ON NATURAL AND CONFORMALLY EQUIVARIANT QUANTIZATIONS

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ABSTRACT. The concept of conformally equivariant quantization was introduced by Duval, Lecomte and Ovsienko in [8] for manifolds endowed with flat conformal structures. They obtained results of existence and uniqueness (up to normalization) of such a quantization procedure. A natural generalization of this concept is to seek for a quantization procedure, over a manifold M, that depends on a pseudo-Riemannian metric, is natural, and is invariant with respect to a conformal change of the metric. The existence of such a procedure was conjectured by P. Lecomte in [15] and proved by C. Duval and V. Ovsienko in [9] for symbols of degree at most 2 and by S. Loubon Djounga in [16] for symbols of degree 3. In two recent papers [18, 17], we investigated the question of existence of projectively equivariant quantizations using the framework of Cartan connections. Here we will show how the formalism developed in these works adapts in order to deal with the conformally equivariant quantization for symbols of degree at most 3. This will allow us to easily recover the results of [9] and [16]. We will then show how it can be modified in order to prove the existence of conformally equivariant quantizations for symbols of degree 4.

1. INTRODUCTION

It is common in the mathematical literature to think of a quantization map as a linear bijection from a space of classical observables to a space of differential operators acting on wave functions (see [20]). In our setting, the observables (also called *Symbols*) are smooth functions on the cotangent bundle T^*M of a manifold M, that are polynomial along the fibres, while the differential operators act on half densities.

The concept of *equivariant quantization* was introduced by Lecomte and Ovsienko in [14] and developed in a series of papers [8, 10, 2, 1]. The idea is that, when a Lie group G acts on M by local diffeomorphisms, it is sometimes natural to require that the quantization map be equivariant with respect to the lifted actions of G on symbols and on differential operators.

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In [14] and [10], the authors considered the case of the projective group $PGL(m+1, \mathbb{R})$ acting on the manifold $M = \mathbb{R}^m$ by linear fractional transformations. This leads to the notion of projectively equivariant quantization or its infinitesimal counterpart, the $sl(m+1, \mathbb{R})$-equivariant quantization. In [8], the authors considered the group $SO(p+1, q+1)$ acting on the space $\mathbb{R}^{p+q}$ or on a manifold endowed with a flat conformal structure. They extended the problem by considering the space $\mathcal{D}_{\lambda,\mu}$ of differential operators mapping λ-densities into μ-densities and a suitable space of symbols $\mathcal{S}_{\mu-\lambda}$.

In both situations, the results were the existence and uniqueness of equivariant quantization provided the shift value $\delta = \mu - \lambda$ does not belong to a set of critical values. These results settled the problem projectively- and conformally- equivariant quantizations in the framework of manifolds endowed with flat structures.

Then the particular case of symbols of degree at most three was considered in detail in [4, 5] in the projective case and in [9, 16] in the conformal case. There the authors showed that the equivariant quantization procedure can be expressed using a connection (the Levi Civita connection associated to a pseudo Riemannian metric in the conformal case), in such a way that it only depends on the projective class of the connection or on the conformal class of the metric.

In [15], P. Lecomte conjectured the existence of a quantization procedure depending on a torsion-free linear connection (resp. on a pseudo-Riemannian metric), that would be natural (in all arguments) and that would be invariant with respect to a projective (resp. conformal) change of connection (resp. metric).

In the projective case, the existence of such a *Natural and equivariant quantization procedure* was proved by M. Bordemann in [3], using the notion of Thomas-Whitehead connection associated to a projective class of connections. His construction was later adapted by S. Hansoul (see [11, 12]) in order to deal with other types of differential operators.

In [18, 17], we analysed the existence problem for a natural and projectively equivariant quantization using the theory of projective Cartan connections. Among our motivations, one was to use the similarity between the theory of projective Cartan connections and the theory of conformal ones (we refer the reader to [13] where both cases are presented concurrently) in order to deal with the conformal case.

In these papers, we showed how the formulae that were obtained in the flat situation could be easily modified in order to obtain a natural and projectively equivariant quantization.

In the present work, we will show that in the conformal situation, the same modifications allow to build a natural and conformally equivariant quantization for symbols of degree at most three, thus recovering more easily the results of [9, 16]. We will show that the same procedure of [17] does not directly work for symbols of degree 4. We will nevertheless exhibit an explicit formula for the conformally equivariant quantization of such symbols, in

terms of the Cartan connection associated to the conformal class of the metric. Moreover the rather simple expression of the solution might show the direction for a solution of the existence problem for symbols of arbitrary degree.

2. Basic notions and problem setting

In this section, we will just fix some notation concerning tensor densities, symbols and differential operators. For more details about these notions, we refer the reader to [18, 17] and references therein. Throughout this paper, we let M denote an m-dimensional Hausdorff, second countable smooth manifold.

2.1. **Tensor densities.** Denote by $\Delta^\lambda(\mathbb{R}^m)$ the one dimensional representation of $GL(m, \mathbb{R})$ given by

$$\rho(A)e = |det A|^{-\lambda}e, \quad \forall A \in GL(m, \mathbb{R}), \ \forall e \in \Delta^\lambda(\mathbb{R}^m).$$

The vector bundle of λ-densities is then defined by

$$P^1M \times_\rho \Delta^\lambda(\mathbb{R}^m) \to M,$$

where P^1M is the linear frame bundle of M.

Recall that the space $\mathcal{F}_\lambda(M)$ of smooth sections of this bundle, the space of λ-densities, can be identified with the space $C^\infty(P^1M, \Delta^\lambda(\mathbb{R}^m))_{GL(m,\mathbb{R})}$ of functions f such that

$$f(uA) = \rho(A^{-1})f(u) \quad \forall u \in P^1M, \ \forall A \in GL(m, \mathbb{R}).$$

Moreover, since the bundle of λ-densities is associated to the linear frame bundle, there are natural actions of local diffeomorphisms and of vector fields on $\mathcal{F}_\lambda$. These actions were detailed for instance in [8, 14].

2.2. **Differential operators and symbols.** As usual, we denote by $\mathcal{D}_{\lambda,\mu}(M)$ the space of differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. The actions of vector fields and of (local) diffeomorphisms on $\mathcal{D}_{\lambda,\mu}(M)$ are induced by the actions on $\mathcal{F}_\lambda(M)$ and $\mathcal{F}_\mu(M)$.

