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Existence of natural and conformally invariant quantizations of arbitrary symbols

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A quantization can be seen as a way to construct a differential operator with prescribed principal symbol. The map from the space of symbols to the space of differential operators is moreover required to be a linear bijection.

In general, there is no natural quantization procedure, that is, spaces of symbols and of differential operators are not equivalent, if the action of local diffeomorphisms is taken into account. However, considering manifolds endowed with additional structures, one can seek for quantizations that depend on this additional structure and that are natural if the dependence with respect to the structure is taken into account.

The existence of such a quantization was proved recently in a series of papers in the context of projective geometry.

Here, we show that the construction of the quantization based on Cartan connections can be adapted from projective to pseudo-conformal geometry to yield the natural and conformally invariant quantization for arbitrary symbols, outside some critical situations.

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1. Introduction

A quantization can be defined as a linear bijection from a space of classical observables, also called symbols (particular functions on the cotangent bundle of a manifold M) to a space of quantum observables (differential operators acting on half densities on M).

It is known that there is no natural quantization procedure : the spaces of classical and quantum observables are not equivalent when the action of diffeomorphisms of M is taken into account. Two possible ways to weaken the naturality condition are considered in the literature. The first one leads to the concept of *G-equivariant quantizations* on manifolds endowed with the (local) action of a Lie group G. The second one is to consider the more general notion of *natural and invariant quantizations.*

The concept of G-equivariant quantization was defined by P. Lecomte and V. Ovsienko in [1] in the following way : if a Lie group G acts on a manifold M as a local transformation group, the action can be lifted to tensor fields, to differential operators and symbols. A G-equivariant quantization is then a bijection from the space of symbols to the space of differential operators that exchanges the actions of G on symbols and differential operators, and satisfies a natural normalization condition (see Equation (2.2)). In [1], these authors considered the space of symbols $\mathcal{S}(\mathbb{R}^m)$ made of functions on the cotangent bundle of $\mathbb{R}^m$ that are polynomial along the fibers, spaces of differential operators $\mathcal{D}_\lambda(\mathbb{R}^m)$ acting on λ-densities, and the projective group $G = PGL(m+1,\mathbb{R})$ acting on $\mathbb{R}^m$ by linear fractional transformations. They showed that there exists a unique *projectively equivariant quantization* in this context.

In [2], C. Duval and V. Ovsienko studied the spaces $\mathcal{D}_{\lambda\mu}(\mathbb{R}^m)$ of differential operators transforming λ-densities into μ-densities. They showed the existence and uniqueness of a projectively equivariant quantization, provided the shift value $\delta = \mu - \lambda$ does not belong to a set of critical values . The authors of [3] considered the group $SO(p+1, q+1)$ acting on the space $\mathbb{R}^{p+q}$. They also showed the existence and uniqueness of a *conformally equivariant quantization* provided the shift value is not critical. They extended directly their results to manifolds endowed with a flat pseudo-conformal structure of signature (p, q). Similar results were obtained for other equivariance conditions in [4].

Independently of physical interpretation, the equivariant quantization, and its inverse, the equivariant symbol map prove to be useful tools in the analysis of spaces of differential operators because they allow to study a filtered space by means of the associated graded space. In [5], a first example of projectively equivariant quantizations for differential operators acting on tensor fields was considered.

The concept of natural and invariant quantization appeared in the conformal case, in [6] and [7], where it was shown that the conformally equivariant quantization procedure for symbols of degree two and three can be expressed using the Levi-Civita connection associated with a pseudo-Riemannian metric in such a way that it only depends on the *conformal class* of the metric.

In the projective case, in [8, 9], it was shown that the formula for the projectively equivariant quantization for differential operators of order two and three could be expressed using a torsion-free linear connection, in such a way that it only depends on the *projective class* of the connection.

In both projective and conformal situations, the problem of natural and invariant quantization was described in [10] :

- In the projective situation, the problem is to associate with every torsion-free linear connection ∇ on a manifold M a quantization procedure $Q_M(\nabla) : \mathcal{S}(M) \to \mathcal{D}_{\lambda,\mu}(M)$, that is natural (when the action of diffeomorphisms on the connection is also taken into account) and is not changed when ∇ is replaced by a projectively equivalent torsion-free connection.
- In the conformal situation, the problem is to associate with every pseudo-Riemannian metric g on a manifold M a quantization procedure $Q_M(g) : \mathcal{S}(M) \to \mathcal{D}_{\lambda,\mu}(M)$, that is natural (see relation (2.3)) and does not change if g is replaced by a conformally equivalent metric.

In the projective setting, the existence of a natural and projectively invariant quantization was first proved by M. Bordemann in [11] (still outside critical situations), using the notion of Thomas-Whitehead connection associated with a projective class of connections. This result was generalized in a series of recent papers. First, S. Hansoul adapted the construction in [12] in order to extend the results of [5]. Then, in [13], we gave an alternative proof of the results of M. Bordemann, using the theory of projective Cartan connections.

In [14, 15], these results were generalized to deal with multilinear operators or linear differential operators acting on arbitrary tensors (i.e. sections of bundles associated with the linear frame bundle, see Section 2.1). At the same time, in [16], we gave an alternative proof of the existence of natural and projectively invariant quantizations for arbitrary tensors using Cartan projective connections.

In the conformal situation, we proved in [17] the existence –outside critical situations– of the conformally invariant quantization for symbols of degree at most four and for differential operators acting between spaces of densities.

In this paper, we give a general result of existence of a conformally invariant natural quantization for differential operators acting on tensors fields (as in the projective situation, sections of bundles associated with the linear frame bundle) and for any order of differentiation, provided the situation is not critical, as usual. We actually give new and more general proofs of some of the results of [16] and we adapt the tools of that paper to the conformal case. The adaptation is not direct because of the effect of the curvature of the Cartan connections appearing in the construction.

2. Problem setting

In this section, we will describe the definitions of the spaces of differential operators acting on tensor fields and of their corresponding spaces of symbols. Then we will set the problem of existence of natural and conformally invariant quantizations. Throughout this work, we let M be a smooth manifold of dimension $m \geq 3$.

