NATURAL AND PROJECTIVELY EQUIVARIANT QUANTIZATIONS BY MEANS OF CARTAN CONNECTIONS

P. MATHONET AND F. RADOUX

Abstract. The existence of a natural and projectively equivariant quantization in the sense of Lecomte [20] was proved recently by M. Bordemann [4], using the framework of Thomas-Whitehead connections. We give a new proof of existence using the notion of Cartan projective connections and we obtain an explicit formula in terms of these connections. Our method yields the existence of a projectively equivariant quantization if and only if an $sl(m+1,\mathbb{R})$-equivariant quantization exists in the flat situation in the sense of [18], thus solving one of the problems left open by M. Bordemann.

Mathematics Subject Classification (2000) : 53B05, 53B10, 53D50, 53C10.
Key words : Projective Cartan connections, differential operators, natural maps, quantization maps.

1. Introduction

Among the different meanings that the word quantization can assume, one (in the framework of geometric quantization) is to think of a quantization procedure as a linear bijection from the space of classical observables to a space of differential operators acting on wave functions (see [27]). More precisely, in our setting, the space of observables (also called the space of *Symbols*) is the space of smooth functions on the cotangent bundle T^*M of a manifold M, that are polynomial along the fibres. The space of differential operators $\mathcal{D}_{\frac{1}{2}}(M)$ is made of differential operators acting on half-densities. It is known that there is no natural quantization procedure. In other words, the spaces of symbols and of differential operators are not isomorphic as representations of $\mathrm{Diff}(M)$.

However, when there is a Lie group G acting on M by local diffeomorphisms, the action can be lifted to symbols and differential operators and one can raise the question of knowing whether these spaces are isomorphic representations of G or not. This leads to the concept of G-equivariant quantization introduced by P. Lecomte and V. Ovsienko in [18] : a G-equivariant

Date: March 31, 2011.
Institute of mathematics, Grande Traverse, 12 - B37, B-4000 Liège, Belgium
email : P.Mathonet@ulg.ac.be, F.Radoux@ulg.ac.be.

quantization is a linear bijection from the space of symbols to the space of differential operators that exchanges the actions of G on these spaces.

In [18], the authors considered the case of the projective group $PGL(m+1,\mathbb{R})$ acting on the manifold $M=\mathbb{R}^m$ by linear fractional transformations. This leads to the notion of projectively equivariant quantization or its infinitesimal counterpart, the $sl(m+1,\mathbb{R})$-equivariant quantization. One of the main results in this case is the existence of a projectively equivariant quantization and its uniqueness, up to some natural normalization condition. The authors also showed that their results could be directly generalized to the case of a manifold endowed with a flat projective structure.

In [11], the authors considered the group $SO(p+1,q+1)$ acting on the space $\mathbb{R}^{p+q}$ or on a manifold endowed with a flat conformal structure. They extended the problem by considering the space $\mathcal{D}_{\lambda,\mu}$ of differential operators mapping λ-densities into μ-densities and a suitable space of symbols $\mathcal{S}_{\mu-\lambda}$. There again, the result was the existence and uniqueness of a conformally equivariant quantization provided the shift value $\delta=\mu-\lambda$ does not belong to a set of critical values. Similar results for other Lie groups G acting on vector spaces or other types of differential operators were obtained in [3, 12, 2].

At that point, all these results were dealing with a manifold endowed with a flat structure. It was remarked in [5, 6] that the formula for the projectively equivariant quantization for differential operators of order two and three could be generalized to an arbitrary manifold. In these papers, S. Bouarroudj showed how to define a quantization map from the space of symbols to the space of differential operators, using a torsion-free connection, in such a way that the quantization map depends only on the *projective class* of the connection (recall that two torsion-free linear connections are projectively equivalent if they define the same paths, that is, the same geodesics up to parametrization).

In [20], P. Lecomte conjectured the existence of a quantization procedure depending on a torsion-free connection, that would be natural (in all arguments) and that would be left invariant by a projective change of connection.

The existence of such a quantization procedure was proved by M. Bordemann in [4]. In order to prove the existence, M. Bordemann used a construction that can be roughly summarized as follows : first he associated to each projective class $[\nabla]$ of torsion-free linear connections on M a unique linear connection $\tilde{\nabla}$ on a principal line bundle $\tilde{M}\to M$, then he showed how to lift the symbols to a suitable space of tensors on $\tilde{M}$, and he eventually applied the so-called *Standard ordering*. This construction was later adapted in [13] in order to deal with differential operators acting on forms.

The study of the projective equivalence of connections goes back to the 1920's. At that time, there were two main approaches to the so-called *Geometry of paths*. The connection used by M. Bordemann is inpired by the approach due to T.Y. Thomas [23], J.H.C. Whitehead [26] and O.Veblen [24](see also [14, 22, 21] for a modern formulation).

The second approach, due to E. Cartan [10], leads to the concept of Cartan projective connection, developed in a modern setting by S. Kobayashi and T. Nagano in [16, 15].

In this paper, we analyse the existence of a natural and projectively equivariant quantization map from the space of symbols $\mathcal{S}_{\mu-\lambda}$ to the space $\mathcal{D}_{\lambda,\mu}$, using Cartan connections. We obtain an explicit formula for the quantization map in terms of the normal Cartan connection associated to a projective equivalence class of torsion free-linear connections. This formula generalizes the one given by M. Bordemann in [4] and is nothing but the formula for the flat case given in [12]. In particular, we show that the natural and projectively equivariant quantization map exists if and only if an $sl(m+1)$-equivariant quantization exists in the flat case, thus solving a problem left open by M. Bordemann.

We believe that our methods will apply in order to solve the problem in the conformal situation or in order to define a projectively equivariant symbol calculus for other types of differential operators.