The space $\mathcal{D}_{\lambda,\mu}$ is filtered by the order of differential operators. This filtration is preserved by the action of local diffeomorphisms and of vector fields. The space of *symbols* is then the associated graded space of $\mathcal{D}_{\lambda,\mu}$. It is also known that the principal operators σ_l ($l \in \mathbb{N}$) allow to identify the space of symbols with the space of contravariant symmetric tensor fields with coefficients in δ-densities where $\delta = \mu - \lambda$ is the shift value.

More precisely, we denote by $S^l_\delta(\mathbb{R}^m)$ or simply S^l_δ the space $S^l\mathbb{R}^m \otimes \Delta^\delta(\mathbb{R}^m)$ endowed with the natural representation ρ of $GL(m, \mathbb{R})$. Then the vector bundle of symbols of degree l is

$$P^1M \times_\rho S^l_\delta(\mathbb{R}^m) \to M.$$

The space $\mathcal{S}_\delta^l(M)$ of symbols of degree l is then the space of smooth sections of this bundle, which can be identified with $C^\infty(P^1 M, S_\delta^l(\mathbb{R}^m))_{GL(m,\mathbb{R})}$. Finally, the whole space of symbols is

$$\mathcal{S}_\delta(M) = \bigoplus_{l=0}^\infty \mathcal{S}_\delta^l(M),$$

endowed with the classical actions of diffeomorphisms and of vector fields.

2.3. **Natural and equivariant quantizations.** A *quantization on M* is a linear bijection Q_M from the space of symbols $\mathcal{S}_\delta(M)$ to the space of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ such that

$$\sigma_l(Q_M(S)) = S, \ \forall S \in \mathcal{S}_\delta^l(M), \ \forall l \in \mathbb{N},$$

where σ_l is the principal symbol operator on the space of operators of order less or equal to l.

In the conformal sense, a *natural quantization* is a collection of quantizations Q_M depending on a pseudo-Riemannian metric such that

- For all pseudo-Riemannian metric g on M, $Q_M(g)$ is a quantization,
- If ϕ is a local diffeomorphism from M to N, then one has

$$Q_M(\phi^* g)(\phi^* S) = \phi^*(Q_N(g)(S)),$$

  for all pseudo-Riemannian metrics g on N, and all $S \in \mathcal{S}_\delta(N)$.

Recall now that two pseudo Riemannian metrics g and g' on a manifold M are conformally equivalent if and only if there exists a positive function f such that $g' = fg$.

A quantization Q_M is then *conformally equivariant* if one has $Q_M(g) = Q_M(g')$ whenever g and g' are conformally equivalent.

2.4. **Conformal structures.** In this section, we will recall the notions of conformal structures over a manifold M and how they are associated to reductions of the second order frame bundle $P^2 M$ to a certain group H. The description of this group as a semi direct product as well as the description of its Lie algebra is one of the main ingredients of our construction. This description was given for the projective and the conformal situation by S. Kobayashi in [13]. Here we will recall the main results of [13] and refer the reader to this book for more details. We will adopt the notation of [18, 17] for the comparison with the projective case to be easy.

A conformal structure on a manifold is classically defined as an equivalence class of pseudo-Riemannian metrics. We now recall the group theoretic point of view.

Given p and q such that $p + q = m$, we consider the bilinear symmetric form of signature $(p+1, q+1)$ on $\mathbb{R}^{m+2}$ defined by

$$B : \mathbb{R}^{m+2} \times \mathbb{R}^{m+2} \to \mathbb{R} : (x, y) \mapsto {}^t y S x,$$

where S is the matrix of order $m + 2$ given by

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & J & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Also the matrix

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

represents a nondegenerate symmetric bilinear form g_0 on $\mathbb{R}^m$, namely

$$g_0 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} : (x, y) \mapsto {}^t y J x.$$

By analogy with the Riemannian setting, we also denote by $|x|^2$ the number $g_0(x, x)$ (even though this expression might be negative).

As we continue, we will use the classical isomorphism between $\mathbb{R}^m$ and $\mathbb{R}^{m*}$ defined by the symmetric bilinear form represented by J :

$$\sharp : \mathbb{R}^m \to \mathbb{R}^{m*} : x \mapsto x^\sharp : x^\sharp(y) = g_0(x, y)$$

and $\flat = \sharp^{-1}$.

The Möbius space is the projection of the light cone associated to this metric on the projective space $\mathbb{R}P^{m+1}$.

The group G is made of linear transformations that leave B invariant, modulo its center, that is,

$$G = \{X \in GL(m + 2, \mathbb{R}) : {}^t X S X = S\}/\{\pm I_{m+2}\}.$$

It acts transitively on the Möbius space $S^p \times S^q$.

The group H is the isotropy subgroup of G at the point $[e_{m+2}]$ of the Möbius space :

$$H = \{\begin{pmatrix} a^{-1} & 0 & 0 \\ a^{-1} A \xi^\sharp & A & 0 \\ \frac{1}{2a} |\xi|^2 & \xi & a \end{pmatrix} : A \in O(p, q), a \in \mathbb{R}_0, \xi \in \mathbb{R}^{m*}\}/\{\pm I_{m+2}\}.$$

As in the projective situation, H is a semi-direct product $G_0 \rtimes G_1$. Here G_0 is isomorphic to $CO(p, q)$ and G_1 is isomorphic to $\mathbb{R}^{m*}$. There is also a projection

$$\pi : H \mapsto CO(p, q) : \left[\begin{pmatrix} a^{-1} & 0 & 0 \\ a^{-1} A \xi^\sharp & A & 0 \\ \frac{1}{2a} |\xi|^2 & \xi & a \end{pmatrix}\right] \mapsto \frac{A}{a}.$$

The Lie algebra of G is $\mathfrak{g} = so(p + 1, q + 1)$, and decomposes as a direct sum of subalgebras

$$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \cong \mathbb{R}^m \oplus co(p, q) \oplus \mathbb{R}^{m*}.$$

The isomorphism is given by

$$\begin{pmatrix} -a & v^\flat & 0 \\ \xi^\sharp & A & v \\ 0 & \xi & a \end{pmatrix} \mapsto (v, A - aI_m, \xi).$$

This correspondence induces a structure of Lie algebra on $\mathbb{R}^m \oplus co(p,q) \oplus \mathbb{R}^{m*}$. It is easy to see that the adjoint actions of $co(p,q)$ on $\mathbb{R}^m$ and on $\mathbb{R}^{m*}$ coincide with the natural actions. Moreover, one has

$$[h,x] = -x \otimes h + h^\sharp \otimes x^\flat - \langle h, x \rangle Id. \tag{1}$$

The Lie algebras corresponding to G_0, G_1 and H are respectively $\mathfrak{g}_0$, $\mathfrak{g}_1$, and $\mathfrak{g}_0 \oplus \mathfrak{g}_1$.