2.1. *Tensor fields*

The arguments of the differential operators that we will consider are classical tensor fields. Recall that one may see them as sections of vector bundles associated with the linear frame bundle P^1M. We consider irreducible representations of the group $GL(m, \mathbb{R})$ defined as follows : let (V, α_D) be the representation of $GL(m, \mathbb{R})$ corresponding to a Young diagram Y_D of depth $n < m$. Fix $\lambda \in \mathbb{R}$ and $z \in \mathbb{Z}$ and set

$$\alpha(A)u = |det(A)|^{\lambda}(det(A))^{z}\alpha_D(A)u, \tag{2.1}$$

for all $A \in GL(m, \mathbb{R})$, and $u \in V$.
If (V, α) is such a representation, we denote by $V(M)$ the vector bundle

$$P^1M \times_\alpha V.$$

We denote by $\mathcal{V}(M)$ the space of smooth sections of $V(M)$. This space can be identified with the space $C^\infty(P^1M, V)_{GL(m,\mathbb{R})}$ of $GL(m, \mathbb{R})$-invariant smooth functions, i.e. functions f such that

$$f(uA) = \alpha(A^{-1})f(u) \quad \forall u \in P^1M, \ \forall A \in GL(m, \mathbb{R}).$$

Finally, since $V(M) \to M$ is associated with P^1M, the group $\mathrm{Diff}(M)$ of diffeomorphisms of M acts in a well-known manner on the space $\mathcal{V}(M)$.

2.2. *Differential operators and symbols*

If (V_1, ρ_1) and (V_2, ρ_2) are representations of $GL(m, \mathbb{R})$, we denote by $\mathcal{D}(\mathcal{V}_1(M), \mathcal{V}_2(M))$ (or simply by $\mathcal{D}(M)$ if there is no risk of confusion) the space of linear differential operators from $\mathcal{V}_1(M)$ to $\mathcal{V}_2(M)$. The actions of $\mathrm{Vect}(M)$ and $\mathrm{Diff}(M)$ are induced by their actions on $\mathcal{V}_1(M)$ and $\mathcal{V}_2(M)$. For instance, one has

$$(\phi \cdot D)(f) = \phi \cdot (D(\phi^{-1} \cdot f)), \quad \forall f \in \mathcal{V}_1(M), D \in \mathcal{D}(M), \text{and}\, \phi \in \mathrm{Diff}(M).$$

The space $\mathcal{D}(M)$ is filtered by the order of differential operators. We denote by $\mathcal{D}^k(M)$ the space of differential operators of order at most k. The space of *symbols*, which we will denote by $\mathcal{S}_{V_1,V_2}(M)$ or simply by $\mathcal{S}(M)$, is then the graded space associated with $\mathcal{D}(M)$.

We denote by $S^l_{V_1,V_2}$ the vector space $S^l\mathbb{R}^m \otimes gl(V_1, V_2)$. We denote by ρ the natural representation of $GL(m, \mathbb{R})$ on this space (the representation of $GL(m, \mathbb{R})$ on symmetric tensors is the natural one). We then denote by $S^l_{V_1,V_2}(M) \to M$ the vector bundle

$$P^1M \times_\rho S^l_{V_1,V_2} \to M,$$

and by $\mathcal{S}^l_{V_1,V_2}(M)$ the space of smooth sections of $S^l_{V_1,V_2}(M) \to M$, that is, the space $C^\infty(P^1M, S^l_{V_1,V_2})_{GL(m,\mathbb{R})}$.

Then the *principal symbol operator* $\sigma_l : \mathcal{D}^l(M) \to \mathcal{S}^l_{V_1,V_2}(M)$ commutes with the action of diffeomorphisms and is a bijection from the quotient space $\mathcal{D}^l(M)/\mathcal{D}^{l-1}(M)$ to $\mathcal{S}^l_{V_1,V_2}(M)$. Hence the space of symbols is nothing but

$$\mathcal{S}(M) = \bigoplus_{l=0}^{\infty} \mathcal{S}^l_{V_1,V_2}(M),$$

endowed with the classical action of $\mathrm{Diff}(M)$.

2.3. *Natural and invariant quantizations*

A *quantization on M* is a linear bijection Q_M from the space of symbols $\mathcal{S}(M)$ to the space of differential operators $\mathcal{D}(M)$ such that

$$\sigma_k(Q_M(S)) = S,\ \forall S \in \mathcal{S}^k_{V_1,V_2}(M),\ \forall k \in \mathbb{N}. \tag{2.2}$$

In the conformal sense, a *natural quantization* is a collection of quantizations Q_M depending on a pseudo-Riemannian metric such that

- For all pseudo-Riemannian metric g on M, $Q_M(g)$ is a quantization,
- If ϕ is a local diffeomorphism from M to N, then one has

$$Q_M(\phi^* g)(\phi^* S) = \phi^*(Q_N(g)(S)), \tag{2.3}$$

  for all pseudo-Riemannian metrics g on N, and all $S \in \mathcal{S}(N)$.

Recall now that two pseudo-Riemannian metrics g and g' on a manifold M are conformally equivalent if and only if there exists a positive function f such that $g' = fg$.

A quantization Q_M is then *conformally invariant* if one has $Q_M(g) = Q_M(g')$ whenever g and g' are conformally equivalent.

Remark : The conformal invariance defined above is a special case of invariance properties that can be considered in the setting of parabolic geometries. It was recently pointed to us by J. Šilhan that such invariance properties where considered in the recent thesis of J. Kroeske [18]. However, our point of view and methods turn out to be completely different.

3. Conformal group and conformal algebra

Let us now recall the definition of the algebraic objects that we will use throughout this work. The gradings of these objects are of special importance.

3.1. *The conformal group*

Given p and q such that $p + q = m$, we consider the bilinear symmetric form of signature $(p+1, q+1)$ on $\mathbb{R}^{m+2}$ defined by

$$B : \mathbb{R}^{m+2} \times \mathbb{R}^{m+2} \to \mathbb{R} : (x, y) \mapsto {}^t y S x,$$

where S is the matrix of order $m + 2$ given by

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & J & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

represents a nondegenerate symmetric bilinear form g_0 on $\mathbb{R}^m$, namely

$$g_0 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} : (x, y) \mapsto {}^t y J x.$$

As we continue, we will use the classical isomorphism between $\mathbb{R}^m$ and $\mathbb{R}^{m*}$ defined by the symmetric bilinear form represented by J :

$$\flat : \mathbb{R}^m \to \mathbb{R}^{m*} : x \mapsto x^\flat : x^\flat(y) = g_0(x,y)$$

and $\sharp = \flat^{-1}$, and we also denote by $|x|^2$ the number $g_0(x,x)$ (although it might be negative).

