties of multiresolution analysis and functions in W_0.

Theorem 2.7 *It follows that the functions*

$$2^{j/2}\psi_s(2^j x - k),\ 2^{j/2}\psi_a(2^j x - k) \quad (j, k \in \mathbb{Z})$$

form a Riesz basis of compactly supported deficient splines of $L^2(\mathbb{R})$ *with symmetry properties.*

To conclude, let us mention a result of Gilson, Faure, Laubin ([3]) concerning an application of splines (deficient splines) in approximation theory of singular functions.

For $q \geq 1$, let $S_{3,q}^N$ the space of classical cubic splines on the interval $[0, 1]$ with respect to the subdivision $(j/N)^q,\ j = 0, \ldots, N$.

Proposition 2.8 *If* $[\alpha_0, \alpha_1] \subset \mathbb{R}$ *with* $\alpha_0 > -1/2$ *and* $q(1 + 2\alpha_0) > 8$*, then there exists* $C > 0$ *such that*

$$\inf_{u \in S_{3,q}^N} \|u - x^\alpha\|_{L^2(]0,1[)} \leq \frac{C}{N^4}$$

for every $N \in \mathbb{N}$ *and* $\alpha \in [\alpha_0, \alpha_1]$.

There is an extension of this result to DEFICIENT SPLINES of odd degree and to Sobolev spaces.

References

[1] F. Bastin and P. Laubin, *Quintic deficient spline wavelets,* to appear (2002) in a volume of the Bulletin de la Société Royale des Sciences de Liège.

[2] C.K. Chui and J.Z. Wang, *On compactly supported spline wavelets and a duality principle,* Transactions of the AMS, 330, 2, 1992, 903-915.

[3] O. Gilson, O. Faure, P. Laubin, *Quasi-optimal convergence using interpolation by non-uniform deficient splines,* submitted for publication

[4] T.N.T. Goodman, S.L. Lee, *Wavelest of multiplicity r,* Trans. Amer. Math. Soc. 342, 1, 1994, 307-324

[5] T.N.T. Goodman, S.L. Lee, W.S. Tang, *Wavelets in wandering subspaces,* Trans. Amer. Math. Soc., 338, 2, 1993, 639-654

[6] S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbb{R})$, Trans. Amer. Math. Soc., 315(1), 1989, 69-88.

[7] Y. Meyer, *Ondelettes et opérateurs,* I,II,III, Hermann, Paris, 1990.

[8] S.S. Rana, Dubey Y.P., *Best error bounds of deficient quintic splines interpolation,* Indian J. Pure Appl. Math., 28, 1997, 1337-1344

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