It is well-known that there is a bijective and natural correspondence between the conformal structures on M and the reductions of $P^1 M$ to the structure group $G_0 \cong CO(p,q)$. In [13], one shows that it is possible to associate at each G_0-structure P_0 a principal H-bundle P on M, this association being natural and conformally invariant. Since H can be considered as a subgroup of G_m^2, this H-bundle can be considered as a reduction of $P^2 M$. The relationship between conformal structures and reductions of $P^2 M$ to H is given by the following proposition.

Proposition 1. *There is a natural one-to-one correspondence between the conformal equivalence classes of pseudo-Riemannian metrics on M and the reductions of $P^2 M$ to H.*

Throughout this work, we will freely identify conformal structures and reductions of $P^2 M$.

2.5. **Lift of equivariant functions.** In the previous section, we recalled how to associate an H-principal bundle P to a conformal structure P_0. We now recall how the densities and symbols can be regarded as equivariant functions on P.

If (V, ρ) is a representation of G_0, then we may extend it to a representation (V, ρ') of H by

$$\rho' = \rho \circ \pi.$$

Now, using the representation ρ', we can recall the relationship between equivariant functions on P_0 and equivariant functions on P (see [7]): if we denote by p the projection $P \to P_0$, we have

Proposition 2. *If (V, ρ) is a representation of G_0, then the map*

$$p^* : C^\infty(P_0, V) \mapsto C^\infty(P, V) : f \mapsto f \circ p$$

defines a bijection from $C^\infty(P_0, V)_{G_0}$ to $C^\infty(P, V)_H$.

2.6. **Cartan connections.** The main ingredient in our construction of natural and equivariant quantization is the normal Cartan connection associated to a conformal structure. In this section we recall the definitions and results about Cartan conformal connections that we will use as we continue. Let us begin with the general definition :

Let L be a Lie group and L_0 a closed subgroup. Denote by $\mathfrak{l}$ and $\mathfrak{l}_0$ the corresponding Lie algebras. Let $N \to M$ be a principal L_0-bundle over M, such that $\dim M = \dim L/L_0$. A Cartan connection on N is an $\mathfrak{l}$-valued one-form ω on N such that

(1) If R_a denotes the right action of $a \in L_0$ on N, then $R_a^* \omega = Ad(a^{-1})\omega$,
(2) If k^* is the vertical vector field associated to $k \in \mathfrak{l}_0$, then $\omega(k^*) = k$,
(3) $\forall u \in N$, $\omega_u : T_u N \mapsto \mathfrak{l}$ is a linear bijection.

When considering in this definition the principal H- bundle P of the previous section, and taking as group L the group G and for L_0 the group H, we obtain the definition of Cartan conformal connections.

We will need some properties of the curvature of Cartan connections.

If ω is a Cartan connection defined on an H-principal bundle P, then its curvature Ω is defined by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{2}$$

The connection ω is constructed (in [13]) in such a way that the projection of curvature on $\mathfrak{g}_{-1}$ is trivial.

Moreover, it is known that the curvature vanishes on vertical vector fields, that is, vector fields of the form h^* for h in $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

It is common to use the function $\kappa \in C^\infty(P, \wedge^2 \mathfrak{g}^* \otimes \mathfrak{g})$ defined as follows

$$\kappa(x, y) = \Omega(\omega^{-1}(x), \omega^{-1}(y)).$$

Since Ω takes non trivial values only in $\mathfrak{g}_0 \oplus \mathfrak{g}_1$, we can decompose κ as $\kappa_0 + \kappa_1$ in an obvious manner.

Finally, since the function κ vanishes on $\mathfrak{g}_0 \oplus \mathfrak{g}_1$, the functions κ_0 and κ_1 actually belong to $C^\infty(P, \wedge^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0)$ and $C^\infty(P, \wedge^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1)$ respectively. Also notice that, since we identified $\mathfrak{g}_0$ to $co(p, q)$, the function κ_0 also belongs to $C^\infty(P, \wedge^2 \mathfrak{g}_{-1}^* \otimes gl(\mathfrak{g}_{-1}))$ while κ_1 belongs to $C^\infty(P, \wedge^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^*)$, since $g_1 \cong \mathbb{R}^{m*}$.

Moreover, if we denote by $(e_1, \cdots, e_m)$ the standard basis of $\mathfrak{g}_{-1} \cong \mathbb{R}^m$, the components of the function κ_0 in this basis can be defined by

$$\kappa_{ijk}^l = (\kappa_0(e_j, e_k))_i^l.$$

Then a *normal Cartan connection* is a Cartan connection whose curvature fulfills the relation

$$\kappa_{jil}^i = 0 \quad \forall j, l.$$

The following result (see [13, 6]) ensures the existence of normal Cartan connections

Proposition 3. *A unique normal conformal Cartan connection with values in the algebra $\mathfrak{g}$ is associated to every conformal structure P. This association is natural.*

We end this section by technical results about Lie derivatives of the curvature. The following lemmas follow easily from the definition of the curvature and from the Ad-invariance of ω.

Lemma 4. *There holds*

$$[\omega^{-1}(x), \omega^{-1}(A)] = -\omega^{-1}(\kappa(x, A)) + \omega^{-1}(L_{\omega^{-1}(x)}A) + \omega^{-1}([x, A]),$$

for every $x \in \mathfrak{g}$ and $A \in C^\infty(P, \mathfrak{g})$.
   In particular there holds

$$[h^*, \omega^{-1}(y)] = \omega^{-1}([h, y]),$$

for every $h \in \mathfrak{h}$ and $y \in \mathfrak{g}$.

   Finally we have

Lemma 5. *For every $h \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$, one has*

$$L_{h^*}\omega = -ad(h) \circ \omega \quad and \quad L_{h^*}\Omega = -ad(h) \circ \Omega.$$

The function κ is $\mathfrak{g}_0$-invariant : for every $h \in \mathfrak{g}_0$ one has

$$L_{h^*}\kappa = -ad(h)\kappa,$$

that is,

$$L_{h^*}\kappa(x, y) = -ad(h)(\kappa(x, y)) + \kappa(ad(h)x, y) + \kappa(x, ad(h)y).$$

Finally, for every $h \in \mathfrak{g}_1$, one has

$$L_{h^*}\kappa_0 = 0 \quad and \quad L_{h^*}\kappa_1(x, y) = [\kappa_0(x, y), h].$$

3. Equivariant quantizations in the flat case

   Our construction of the natural and conformally equivariant quantization is based on an adaptation of two previous techniques : the first one is the conformally equivariant quantization in the flat case (see [8]) and the second is our construction of the projectively equivariant natural quantization in general. Here we recall the main features of the construction of the quantization in the flat case. The details and the proofs can be found in [8, 2, 17].