The Möbius space is the projection of the light cone defined by the metric B on the projective space $\mathbb{R}P^{m+1}$.

We consider the group G made of linear transformations that leave B invariant, modulo its center, that is,

$$G = \{X \in GL(m+2,\mathbb{R}) : {}^tXSX = S\}/\{\pm I_{m+2}\}.$$

It acts transitively on the Möbius space $S^p \times S^q$.

The group H is the isotropy subgroup of G at the point $[e_{m+2}]$ of the Möbius space :

$$H = \{ \begin{pmatrix} a^{-1} & 0 & 0 \\ a^{-1}A\xi^\sharp & A & 0 \\ \frac{1}{2a}|\xi|^2 & \xi & a \end{pmatrix} : A \in O(p,q), a \in \mathbb{R}_0, \xi \in \mathbb{R}^{m*}\}/\{\pm I_{m+2}\}.$$

As in the projective situation, H is a semi-direct product $G_0 \rtimes G_1$. Here G_0 is isomorphic to $CO(p,q)$ and G_1 is isomorphic to $\mathbb{R}^{m*}$. There is also a projection

$$\pi : H \mapsto CO(p,q) : \left[\begin{pmatrix} a^{-1} & 0 & 0 \\ a^{-1}A\xi^\sharp & A & 0 \\ \frac{1}{2a}|\xi|^2 & \xi & a \end{pmatrix}\right] \mapsto \frac{A}{a}. \tag{3.1}$$

3.2. *The conformal algebra*

The Lie algebra of G is $\mathfrak{g} = so(p+1,q+1)$. It decomposes as a direct sum of subalgebras :

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{3.2}$$

where $\mathfrak{g}_{-1} \cong \mathbb{R}^m$, $\mathfrak{g}_0 \cong co(p,q)$, and $\mathfrak{g}_1 \cong \mathbb{R}^{m*}$. The isomorphism is given explicitly by

$$\begin{pmatrix} -a & v^\flat & 0 \\ \xi^\sharp & A & v \\ 0 & \xi & a \end{pmatrix} \mapsto (v, A - aI_m, \xi).$$

This correspondence induces a structure of Lie algebra on $\mathbb{R}^m \oplus co(p,q) \oplus \mathbb{R}^{m*}$. It is easy to see that the adjoint actions G_0 and of $co(p,q)$ on $\mathfrak{g}_{-1} = \mathbb{R}^m$ and on $\mathfrak{g}_1 = \mathbb{R}^{m*}$ coincide with the natural actions of $CO(p,q)$ and of $co(p,q)$. The Lie algebras corresponding to G_0, G_1 and H are respectively $\mathfrak{g}_0$, $\mathfrak{g}_1$, and $\mathfrak{g}_0 \oplus \mathfrak{g}_1$.

Actually, the simple Lie algebras carrying a grading (3.2) are known as simple $|1|$-graded algebras or Irreducible Filtered Lie algebras of Finite Type (IFFT algebras for short). Hence the algebra $so(p+1,q+1)$ is a particular case of such an algebra. The classification of these algebras was obtain in [19]. Recall also that for every simple $|1|$-graded algebra, the subalgebra $\mathfrak{g}_0$ is reductive and decomposes as

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathbb{R}\mathcal{E} \tag{3.3}$$

where $\mathfrak{h}_0$ is semi-simple and where the *grading* or *Euler* element $\mathcal{E}$ is defined by $ad(\mathcal{E})|_{\mathfrak{g}_k} = kId$ $(k \in \{-1, 0, 1\})$ and is therefore in the center of $\mathfrak{g}_0$. It is also noteworthy that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are always dual of each other because of the non-degeneracy of the Killing form K of $\mathfrak{g}$.

Let us now close this section by two technical results about Killing-dual bases in such algebras. The first one is taken directly from [4].

Proposition 1. *For every IFFT-algebra $\mathfrak{g}$, there exists a basis $(e_i, \mathcal{E}, A_j, \varepsilon^i)$ of $\mathfrak{g}$ such that $(e_i : 1 \leq i \leq d)$ is a basis of $\mathfrak{g}_{-1}$, $(A_j : 1 \leq j \leq \dim \mathfrak{h}_0)$ is a basis of $\mathfrak{h}_0$ and $(\varepsilon^i : 1 \leq i \leq d)$ is a basis of $\mathfrak{g}_1$ and such that the Killing dual basis writes $(\varepsilon^i, \frac{1}{2d}\mathcal{E}, A_j^*, e_i)$, where A_j^* is in $\mathfrak{h}_0$ for all $j \leq \dim \mathfrak{h}_0$. Moreover we have*

$$\sum_{i=1}^{d} [\varepsilon^i, e_i] = \frac{1}{2}\mathcal{E}.$$

As we continue, we will need other relations concerning these bases.

Proposition 2. *Denote respectively by a_j and a_j^* the matrix representations of $ad(A_j)|_{\mathfrak{g}_{-1}}$ and $ad(A_j^*)|_{\mathfrak{g}_{-1}}$ in the basis $(e_i : i \leq d)$. Then we have*

$$\begin{aligned} [\varepsilon^r, e_i] &= \tfrac{1}{2d}\delta_i^r \mathcal{E} - \textstyle\sum_{j=1}^{\dim \mathfrak{h}_0} a_{ji}^r A_j^* \\ &= \tfrac{1}{2d}\delta_i^r \mathcal{E} - \textstyle\sum_{j=1}^{\dim \mathfrak{h}_0} a_{ji}^{*r} A_j \end{aligned}$$

for every $i, r \leq d$. Moreover we have

$$[A_j, \varepsilon^r] = -\sum_{k=1}^{d} a_{jk}^r \varepsilon^k \quad \text{and} \quad [A_j^*, \varepsilon^r] = -\sum_{k=1}^{d} a_{jk}^{*r} \varepsilon^k.$$

Proof. Let us prove the first relation. The other ones are obtained in a similar way. We know that $[\varepsilon^r, e_i]$ belongs to $\mathfrak{g}_0$ since the algebra $\mathfrak{g}$ is $|1|$-graded. Hence we have a decomposition

$$[\varepsilon^r, e_i] = b_i^r \mathcal{E} + \sum_{j=1}^{\dim \mathfrak{h}_0} b_{ji}^r A_j^*.$$

We compute the coefficients using the Killing dual basis given in Proposition 1 : we have

$$b_i^r = K([\varepsilon^r, e_i], \frac{1}{2d}\mathcal{E}) = -K(\varepsilon^r, \frac{1}{2d}[\mathcal{E}, e_i]) = \frac{1}{2d}K(\varepsilon^r, e_i),$$

by the invariance of the Killing form K and the definition of $\mathcal{E}$. In the same way we obtain

$$b_{ji}^r = K([\varepsilon^r, e_i], A_j) = -K(\varepsilon^r, [A_j, e_i]) = -K(\varepsilon^r, \sum_s a_{ji}^s e_s) = -a_{ji}^r.$$

□

4. Cartan fiber bundles and connections

4.1. *Cartan fiber bundles*

It is well-known that there is a bijective and natural correspondence between the conformal structures on M and the reductions of P^1M to the structure group $G_0 \cong CO(p,q)$. The representations (V,α) of $GL(m,\mathbb{R})$ defined so far can be restricted to the group $CO(p,q)$. Therefore, once a conformal structure is given, i.e. a reduction P_0 of P^1M to G_0, we can identify tensors fields of type V as G_0-invariant functions on P_0.

In [20], one shows that it is possible to associate with each G_0-structure P_0 a principal H-bundle P on M, this association being natural and obviously conformally invariant. Since H can be considered as a subgroup of G^2_m, this H-bundle can be considered as a reduction of P^2M. The relationship between conformal structures and reductions of P^2M to H is given by the following proposition.

Proposition 3. *There is a natural one-to-one correspondence between the conformal equivalence classes of pseudo-Riemannian metrics on M and the reductions of P^2M to H.*

Throughout this work, we will freely identify conformal structures and reductions of P^2M to H.

4.2. *Cartan connections*

Let L be a Lie group and L_0 a closed subgroup. Denote by $\mathfrak{l}$ and $\mathfrak{l}_0$ the corresponding Lie algebras. Let $N \to M$ be a principal L_0-bundle over M, such that $\dim M = \dim L/L_0$. A Cartan connection on N is an $\mathfrak{l}$-valued one-form ω on N such that

(1) If R_a denotes the right action of $a \in L_0$ on N, then $R_a^*\omega = Ad(a^{-1})\omega$,
(2) If k^* is the vertical vector field corresponding to to $k \in \mathfrak{l}_0$, then $\omega(k^*) = k$,
(3) $\forall u \in N$, $\omega_u : T_uN \mapsto \mathfrak{l}$ is a linear bijection.

When considering in this definition a principal H-bundle P, and taking as group L the group G and for L_0 the group H defined above, we obtain the definition of Cartan conformal connections.

If ω is a Cartan connection defined on an H-principal bundle P, then its curvature Ω is defined by

$$\Omega = d\omega + \frac{1}{2}[\omega,\omega]. \tag{4.1}$$

The notion of *Normal* Cartan connection is defined by natural conditions imposed on the components of the curvature.

Now, the following result ([20, p. 135]) gives the relationship between conformal structures and Cartan connections :

Proposition 4. *A unique normal Cartan conformal connection is associated with every conformal structure P. This association is natural.*

The connection associated with a conformal structure P is called the normal conformal connection of the conformal structure.

5. Lift of equivariant functions and the application Q_ω

In the previous section, we recalled how to associate an H-principal bundle P with a conformal structure P_0. We now recall how tensor fields, and in particular symbols, can be regarded as equivariant functions on P.

If (V,α) is a representation of G_0, then we may extend it to a representation (V,α') of H by

$$\alpha' = \alpha \circ \pi,$$

where π is defined in (3.1). Now, using the representation α', we can recall the relationship between equivariant functions on P_0 and equivariant functions on P (see [21]): if we denote by p the projection $P \to P_0$, we have

Proposition 5. *If (V,α) is a representation of G_0, then the map*

$$p^* : C^\infty(P_0, V) \mapsto C^\infty(P, V) : f \mapsto f \circ p$$

defines a bijection from $C^\infty(P_0,V)_{G_0}$ to $C^\infty(P,V)_H$.

One of the main ingredients in the construction of the invariant quantization is the application Q_ω. Its definition is based on the concept of invariant differentiation developed in [21, 22]. Let us recall the definition :

Definition 1. If $f \in C^\infty(P,V)$ then $(\nabla^\omega)^k f \in C^\infty(P, \otimes^k \mathbb{R}^{m*} \otimes V)$ is defined by

$$(\nabla^\omega)^k f(u)(X_1,\dots,X_k) = L_{\omega^{-1}(X_1)} \circ \dots \circ L_{\omega^{-1}(X_k)} f(u)$$

for $X_1,\dots,X_k \in \mathbb{R}^m$.

Now, since $\mathfrak{g}_{-1} \cong \mathbb{R}^m$ and $\mathfrak{g}_1 \cong \mathbb{R}^{m*}$ are natural representations of $G_0 \cong CO(p,q)$, they become representations of H and we can state an important property of the invariant differentiation, which is a direct consequence of the Ad-invariance of the Cartan connection.

Proposition 6. *If f belongs to $C^\infty(P,V)_{G_0}$ then $\nabla^\omega f \in C^\infty(P, \mathbb{R}^{m*} \otimes V)_{G_0}$.*

The main point that we will discuss in the next sections is that this result is not true in general for H-equivariant functions : for an H-equivariant function f, the function $\nabla^\omega f$ is in general not G_1-equivariant.