3.1. **The flat case.** Let us first recall the problem of equivariant quantizations over the Euclidean space $\mathbb{R}^m$.
   In this framework, we make the following identifications :

$$\begin{array}{rcl} \mathcal{F}_\lambda(\mathbb{R}^m) & \cong & C^\infty(\mathbb{R}^m, \Delta^\lambda(\mathbb{R}^m)), \\ \mathcal{S}^k_\delta(\mathbb{R}^m) & \cong & C^\infty(\mathbb{R}^m, S^k_\delta(\mathbb{R}^m)). \end{array}$$

The Lie algebra $Vect(\mathbb{R}^m)$ acts on these spaces in a well-known manner : for every $X \in Vect(\mathbb{R}^m)$ and any symbol s, one has

$$L_X s(x) = X.s(x) - \rho_*(D_x X)s(x)$$

where ρ is the natural action of $GL(m, \mathbb{R})$ on the fibre. The space $\mathcal{D}_{\lambda\mu}(\mathbb{R}^m)$ of differential operators is equipped with the Lie derivative given by the commutator.

3.1.1. *Conformal algebra of vector fields.* The data of section 2.4 allow to define Lie algebras of vector fields. The Lie group G acts on the homogeneous space G/H. Since this space can locally be identified with $\mathbb{R}^m$, there is a local action of G on $\mathbb{R}^m$. This action allows to define algebras of vector fields which are obviously isomorphic to $so(p+1, q+1, \mathbb{R})$. As mentioned in [2], the realization of $\mathfrak{g} \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ in vector fields is the following (we denote by X^h the vector field associated to h) : for every $x \in \mathfrak{g}_{-1} \cong \mathbb{R}^m$,

$$\begin{cases} X_x^h & = & -h & \text{if } h \in \mathfrak{g}_{-1} \\ X_x^h & = & -[h,x] & \text{if } h \in \mathfrak{g}_0 \\ X_x^h & = & -\frac{1}{2}[[h,x],x] & \text{if } h \in \mathfrak{g}_1 \end{cases} \tag{3}$$

3.1.2. *Equivariant quantizations.* A conformally equivariant quantization (in the sense of [8]) is a quantization $Q : \mathcal{S}_\delta(\mathbb{R}^m) \to \mathcal{D}_{\lambda\mu}(\mathbb{R}^m)$ (where $\delta = \mu - \lambda$ is the shift value) such that, for every $X^h \in \mathfrak{g}$, one has

$$\mathcal{L}_{X^h} \circ Q = Q \circ L_{X^h}.$$

The existence and uniqueness of such quantizations were proved in [14, 8] provided the shift value does not belong to a set of *Critical values*.

From now on to the end of this section, we will present the tools that were used by Duval, Lecomte and Ovsienko, and generalized in [2] in order to obtain this result.

3.1.3. *The affine quantization map.* There exists a well-known bijection from symbols to differential operators over $\mathbb{R}^m$: the so-called *Standard ordering* Q_{Aff}. If a symbol $T \in \mathcal{S}_\delta^k(\mathbb{R}^m)$ writes

$$T(x, \xi) = \sum_{|\alpha|=k} C_\alpha(x) \xi^\alpha,$$

where α is a multiindex, then one has

$$Q_{Aff}(T) = \sum_{|\alpha|=k} C_\alpha(x) (\frac{\partial}{\partial x})^\alpha.$$

It is easily seen that this map exchanges the actions of the affine algebra on the space of symbols and of differential operators. This is why we call it the affine quantization map.

Formula 3 allows to express this quantization map in a coordinate free manner :

Proposition 6. *If $h_1, \cdots, h_k \in \mathbb{R}^m \cong \mathfrak{g}_{-1}$, $A \in \Delta^\delta(\mathbb{R}^m)$, $s \in C^\infty(\mathbb{R}^m)$, and*

$$T(x) = s(x) A \otimes h_1 \vee \cdots \vee h_k,$$

one has

$$Q_{Aff}(T) = (-1)^k s \circ A \circ L_{X^{h_1}} \circ \cdots \circ L_{X^{h_k}},$$

where A is understood as a linear map from $\Delta^\lambda(\mathbb{R}^m)$ to $\Delta^\mu(\mathbb{R}^m)$.

3.1.4. *The map γ.* Using the affine quantization map, one can endow the space of symbols with a structure of representation of $Vect(\mathbb{R}^m)$ (or $\mathfrak{g}$), isomorphic to $\mathcal{D}_{\lambda\mu}(\mathbb{R}^m)$. Explicitly, we set

$$\mathcal{L}_X T := Q_{Aff}^{-1} \circ \mathcal{L}_X \circ Q_{Aff}(T),$$

for every $T \in \mathcal{S}_\delta(\mathbb{R}^m)$ and $X \in Vect(\mathbb{R}^m)$.

An equivariant quantization is then a $so(p+1, q+1)$-isomorphism from the representation $(\mathcal{S}_\delta(\mathbb{R}^m), L)$ to $(\mathcal{S}_\delta(\mathbb{R}^m), \mathcal{L})$.

In order to measure the difference between these representations the map

$$\gamma : \mathfrak{g} \to gl(\mathcal{S}_\delta(\mathbb{R}^m), \mathcal{S}_\delta(\mathbb{R}^m)) : h \mapsto \gamma(h) = \mathcal{L}_{X^h} - L_{X^h}$$

was introduced in [2], where its most important properties were listed.

In [18], we obtained a coordinate free expression of γ.

Proposition 7. *For every $h_1, \cdots, h_k \in \mathbb{R}^m \cong \mathfrak{g}_{-1}$, $A \in \Delta^\delta(\mathbb{R}^m)$ and $h \in \mathfrak{g}_1 \cong \mathbb{R}^{m*}$ we have*

$$
\begin{aligned}
\gamma(h)(h_1 \vee \cdots \vee h_k \otimes A) &= -\lambda \sum_{i=1}^k tr([h_i, h])\, h_1 \vee \overset{(i)}{\cdots} \vee h_k \otimes A \\
&+ \sum_{i=1}^k \sum_{j<i} h_1 \vee \overset{(i,j)}{\cdots} \vee h_k \vee [h_i, [h_j, h]] \otimes A
\end{aligned}
$$

3.1.5. *Casimir operators.* In [8, 2], the construction of the quantization is based on the comparison of the spectra and of the eigenvectors of some (second order) Casimir operators. These operators are on the one hand the Casimir operator C associated to the representation $(\mathcal{S}_\delta(\mathbb{R}^m), L)$ and on the other hand the Casimir operator $\mathcal{C}$ associated to the representation $(\mathcal{S}_\delta(\mathbb{R}^m), \mathcal{L})$.