As we continue, starting with a representation (V,α) of G_0 as above, we will use the representation α'_* of the Lie algebra of H on V. If we recall that this algebra is isomorphic to $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ then we have

$$\alpha'_*(A,\xi) = \alpha_*(A), \quad \forall A \in \mathfrak{g}_0, \xi \in \mathfrak{g}_1. \tag{5.1}$$

In our computations, we will make use of the infinitesimal version of the equivariance relation : If $f \in C^\infty(P,V)_H$ then one has

$$L_{h^*} f(u) + \alpha'_*(h) f(u) = 0, \quad \forall h \in \mathfrak{g}_0 \oplus \mathfrak{g}_1, \forall u \in P. \tag{5.2}$$

Definition 2. The map Q_ω is defined by its restrictions to $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1,V_2))$, $(k \in \mathbb{N})$: we set

$$Q_\omega(T)(f) = \langle T, (\nabla^\omega)^k f \rangle, \tag{5.3}$$

for all $T \in C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2))$ and $f \in C^\infty(P, V_1)$.

Explicitly, when the symbol T writes $tA \otimes h_1 \otimes \cdots \otimes h_k$ for $t \in C^\infty(P)$, $A \in V_1^* \otimes V_2$ and $h_1, \cdots, h_k \in \mathbb{R}^m \cong \mathfrak{g}_{-1}$ then one has

$$Q_\omega(T)f = tA \circ L_{\omega^{-1}(h_1)} \circ \cdots \circ L_{\omega^{-1}(h_k)} f,$$

where t is considered as a multiplication operator.

Remark 1. If $T \in C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2))$ is $H-$equivariant, the differential operator $Q_\omega(T)$ does not transform $H-$equivariant functions into $H-$equivariant functions. Indeed, when f is $H-$equivariant, the function $(\nabla^\omega)^k f$ is only G_0-equivariant. Hence the function $Q_\omega(T)f$ does not correspond to a section of $\mathcal{V}_2(M)$. As we continue, we will show that one can modify the symbol T by lower degree correcting terms in order to solve this problem.

6. Measuring the default of equivariance

Throughout this section, T will denote an element of $C^\infty(P, S^k_{V_1,V_2})_{G_0}$ and $f \in C^\infty(P, V_1)_{G_0}$ (remark that this ensures that $Q_\omega(T)(f)$ is in $C^\infty(P, V_2)_{G_0}$). Now, in order to analyze the invariance of functions, we have this first easy result, which follows from the fact that $\mathfrak{g}_1$ is a vector space.

Proposition 7 ([16]). *If (V, α) is a representation of G_0 and becomes a representation of H as stated in section 5, then a function $v \in C^\infty(P, V)$ is $H-$equivariant iff*

$$\begin{cases} v \text{ is } G_0-\text{equivariant} \\ \text{One has } L_{h^*}v = 0 \text{ for every } h \text{ in } \mathfrak{g}_1 \end{cases}$$

6.1. *The map γ*

As we continue, we are interested in measuring the failure of equivariance of the map Q_ω. To this aim, we compute the Lie derivative of the differential operator $Q_\omega(T)$ in the direction of a field L_{h^*}, $h \in \mathfrak{g}_1$. We already defined a map γ in the projective situation :

Definition 3. We define γ on $\otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2)$ by

$$\begin{aligned} \gamma(h)(x_1 \otimes \cdots \otimes x_k \otimes l) = &-\textstyle\sum_{i=1}^k x_1 \otimes \cdots (i) \cdots \otimes x_k \otimes (l \circ \rho_{1_*}([h, x_i])) \\ &+\textstyle\sum_{i=1}^k \sum_{j>i} x_1 \otimes \cdots (i) \cdots \otimes \underbrace{[[h, x_i], x_j]}_{(j)} \otimes \cdots \otimes x_k \otimes l. \end{aligned}$$

for every $x_1, \cdots, x_k \in \mathfrak{g}_{-1}$, $l \in V_1^* \otimes V_2$ and $h \in \mathfrak{g}_1$. Then we extend it to $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2))$ by $C^\infty(P)$-linearity.

The main property of this map is given by the following result.

Proposition 8. *For every $T \in C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2))$, one has*

$$L_{h^*} \circ Q_\omega(T) - Q_\omega(T) \circ L_{h^*} = Q_\omega(L_{h^*}T + \gamma(h)T)$$

on G_0-equivariant functions and for every $h \in \mathfrak{g}_1$.

Proof. The proof is straightforward an is similar to the corresponding one in [16]. □

Moreover, we can write γ in a very compact way. To this aim, we define a new representation of G_0 and $\mathfrak{g}_0$ on $\otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2)$:

Definition 4. The representation ρ_r of G_0 on $\otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2)$ is defined by

$$\rho_r(a)(x_1 \otimes \cdots \otimes x_k \otimes l) = Ad(a)x_1 \otimes \cdots \otimes Ad(a)x_k \otimes (l \circ \rho_1(a^{-1})).$$

Note that the adjoint action of G_0 on $\mathfrak{g}_{-1}$ identifies with the natural action of $CO(p, q)$.

The corresponding representation of $\mathfrak{g}_0$ on the same space is

$$\begin{aligned}\rho_{r_*}(A)(x_1 \otimes \cdots \otimes x_k \otimes l) = &\textstyle\sum_{i=1}^k x_1 \otimes \cdots \otimes [A, x_i] \otimes \cdots \otimes x_k \otimes l \\ &-x_1 \otimes \cdots \otimes x_k \otimes l \circ \rho_{1_*}(A).\end{aligned}$$

for every $A \in \mathfrak{g}_0$, $x_1, \ldots, x_k \in \mathfrak{g}_{-1}$ and $l \in gl(V_1, V_2)$.

Then we have the following immediate result.

Proposition 9. *One has*

$$\gamma(h)(x_1 \otimes \cdots \otimes x_k \otimes l) = \sum_{i=1}^{k} x_1 \otimes \cdots \otimes x_{i-1} \otimes \rho_{r_*}([h, x_i])(x_{i+1} \otimes \cdots \otimes x_k \otimes l).$$

Now, we are interested in the commutation relations of γ and the representations ρ and ρ_r.