From [2], we know that the difference between C and $\mathcal{C}$ can be expressed in terms of γ : fixing a basis (e_i) in $\mathfrak{g}_{-1}$ and denoting by (ϵ^i) the Killing-dual basis in $\mathfrak{g}_1$, one has

$$\mathcal{C} = C + N, \tag{4}$$

where

$$N = 2 \sum_i \gamma(\epsilon^i) L_{X^{e_i}}.$$

Moreover, it was also shown in [8] that $\mathcal{S}_\delta(\mathbb{R}^m)$ is the direct sum of eigenspaces of C. Indeed, if

$$S^k(\mathbb{R}^m) = \oplus_{s \le \frac{k}{2}} S_{(k,s)}$$

is the decomposition of $S^k(\mathbb{R}^m)$ into irreducible representations of $\mathfrak{h}_0 \cong so(p,q) \subset \mathfrak{g}_0$ and if we set $\mathcal{S}_{(k,s)} = C^\infty(\mathbb{R}^m, S_{(k,s)} \otimes \Delta^\delta(\mathbb{R}^m))$, then we have

$$C|_{\mathcal{S}_{(k,s)}} = \alpha_{k,s} \mathrm{Id}.$$

3.1.6. *Construction of the quantization.* It turns out that the problem of existence of an $so(p+1, q+1)$-equivariant quantization can be reduced to the following question :

Can we associate to every $T \in \mathcal{S}_{(k,s)}$ a *unique* symbol $\hat{T} = T_k + \cdots + T_0$ $(T_l \in \mathcal{S}^l_\delta(\mathbb{R}^m), \forall l \in \{0, \cdots, k\})$ such that

$$\begin{cases} T_k & = & T \\ \mathcal{C}(\hat{T}) & = & \alpha_{k,s}\hat{T} \end{cases} ? \tag{5}$$

In view of (4), the last equation in (5) can be rewritten as

$$(C - \alpha_{k,s}\mathrm{Id})T_l = N(T_{l+1}) \quad \forall l \in \{0, \cdots, k-1\} \tag{6}$$

In order to analyse this latter equation, the authors of [8] introduced the tree-like subspace associated to $\mathcal{S}_{(k,s)}$, namely

$$\widetilde{\mathcal{S}_{(k,s)}} = \bigoplus_{0 \leq s-t \leq k-l} \mathcal{S}_{(l,t)}.$$

They indeed showed that N maps $\mathcal{S}_{(l,t)}$ into $\mathcal{S}_{(l-1,t)} \oplus \mathcal{S}_{(l-1,t-1)}$. Hence, equation (6) admits a unique solution inside $\widetilde{\mathcal{S}_{(k,s)}}$ if $\alpha_{k,s}$ does not belong to the spectrum of the restriction of C to $\widetilde{\mathcal{S}_{(k,s)}}$. This leads to the definition of critical values (see [8], formulas 3.7, 3.8 and 3.10 for an explicit description of the set Σ_0 of critical values). Now the association $Q : T \mapsto \hat{T}$ defines an equivariant quantization because it is a bijection and fulfills

$$Q \circ L_{X^h} = \mathcal{L}_{X^h} \circ Q \quad \forall h \in \mathfrak{g}.$$

Indeed, for all $T \in \mathcal{S}_{(k,s)}$, $Q(L_{X^h}T)$ and $\mathcal{L}_{X^h}(Q(T))$ share the following properties

- They are eigenvectors of $\mathcal{C}$ of eigenvalue $\alpha_{k,s}$ because on the one hand $\mathcal{C}$ commutes with $\mathcal{L}_{X^h}$ for all h and on the other hand $L_{X^h}T$ belongs to $\mathcal{S}_{(k,s)}$
- their term of degree k is exactly $L_{X^h}T$,
- they both belong to $\widetilde{\mathcal{S}_{(k,s)}}$.

4. Natural and conformally equivariant quantizations up to degree 3

We will show in this section that the construction of the projectively equivariant and natural quantization of [17] adapts in the conformal case. We will recall the sequence of propositions that leads to the existence result in [17] and omit the proofs when the modifications are obvious. We will focus our attention to the key result that does not work for symbols of degree higher than 3.

We begin with the definition of the curved affine quantization map Q_ω. It is based on an iterated an symmetrized version of the invariant differentiation

Definition 1. If $f \in C^\infty(P, V)$ then $(\nabla^\omega)^k f \in C^\infty(P, S^k \mathbb{R}^{m*} \otimes V)$ is defined by

$$(\nabla^\omega)^k f(u)(X_1, \ldots, X_k) = \frac{1}{k!} \sum_\nu L_{\omega^{-1}(X_{\nu_1})} \circ \ldots \circ L_{\omega^{-1}(X_{\nu_k})} f(u)$$

for $X_1, \ldots, X_k \in \mathbb{R}^m$.

Definition 2. For every symbol $T = t \otimes h_1 \vee \cdots \vee h_k$, ($t \in C^\infty(P, \Delta^\delta(\mathbb{R}^m))$ and $h_1, \cdots, h_k \in \mathbb{R}^m \cong \mathfrak{g}_{-1}$) we set

$$Q_\omega(T)f = \langle T, (\nabla^\omega)^k f \rangle = \frac{1}{k!} \sum_\nu t \circ L_{\omega^{-1}(h_{\nu_1})} \circ \cdots \circ L_{\omega^{-1}(h_{\nu_k})} f, \quad (7)$$

where ν runs over all permutations of the indices $\{1, \cdots, k\}$,t is considered as a multiplication operator and f lies in $C^\infty(P, \Delta^\lambda(\mathbb{R}^m))$.

Remark 1. If $T \in C^\infty(P, S^k_\delta)$ is $H-$equivariant, the differential operator $Q_\omega(T)$ does not transform $H-$equivariant functions into $H-$equivariant functions; indeed, the function $Q_\omega(T)f$ in (7) is only G_0-equivariant. The idea is then to modify the symbol T by lower degree correcting terms in order to solve this problem.