Proposition 10. *The following holds on $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2))$, for $\alpha \in \{\rho, \rho_r\}$:*

$$\alpha(a) \circ \gamma(h) = \gamma(Ad(a)h) \circ \alpha(a),$$

and

$$\alpha_*(A) \circ \gamma(h) = \gamma([A, h]) + \gamma(h) \circ \alpha_*(A)$$

for all $a \in G_0$, $A \in \mathfrak{g}_0$ and all $h \in \mathfrak{g}_1$.

Proof. First notice that, since all operators under consideration are $C^\infty(P)$-linear, we only have to prove that the desired relations hold on $\otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2)$.

In order to link the relations for ρ and ρ_r, we extend the representation ρ_2 of G_0 (see section 2.2) from V_2 to $\otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2)$ in a natural way by setting

$$\rho_2(a)(x_1 \otimes \cdots \otimes x_k \otimes L) = x_1 \otimes \cdots \otimes x_k \otimes \rho_2(a) \circ L.$$

It is then obvious that the operators $\rho_2(a)$ and $\rho_r(b)$ commute for all a and b in G_0 and that $\rho(a) = \rho_2(a) \circ \rho_r(a)$. Therefore, we directly get

$$\alpha(a) \circ \rho_r(\exp tB) = \rho_r(a \exp tB\, a^{-1}) \circ \alpha(a)$$

for every $a \in G_0$ and $B \in \mathfrak{g}_0$ and $t \in \mathbb{R}$. Differentiating this expression, we obtain

$$\alpha(a) \circ \rho_{r_*}(B) = \rho_{r_*}(Ad(a)B) \circ \alpha(a).$$

Now we proceed by induction on k. For $k = 0$, the result is obvious. Then we set $T_1 = x_1 \otimes \cdots \otimes x_k \otimes l$ and $T = x_0 \otimes T_1$ for $x_0, \ldots, x_k \in \mathfrak{g}_{-1}$ and $l \in gl(V_1, V_2)$. We use Proposition

9 to obtain

$$\gamma(h)T = \rho_{r_*}([h, x_0])T_1 + x_0 \otimes \gamma(h)T_1, \tag{6.1}$$

and we have

$$\begin{aligned}\alpha(a) \circ \gamma(h)T &= \alpha(a) \circ \gamma(h)(x_0 \otimes T_1) \\ &= \alpha(a)(\rho_{r_*}([h, x_0])T_1 + x_0 \otimes \gamma(h)T_1) \\ &= \rho_{r_*}(Ad(a)[h, x_0])\alpha(a)T_1 + \alpha(a)(x_0 \otimes \gamma(h)T_1).\end{aligned}$$

The last term is equal to

$$Ad(a)x_0 \otimes \alpha(a) \circ \gamma(h)T_1,$$

that is to

$$Ad(a)x_0 \otimes \gamma(Ad(a)h) \circ \alpha(a)T_1$$

by induction. We then have

$$\begin{aligned}\alpha(a) \circ \gamma(h)T &= \rho_{r_*}([Ad(a)h, Ad(a)x_0])\alpha(a)T_1 + Ad(a)x_0 \otimes \gamma(Ad(a)h) \circ \alpha(a)T_1 \\ &= \gamma(Ad(a)h)\alpha(a)T,\end{aligned}$$

by using (6.1) and noticing that $\alpha(a)T = Ad(a)x_0 \otimes \alpha(a)T_1$. □

Proposition 11. *One has*

$$[\gamma(h), \gamma(h')] = 0$$

on $\otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2)$ *for all* $k \in \mathbb{N}$ *and* $h, h' \in \mathfrak{g}_1$.

Proof. For $k = 0$ or $k = 1$, there is nothing to prove, since $\gamma(h)$ lowers the degree of tensors. Let us now proceed by induction. We use the notation of Proposition 10. One has then, iterating (6.1),

$$\gamma(h')\gamma(h)T = \gamma(h')\rho_{r_*}([h, x_0])T_1 + \rho_{r_*}([h', x_0])\gamma(h)T_1 + x_0 \otimes \gamma(h')\gamma(h)T_1. \tag{6.2}$$

Using Proposition 10, we obtain

$$\gamma(h')\rho_{r_*}([h, x_0])T_1 = \rho_{r_*}([h, x_0])\gamma(h')T_1 - \gamma([[h, x_0], h'])T_1.$$

It is then obvious that (6.2) is symmetric in h and h' by induction. □

7. Casimir-like operators

In the papers dealing with equivariant quantization (see for instance [3–5]), the existence of quantizations over vector spaces was ruled by the properties of some Casimir operators associated with the equivariance algebra. In this section we will generalize these operators to our setting. Unfortunately, we have to define them by analogy and not as true Casimir operators. Therefore, we will have to check their properties by direct computations. Hopefully, these computations are quite nice. Let us begin by what we call the flat Casimir operator. We use the basis of the algebra $\mathfrak{g}$ defined by Proposition 1 in section 3.2.

Definition 5. The operator $C^\flat$ is defined on $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2))$ by

$$C^\flat = -\frac{1}{2}\rho_*(\mathcal{E}) + \frac{1}{2d}\rho_*(\mathcal{E})^2 + \sum_{j=1}^{\dim \mathfrak{h}_0} \rho_*(A_j)\rho_*(A_j^*).$$

The main property of this operator is the following.

Proposition 12. *The Casimir operator $C^\flat$ is semi-simple. More precisely, the vector space $\otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2)$ can be decomposed as an $\mathfrak{h}_0$-representation (see section 3.2) into irreducible components (since $\mathfrak{h}_0$ is semi-simple):*

$$\otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2) = \oplus_{s=1}^{n_k} I_{k,s}.$$

The restriction of $C^\flat$ to $C^\infty(P, I_{k,s})$ is then a scalar multiple of the identity.

Proof. The proof goes as in [4] and [16]. Just notice that $C^\flat$ is $C^\infty(P)$-linear. Thus, we only have to compute it on $I_{k,s}$ for every s. Then, it is easy to see that the operator $\rho_*(\mathcal{E})$ is a scalar multiple of the identity, by using the definition of $\mathcal{E}$ and of ρ. It was proved in [4] that the last term in the expression of $C^\flat$ is a scalar multiple of the Casimir operator of $\mathfrak{h}_0$, if $\mathfrak{h}_0$ is absolutely simple. Finally, the restriction of the Casimir operator of $\mathfrak{h}_0$ to every irreducible representation is a scalar multiple of the identity, by Schur's Lemma. □

Using the same basis, we define another operator.

Definition 6. The operator N^ω is defined on $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2))$ by

$$N^\omega = -2\sum_{i=1}^{d} \gamma(\varepsilon^i) L_{\omega^{-1}}(e_i),$$

and we set

$$C^\omega = C^\flat + N^\omega.$$

The operator N^ω has an important property of invariance :

Proposition 13. *The operator N^ω preserves the G_0-equivariance of functions.*

Proof. The proof is exactly the same as in [16]. This property is a consequence of the proposition 10 and of the fact that the invariant differentiation preserves the G_0-equivariance. □

We now have a technical lemma.

Lemma 14. *One has on $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2))$*

$$C^\flat \circ \gamma(h) - \gamma(h) \circ C^\flat = 2\sum_{i=1}^{d} \gamma(\varepsilon^i)\rho_*([h, e_i])$$

for all h in $\mathfrak{g}_1$.

Proof. Using the definition of $C^\flat$ and Proposition 10, we directly obtain the relation

$$\begin{aligned} C^\flat\gamma(h) &= \gamma(h)C^\flat + \gamma((-\tfrac{1}{2}ad(\mathcal{E}) + \tfrac{1}{2d}ad(\mathcal{E})^2 + \textstyle\sum_{j=1}^{\dim \mathfrak{h}_0} ad(A_j)ad(A_j^*))h) \\ &+\tfrac{2}{2d}\gamma([\mathcal{E},h])\rho_*(\mathcal{E}) + \textstyle\sum_{j=1}^{\dim \mathfrak{h}_0}(\gamma([A_j,h])\rho_*(A_j^*) + \gamma([A_j^*,h])\rho_*(A_j)). \end{aligned}$$

On the other hand, using Proposition 2, we may compute

$$[\varepsilon^r, e_i] = \frac{1}{2d}\delta_i^r\mathcal{E} - \frac{1}{2}\sum_{j=1}^{\dim h_0}(a_{ji}^r A_j^* + a_{ji}^{*r}A_j)$$

so that

$$\begin{aligned} 2\textstyle\sum_{i=1}^d \gamma(\varepsilon^i)\rho_*([h,e_i]) &= 2\textstyle\sum_{i,j} h_r\gamma(\varepsilon^i)\rho_*([\varepsilon^r,e_i]) \\ &= \tfrac{2}{2d}\gamma(h)\rho_*(\mathcal{E}) - \textstyle\sum_{i,j} h_r(a_{ji}^r\gamma(\varepsilon^i)\rho_*(A_j^*) + a_{ji}^{*r}\gamma(\varepsilon^i)\rho_*(A_j)) \\ &= \tfrac{2}{2d}\gamma(h)\rho_*(\mathcal{E}) + \textstyle\sum_{j=1}^{\dim h_0}(\gamma([A_j,h]\rho_*(A_j^*) + \gamma([A_j^*,h]\rho_*(A_j)), \end{aligned}$$

by using again Proposition 2.

To sum up, we now have proved the relation

$$\begin{aligned} C^\flat\gamma(h) &= \gamma(h)C^\flat + 2\textstyle\sum_{i=1}^d \gamma(\varepsilon^i)\rho_*([h,e_i]) \\ &+\gamma((-\tfrac{1}{2}ad(\mathcal{E}) + \tfrac{1}{2d}ad(\mathcal{E})^2 + \textstyle\sum_{j=1}^{\dim \mathfrak{h}_0} ad(A_j)ad(A_j^*))h). \end{aligned}$$

We now prove that the last term vanishes by looking at the Casimir operator of the adjoint action of $\mathfrak{g}$: It is given by

$$C_{ad} = \sum_{i=1}^d (ad(e_i)ad(\varepsilon^i) + ad(\varepsilon^i)ad(e_i)) + \frac{1}{2d}ad(\mathcal{E})^2 + \sum_{j=1}^{\dim \mathfrak{h}_0} ad(A_j)ad(A_j^*)$$

Since $\mathfrak{g}_1$ is an abelian subalgebra, the restriction of this operator to $\mathfrak{g}_1$ is

$$\begin{aligned} C_{ad}|_{\mathfrak{g}_1} &= \textstyle\sum_{i=1}^d ad([\varepsilon^i,e_i]) + \tfrac{1}{2d}ad(\mathcal{E})^2 + \textstyle\sum_{j=1}^{\dim \mathfrak{h}_0} ad(A_j)ad(A_j^*) \\ &= \tfrac{1}{2}ad(\mathcal{E}) + \tfrac{1}{2d}ad(\mathcal{E})^2 + \textstyle\sum_{j=1}^{\dim \mathfrak{h}_0} ad(A_j)ad(A_j^*). \end{aligned}$$

Hence, we just need to prove

$$C_{ad}|_{\mathfrak{g}_1} - ad(\mathcal{E})|_{\mathfrak{g}_1} = 0,$$

i.e., $C_{ad}|_{\mathfrak{g}_1} = Id_{\mathfrak{g}_1}$. We compute C_{ad} on $\mathfrak{g}$ in the following way : on the one hand, since $\mathcal{E}$ is in the center of $\mathfrak{g}_0$, we have

$$C_{ad}\mathcal{E} = \sum_{i=1}^d (ad(e_i)ad(\varepsilon^i) + ad(\varepsilon^i)ad(e_i))\mathcal{E} = 2\sum_{i=1}^d [\varepsilon^i, e_i] = \mathcal{E},$$

by the definition of $\mathcal{E}$ and Proposition 1. On the other hand, since $\mathfrak{g}$ is simple, we may apply Schur's Lemma. We obtain $C_{ad} = Id$ and the result follows. □

Now, we can come to the main result about the second Casimir operator.

Proposition 15. *The relation*

$$[C^\omega, L_{h^*} + \gamma(h)] = 0$$

holds on $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1,V_2))_{G_0}$, *for all* $h \in \mathfrak{g}_1$.