The following proposition shows how to measure the default of equivariance of a function defined on P.

Proposition 8. *If (V, ρ) is a representation of G_0 and becomes a representation of H as stated in section 2.5, then a function $v \in C^\infty(P, V)$ is $H-$equivariant iff*

$$\begin{cases} v \text{ is } G_0-\text{equivariant}; \\ \text{one has } L_{h^*} v = 0 \text{ for every } h \text{ in } \mathfrak{g}_1 \end{cases}$$

Since basically, our tools preserve the G_0-equivariance, we are mostly interested in the $\mathfrak{g}_1$-equivariance. The following result is the keystone of our method. It works in the projective case for symbols of arbitrary degree but does not hold in the conformal case for symbols of degree higher than 3.

Proposition 9. *The relation*

$$L_{h^*} Q_\omega(T)(f) - Q_\omega(T)(L_{h^*} f) = Q_\omega((L_{h^*} + \gamma(h))T)(f) \quad (8)$$

holds for all $f \in C^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$, $h \in \mathfrak{g}_1$, and $T \in C^\infty(P, S^k_\delta)$, for $k \leq 3$.

Proof. We may check this relation on symbols of the form $T = tX^k$ where $t \in C^\infty(P, \Delta^\delta(\mathbb{R}^m))$ and X is in $\mathfrak{g}_{-1}$.

Then the left-hand side is equal to

$$L_{h^*}(tL_{\omega^{-1}(X)} \ldots L_{\omega^{-1}(X)} f) - tL_{\omega^{-1}(X)} \ldots L_{\omega^{-1}(X)} L_{h^*} f \quad (9)$$

while Proposition 7 shows that the right-hand side is equal to

$$(L_{h^*} t)L_{\omega^{-1}(X)} \ldots L_{\omega^{-1}(X)} f + k\lambda tr([h, X])Q_\omega(tX^{k-1})f$$

$$+ Q_\omega(t\frac{k(k-1)}{2}[X, [X, h]] \vee X^{k-2})f \quad (10)$$

Expression (9) is equal to

$$(L_{h^*}t)L_{\omega^{-1}(X)}\ldots L_{\omega^{-1}(X)}f + tL_{h^*}L_{\omega^{-1}(X)}\ldots L_{\omega^{-1}(X)}f$$
$$- tL_{\omega^{-1}(X)}\ldots L_{\omega^{-1}(X)}L_{h^*}f. \quad (11)$$

The first term in this latter formula is in expression (10), and using the vanishing of the curvature on vertical vector fields, we see that the other ones are equal to

$$\sum_{j=1}^{k} tL_{\omega^{-1}(X)}\ldots \overset{(j)}{L_{[h,X]^*}}\ldots L_{\omega^{-1}(X)}f$$
$$= \sum_{j=1}^{k-1}\sum_{i>j} tL_{\omega^{-1}(X)}\overset{(j)}{\ldots}L_{\omega^{-1}([[h,X],X])}\overset{(i)}{\ldots} L_{\omega^{-1}(X)}f$$
$$+\lambda t\sum_{i=1}^{k} tr([h,X])L_{\omega^{-1}(X)}\overset{(i)}{\ldots}L_{\omega^{-1}(X)}f.$$

The first part in the latter expression is equal to

$$\frac{k}{2}\sum_{j=1}^{k-1} tL_{\omega^{-1}(X)}\ldots \overset{(j)}{L_{\omega^{-1}([[h,X],X])}}\ldots L_{\omega^{-1}(X)}f$$

$$+\sum_{j=1}^{k-1}\frac{2j-k}{2}tL_{\omega^{-1}(X)}\ldots \overset{(j)}{L_{\omega^{-1}([[h,X],X])}}\ldots L_{\omega^{-1}(X)}f. \quad (12)$$

The first sum in this expression is equal to

$$\frac{k(k-1)}{2}Q_{\omega}(t[[h,X],X]\vee X^{k-2}).$$

Hence, in order to obtain the desired result, we just have to show that the expression (12) vanishes.

For k less or equal to 2, the result is then obvious.

For $k=3$, the term (12) is equal to

$$\frac{|X|^2}{2}t(L_{\omega^{-1}(X)}L_{\omega^{-1}(h^\sharp)}f - L_{\omega^{-1}(h^\sharp)}L_{\omega^{-1}(X)}f),$$

i.e. to

$$\frac{t|X|^2}{2}L_{\omega^{-1}(\kappa(X,h^\sharp))}f.$$

Using the fact that κ has its non trivial values in $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ and the G_1-equivariance of f, this term is equal to

$$\frac{t|X|^2}{2}L_{\omega^{-1}(\kappa_0(X,h^\sharp))}f,$$

i.e., using the G_0-equivariance of f, to

$$-\frac{t|X|^2}{2}\rho_*(\kappa_0(X,h^\sharp))f.$$

This term vanishes because of the normality of ω (the traces of κ_0 vanish).

$\square$

4.1. **Casimir operators in the curved situation.** We first define the analog of N by setting

$$N^\omega : C^\infty(P, S_\delta^k) \to C^\infty(P, S_\delta^{k-1}) : T \mapsto -2\sum_i \gamma(\epsilon^i) L_{\omega^{-1}(e_i)} T.$$

Then we can define the operators C^ω and $\mathcal{C}^\omega$ by their restrictions to the spaces $C^\infty(P, S_{(k,s)})$: for all $T \in C^\infty(P, S_{(k,s)})$, we set

$$\begin{cases} C^\omega(T) & = & \alpha_{k,s} T \\ \mathcal{C}^\omega(T) & = & C^\omega(T) + N^\omega(T), \end{cases}$$

where $\alpha_{k,s}$ is the eigenvalue of C on $\mathcal{S}_{(k,s)}$.

The operator $\mathcal{C}^\omega$ has the following property

Proposition 10. *For every $h \in \mathfrak{g}_1$, one has*

$$[L_{h^*} + \gamma(h), \mathcal{C}^\omega] = 0$$

on $C^\infty(P, S_\delta^k)_{G_0}$.

For the operator N^ω, we have the following result :

Proposition 11. *The operator N^ω preserves the G_0-equivariance of functions.*

5. Construction of the quantization

First remark that the construction of section 3.1.6 is still valid in the curved case.