Proof. By the very definition of C^ω, we have

$$\begin{aligned}[C^\omega, L_{h^*} + \gamma(h)] &= [C^\flat - 2\textstyle\sum_{i=1}^d \gamma(\varepsilon^i)L_{\omega^{-1}}(e_i), L_{h^*} + \gamma(h)] \\ &= [C^\flat, L_{h^*}] + [C^\flat, \gamma(h)] + [-2\textstyle\sum_{i=1}^d \gamma(\varepsilon^i)L_{\omega^{-1}}(e_i), L_{h^*}] \\ &\quad + [-2\textstyle\sum_{i=1}^d \gamma(\varepsilon^i)L_{\omega^{-1}}(e_i), \gamma(h)].\end{aligned}$$

The first and last term vanish in view of the definition of $C^\flat$, γ and of Proposition 11. We already computed the second one, and the third one is obviously equal to

$$-2\sum_{i=1}^d \gamma(\varepsilon^i)L_{[e_i,h]^*} = -2\sum_{i=1}^d \gamma(\varepsilon^i)\rho_*([h, e_i]),$$

hence the result. □

8. Construction of the quantization

The construction is based on the eigenvalue problem for the Casimir-like operators $C^\flat$ and C^ω. The construction was given in [16] in the projective case, and based on the original computations of [3]. Actually, this construction applies to our setting. The main point is that we modified the definitions of Q_ω, γ, $C^\flat$ and C^ω so that Propositions 8, 13 and 15 hold true and $C^\flat$ is semi-simple. We recall here the key results of the construction and we refer the reader to [16] for the proofs.

Recall that $\otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2)$ is decomposed as a representation of $\mathfrak{h}_0$ as the direct sum of irreducible components $I_{k,s}$. Denote by $E_{k,s}$ the space $C^\infty(P, I_{k,s})$ and by $\alpha_{k,s}$ the eigenvalue of $C^\flat$ restricted to $E_{k,s}$.

As in [3, 4], the tree-like subspace $\mathcal{T}_\gamma(I_{k,s})$ associated with $I_{k,s}$ is defined by

$$\mathcal{T}_\gamma(I_{k,s}) = \bigoplus_{l\in\mathbb{N}} \mathcal{T}^l_\gamma(I_{k,s}),$$

where $\mathcal{T}^0_\gamma(I_{k,s}) = I_{k,s}$ and $\mathcal{T}^{l+1}_\gamma(I_{k,s}) = \gamma(\mathfrak{g}_1)(\mathcal{T}^l_\gamma(I_{k,s}))$, for all $l \in \mathbb{N}$. The space $\mathcal{T}^l_\gamma(E_{k,s})$ is then defined in the same way. Since γ is $C^\infty(P)$-linear, this space is equal to $C^\infty(P, \mathcal{T}^l_\gamma(I_{k,s}))$.

The following definition is a direct generalization of the ones of [3, 4] :

Definition 7. An ordered pair of representations (V_1, V_2) is *critical* if there exists k, s such that the eigenvalue $\alpha_{k,s}$ corresponding to an irreducible component $I_{k,s}$ of $S^k_{V_1,V_2}$ belongs to the spectrum of the restriction of $C^\flat$ to $\bigoplus_{l\geq 1} \mathcal{T}^l_\gamma(E_{k,s})$.

We can now analyze the eigenvalue problem for the operator C^ω.

Theorem 16. *If the pair (V_1, V_2) is not critical, for every T in $C^\infty(P, I_{k,s})$, (where $I_{k,s}$ is an irreducible component of $S^k_{V_1,V_2}$) there exists a unique function $\hat{T}$ in $C^\infty(P, \mathcal{T}_\gamma(I_{k,s}))$ such that*

$$\begin{cases} \hat{T} &= T_k + \cdots + T_0, \quad T_k = T \\ C^\omega(\hat{T}) &= \alpha_{k,s}\hat{T}. \end{cases} \tag{8.1}$$

Moreover, if T is G_0-invariant, then $\hat{T}$ is G_0-invariant.

This result allows to define the main ingredient in order to define the quantization : The "modification map", acting on symbols.

Definition 8. Suppose that the pair (V_1, V_2) is not critical. Then the map

$$R : \oplus_{k=0}^{\infty} C^{\infty}(P, S^k_{V_1,V_2}) \to \oplus_{k=0}^{\infty} C^{\infty}(P, \otimes^k \mathfrak{g}_{-1} \otimes gl(V_1, V_2))$$

is the linear extension of the association $T \mapsto \hat{T}$.

The map R has the following nice property :

Proposition 17. *One has*

$$(L_{h^*} + \gamma(h))R(T) = R(L_{h^*}T), \tag{8.2}$$

for every $h \in \mathfrak{g}_1$ *every* $T \in C^{\infty}(P, S^k_{V_1,V_2})_{G_0}$ *and every* $k \in \mathbb{N}$.

And finally, the main result :

Theorem 18. *If the pair* (V_1, V_2) *is not critical, then the formula*

$$Q_M : (\nabla, T) \mapsto Q_M(\nabla, T)(f) = (p^*)^{-1}[Q_\omega(R(p^*T))(p^*f)],$$

(where Q_ω *is given by (5.3)) defines a natural and conformally invariant quantization.*

8.1. *Final remarks*

Throughout the computations, we did not use explicitly the bracket of the algebra $so(p+1, q+1)$, we only used the $|1|$-grading of this algebra and the subsequent properties. Another ingredient is the existence of a Cartan bundle associated with the G_0-bundle P_0, and of a normal Cartan connection to this bundle. Therefore, our construction can be generalized to the construction of an invariant quantization, once these data are given.

In this respect, after a preliminary version of this paper was posted in 2008 on the arXiv site [arXiv:0811.3710v1], an explicit construction of quantization for AHS-structures was given in 2009 by A. Čap and J. Šilhan [arxiv:0904.3278v1, [23]].

Finally, we did not address explicitly the uniqueness problem of the quantization. But it was proved by F. Radoux in [24] that, even in the projective case, that is the most simple case, the quantization is not unique in general, due to the presence of the Weyl curvature tensor. Therefore, we conclude that the quantization is not unique in general, since the most simple example given by F. Radoux holds for any other geometry. It would be interesting to find a natural condition to impose to the quantization procedure in order to obtain the uniqueness that was one of the main features of the equivariant quantization problem in the flat situation.

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