Theorem 12. *If δ does not belong to the set Σ_0 of critical values , for every T in $C^\infty(P, S_{(k,s)})$, there exists a unique function $\hat{T}$ in $C^\infty(P, \widetilde{S_{(k,s)}})$ such that*

$$\begin{cases} \hat{T} & = & T_k + \cdots + T_0, \quad T_k = T \\ \mathcal{C}^\omega(\hat{T}) & = & \alpha_{k,s}\hat{T}. \end{cases} \tag{13}$$

Moreover, if T is G_0-invariant, then $\hat{T}$ is G_0-invariant.

This result allows to define the main ingredient in order to define the quantization.

Definition 3. Suppose that δ is not critical. Then the map

$$Q : C^\infty(P, S_\delta) \to C^\infty(P, S_\delta)$$

is the linear extension of the association $T \mapsto \hat{T}$.

The map Q has the following property :

Proposition 13. *There holds*

$$(L_{h^*} + \gamma(h))Q(T) = Q(L_{h^*}T), \tag{14}$$

for every $h \in \mathfrak{g}_1$ and every $T \in C^\infty(P, S_\delta)_{G_0}$.

Finally, we obtain

Theorem 14. *If δ does not belong to the set Σ_0 of critical values ([8, Formula 3.10]), then the formula*

$$Q_M : (g, T) \mapsto Q_M(g, T)(f) = (p^*)^{-1}[Q_\omega(Q(p^*T))(p^*f)], \qquad (15)$$

(where Q_ω is given by (7)) defines a natural and conformally equivariant quantization on symbols of degree at most three.

6. Quantization of symbols of degree four

In this section, we will first show how proposition 9 has to be adapted for symbols of degree four in the presence of a nontrivial curvature. We will then introduce the corresponding correcting terms to the map Q of definition 3 in order to prove the existence of the natural and conformally equivariant quantization for symbols of degree four.

6.1. **Modification of the operator γ.** The key result in the construction of the quantization is proposition 9. For symbols of degree four we introduce the following operators :

Definition 4. We define the operators

$$\gamma_3 : \mathfrak{g}_1 \otimes C^\infty(P, S_\delta^4) \to C^\infty(P, S_\delta^1) \quad \text{and} \quad \gamma_4 : \mathfrak{g}_1 \otimes C^\infty(P, S_\delta^4) \to C^\infty(P, S_\delta^0)$$

on every $T \in C^\infty(P, S_\delta^4)$ by

$$\begin{cases} \gamma_3(h)(T) & = & \langle \kappa_0, h^\sharp \otimes i(g_0)T \rangle \\ \gamma_4(h)(T) & = & -\lambda m \langle \kappa_1, h^\sharp \otimes i(g_0)T \rangle \end{cases}$$

Here we view respectively κ_0 and κ_1 as elements of $C^\infty(P, \otimes^3 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1})$ and of $C^\infty(P, \otimes^3 \mathfrak{g}_{-1}^*)$. On the spaces of symbols of degree less or equal to 3, we also set $\gamma_3(h) = \gamma_4(h) = 0$.

Finally we define the modified operator γ' on symbols of degree less or equal to four :

Definition 5. The map γ' is defined on $\oplus_{k=0}^4 \mathfrak{g}_1 \otimes C^\infty(P, S_\delta^k)$ by

$$\gamma'(h) = \gamma(h) + \gamma_3(h) + \gamma_4(h),$$

for every h in $\mathfrak{g}_1$.

We then have the following result, which is the adaptation of Proposition 9.

Proposition 15. *There holds*

$$L_{h^*}Q_\omega(T)(f) - Q_\omega(T)(L_{h^*}f) = Q_\omega((L_{h^*} + \gamma'(h))T)(f) \qquad (16)$$

for all $f \in C^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$, $h \in \mathfrak{g}_1$, and $T \in C^\infty(P, S_\delta^k)$, for $k \leq 4$.

Proof. The first part of the proof of proposition 9 is still valid for symbols of degree 4. It now remains to compute the expression (12). It is equal to

$$|X|^2 \left[L_{\omega^{-1}(X)} L_{\omega^{-1}(X)} L_{\omega^{-1}(h^\sharp)} f - L_{\omega^{-1}(h^\sharp)} L_{\omega^{-1}(X)} L_{\omega^{-1}(X)} f \right].$$

The next step is to use Lemma 4 to compute this expression. We get that expression (12) is equal to

$$|X|^2 \left[L_{\omega^{-1}(X)} L_{\omega^{-1}(\kappa(h^\sharp, X))} f + L_{\omega^{-1}(\kappa(h^\sharp, X))} L_{\omega^{-1}(X)} f \right].$$

The first term of this expression vanishes due to the H-invariance of f (recall that κ has only non trivial values in $\mathfrak{h}$). The second term is easy to compute using Lemma 4 and the invariance of f. □

6.2. **Modification of the application Q.** In the construction of the quantization up to degree 3, the main interest of the application Q (see Definition 3) is that the operator $Q_\omega(Q(T))$, for all H-equivariant functions T, transforms H-equivariant functions into H-equivariant functions. This ensures that formula (15) makes sense. The following proposition shows how this property fails for symbols of degree 4.

Proposition 16. *There holds*

$$L_{h^*}[(Q_\omega(Q(T)))(f)] = Q_\omega(\gamma_3(h)T + \gamma_4(h)T)(f)$$

for all $h \in \mathfrak{g}_1$, $T \in C^\infty(P, S^k_\delta)_H$ ($k \leq 4$) and $f \in C^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$.

Proof. By Proposition 15, we have

$$L_{h^*}[(Q_\omega(Q(T)))(f)] = Q_\omega(Q(T))(L_{h^*}f) + Q_\omega((L_{h^*} + \gamma'(h))Q(T))(f). \quad (17)$$

The first term is vanishing. In view of Proposition 13, the second one is equal to

$$Q_\omega(Q(L_{h^*}T)) + Q_\omega((\gamma_3(h) + \gamma_4(h))Q(T)).$$

The result follows from the H-invariance of T and from the vanishing of $\gamma_3(h)$ and $\gamma_4(h)$ on symbols of degree less or equal to 3. □

We will now introduce some modifications to the map Q in order to annihilate the right hand side of equation (17). In the next definition, we still denote by (e_j) a basis of $\mathfrak{g}_{-1} \cong \mathbb{R}^m$ and we denote by (η^j) the dual basis in the usual sense.

Recall that the divergence associated to a Cartan connection is then defined by

$$div^\omega : C^\infty(P, S^k_\delta) \to C^\infty(P, S^{k-1}_\delta) : T \mapsto \sum_j L_{\omega^{-1}(e_j)} i(\eta^j) T.$$

Definition 6. When $\delta \notin \{\frac{m+1}{m}, \frac{m+2}{m}\}$, we define the maps Q_3 and Q_4 on $C^\infty(P, S^4_\delta)$ by

$$Q_3(T) = - \left[\langle \kappa_1, \eta^{j^\sharp} \otimes i(g_0)T \rangle e_j + \frac{2}{m+2-m\delta} div^\omega(\langle \kappa_0, \eta^{j^\sharp} \otimes i(g_0)T \rangle \vee e_j) \right]$$

$$Q_4(T) = \frac{-m\lambda}{(m+1-m\delta)(m+2-m\delta)} div^{\omega^2}(\langle \kappa_0, \eta^{j^\sharp} \otimes i(g_0)T \rangle \vee e_j).$$

We also set $Q' = Q + Q_3 + Q_4$.

Remark : It is easy to check that the values $\frac{m+1}{m}$ and $\frac{m+2}{m}$ belong to the set Σ_0 of critical values, so that the formulas for Q_3 and Q_4 make sense if δ is not critical.

In order to obtain the analog of Proposition 13, we need the following technical lemma.

Lemma 17. *For every $T \in C^\infty(P, S_\delta^k)_{G_0}$ and $h \in \mathfrak{g}_1$, one has*

$$L_{h^*} div^\omega T - div^\omega L_{h^*} T = (m + 2k - 2 - m\delta)i(h)T - (k-1)i(\eta^j)i(e_j^\flat)T \vee h^\sharp.$$

For $T \in C^\infty(P, S_\delta^2)_{G_0}$ and $h \in \mathfrak{g}_1$, one has

$$L_{h^*} div^{\omega^2} T - div^{\omega^2} L_{h^*} T = 2(m+1-m\delta)i(h)div^\omega T - L_{\omega^{-1}(h^\sharp)}i(\eta^j)i(e_j^\flat)T.$$

Proof. We proceed as in [18, Lemma 8] : we have for $\xi^1, \cdots, \xi^{k-1} \in \mathbb{R}^{m^*}$

$$\begin{aligned}
&L_{h^*} div^\omega T(\xi^1, \cdots, \xi^{k-1}) - div^\omega L_{h^*} T(\xi^1, \cdots, \xi^{k-1}) = \\
&(\rho(h \otimes e_j + h_j Id - e_j^\flat \otimes h^\sharp)T)(\eta^j, \xi^1, \cdots, \xi^{k-1}) = \\
&[(m + 2k - 1 - (m+1)\delta) + (\delta - 1)]\, i(h)T(\xi^1, \cdots, \xi^{k-1}) \\
&- \sum_l \langle \xi^l, h^\sharp \rangle T(\eta^j, e_j^\flat, \xi^1 \overset{(l)}{\cdots}, \eta^{k-1}).
\end{aligned}$$

This yields the first part of the result. The second part follows by induction. $\qquad\square$

Proposition 18. *There holds*

$$(L_{h^*} + \gamma(h))(Q_3 + Q_4)(T) = -(\gamma_3(h) + \gamma_4(h))T$$

for every $h \in \mathfrak{g}_1$ and $T \in C^\infty(P, S_\delta^4)_H$.

Proof. Let us compute the left hand side. It is clear that $\gamma(h)Q_4(T) = 0$, since $Q_4(T)$ has degree zero. Now, since $\gamma(h)$ reduces to $-\lambda m\, i(h)$ on symbols of degree 1, we have

$$\gamma(h)Q_3(T) = \lambda m \langle \kappa_1, h^\sharp \otimes i(g_0)T \rangle + \frac{2\lambda m}{m+2-m\delta} i(h)div^\omega(\langle \kappa_0, \eta^{j^\sharp} \otimes i(g_0)T \rangle \vee e_j).$$

Moreover, we use Lemma 17 to obtain

$$\begin{aligned}
L_{h^*}Q_3(T) = &-\langle L_{h^*}\kappa_1, \eta^{j^\sharp} \otimes i(g_0)T \rangle e_j - 2i(h)(\langle \kappa_0, \eta^{j^\sharp} \otimes i(g_0)T \rangle \vee e_j) \\
&+ \frac{2}{m+2-m\delta} i(\eta^r)i(e_r^\flat)(\langle \kappa_0, \eta^{j^\sharp} \otimes i(g_0)T \rangle \vee e_j) \vee h^\sharp.
\end{aligned}$$

By Lemma 5, the first term is equal to

$$\langle h, \langle \kappa_0, \eta^{j^\sharp} \otimes i(g_0)T \rangle \rangle e_j;$$

and the last term is equal to (up to a constant factor)

$$\langle \eta^j, \langle \kappa_0, e_j \otimes i(g_0)T \rangle \rangle \vee h^\sharp$$

and vanishes because of the normality of the Cartan connection.

In the same way, still using Lemma 17, we have

$$L_{h^*}Q_4(T) = \frac{-2\lambda m}{m + 2 - m\delta}i(h)div^\omega(\langle\kappa_0, \eta^{j^\sharp} \otimes i(g_0)T\rangle \vee e_j),$$

and the result easily follows.   □

We then have an important corollary.

Corollary 19. *For every $f \in C^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$ and $T \in C^\infty(P, S^4_\delta)_H$, the function $Q_\omega(Q'(T))(f)$ is H-invariant.*

Proof. First notice that the map Q' transforms G_0-invariant functions into G_0-invariant functions. Indeed, this is certainly true for Q as quoted in [17], and holds true for Q_3 and Q_4, because of the properties of invariance of κ_0 and κ_1 (see [6]). Therefore, the operator $Q_\omega(Q'(T))$ also transforms G_0-equivariant functions into G_0 equivariant functions. We just need to show that $Q_\omega(Q'(T))(f)$ is $\mathfrak{g}_1$-invariant. This follows immediately from Propositions 16 and 18.   □

Finally, we have the main result :

Theorem 20. *If δ is not critical, then the formula*

$$Q_M : (g, T) \mapsto Q_M(g, T)(f) = (p^*)^{-1}[Q_\omega(Q'(p^*T))(p^*f)],$$

$$\forall T \in \oplus^4_{i=0}\mathcal{S}^i_\delta(M), \forall f \in \mathcal{F}_\lambda(M) \quad (18)$$

defines a natural and conformally equivariant quantization on symbols of degree four.

Proof. The proof goes as in [17]. The main point is that the formula makes sense. But this is a consequence of corollary 19.   □

Remark : It was proved in [19] that the quantization of symbols of degree four is not unique in the projective setting. The arguments of [19] adapt to the conformal situation, so that the quantization defined above is not unique.

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