2. Problem setting

For the sake of completeness, we briefly recall in this section the definitions of tensor densities, differential operators and symbols. Then we set the problem of existence of projectively equivariant natural quantizations. Throughout this note, we denote by M a smooth, Hausdorff and second countable manifold of dimension m.

2.1. **Tensor densities.** The vector bundle of tensor densities $F_\lambda(M) \to M$ is a line bundle associated to the linear frame bundle :

$$F_\lambda(M) = P^1 M \times_\rho \Delta^\lambda(R^m),$$

where the representation ρ of the group $GL(m,\mathbb{R})$ on the one-dimensional vector space $\Delta^\lambda(R^m)$ is given by

$$\rho(A)e = |detA|^{-\lambda}e, \quad \forall A \in GL(m,\mathbb{R}),\ \forall e \in \Delta^\lambda(\mathbb{R}^m).$$

As usual, we denote by $\mathcal{F}_\lambda(M)$ the space of smooth sections of this bundle. This is the space $C^\infty(P^1M, \Delta^\lambda(R^m))_{GL(m,\mathbb{R})}$ of functions f such that

$$f(uA) = \rho(A^{-1})f(u) \quad \forall u \in P^1M,\ \forall A \in GL(m,\mathbb{R}).$$

Since $F_\lambda(M) \to M$ is associated to P^1M, there are natural actions of $\mathrm{Diff}(M)$ and of $\mathrm{Vect}(M)$ on $\mathcal{F}_\lambda(M)$. For more details, we refer the reader to [11, 18].

2.2. **Differential operators and symbols.** As in [18, 4], we denote by $\mathcal{D}_{\lambda,\mu}(M)$ the space of differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. The actions of $\mathrm{Vect}(M)$ and $\mathrm{Diff}(M)$ are induced by the actions on tensor densities : One has

$$(\phi \cdot D)(f) = \phi \cdot (D(\phi^{-1} \cdot f)), \quad \forall f \in \mathcal{F}_\lambda(M), D \in \mathcal{D}_{\lambda,\mu}, \text{and } \phi \in \mathrm{Diff}(M).$$

The space $\mathcal{D}_{\lambda,\mu}$ is filered by the order of differential operators. We denote by $\mathcal{D}^k_{\lambda,\mu}$ the space of differential operators of order at most k. It is well-known that this filtration is preserved by the action of local diffeomorphisms. The space of *symbols* is then the associated graded space of $\mathcal{D}_{\lambda,\mu}$.

We denote by $S^l_\delta(\mathbb{R}^m)$ the vector space $S^l\mathbb{R}^m \otimes \Delta^\delta(\mathbb{R}^m)$. There is a natural representation ρ of $GL(m,\mathbb{R})$ on this space (the representation of $GL(m,\mathbb{R})$ on symmetric tensors is the natural one). We then denote by $S^l_\delta(M) \to M$ the vector bundle

$$P^1M \times_\rho S^l_\delta(\mathbb{R}^m) \to M,$$

and by $\mathcal{S}^l_\delta(M)$ the space of smooth sections of $S^l_\delta(M) \to M$, that is, the space $C^\infty(P^1M, S^l_\delta(\mathbb{R}^m))_{GL(m,\mathbb{R})}$.

Then if $\delta = \mu - \lambda$ the *principal symbol operator* $\sigma : \mathcal{D}^l_{\lambda,\mu}(M) \to \mathcal{S}^l_\delta(M)$ commutes with the action of diffeomorphisms and is a bijection from the quotient space $\mathcal{D}^l_{\lambda,\mu}(M)/\mathcal{D}^{l-1}_{\lambda,\mu}(M)$ to $\mathcal{S}^l_\delta(M)$. Hence the space of symbols is nothing but

$$\mathcal{S}_\delta(M) = \bigoplus_{l=0}^{\infty} \mathcal{S}^l_\delta(M),$$

endowed with the classical actions of $\mathrm{Diff}(M)$ and of $\mathrm{Vect}(M)$.

2.3. **Projective equivalence of connections.** We denote by $\mathcal{C}_M$ the space of torsion-free linear connections on M. Two such connections are *Projectively equivalent* if there exists a one form α on M such that their associated covariant derivatives ∇ and ∇' fulfill the relation

$$\nabla'_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X.$$

This equation was introduced by H. Weyl in [25]. He showed that it was a necessary and sufficient condition for the two connections to define the same *paths*, that is, the same geodesics up to parametrization.

2.4. **Problem setting.** A *quantization* on M is a linear bijection Q_M from the space of symbols $\mathcal{S}_\delta(M)$ to the space of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ such that

$$\sigma(Q_M(S)) = S, \quad \forall S \in \mathcal{S}^k_\delta(M),\ \forall k \in \mathbb{N}.$$

Roughly speaking, a *natural quantization* is a quantization which depends on a torsion-free connection and commutes with the action of diffeomorphisms. More precisely, a natural quantization is a collection of maps (defined for every manifold M)

$$Q_M : \mathcal{C}_M \times \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M)$$

such that

- For all ∇ in $\mathcal{C}_M$, $Q_M(\nabla)$ is a quantization,
- If ϕ is a local diffeomorphism from M to N, then one has

$$Q_M(\phi^*\nabla)(\phi^*S) = \phi^*(Q_N(\nabla)(S)), \quad \forall \nabla \in \mathcal{C}_N, \forall S \in \mathcal{S}_\delta(N).$$

A quantization Q_M is *projectively equivariant* if one has $Q_M(\nabla) = Q_M(\nabla')$ whenever ∇ and ∇' are projectively equivalent torsion-free linear connections on M.

Remark : The definition of a natural quantization was set by M. Bordemann in functorial terms and relates in this sense to the concept of natural operators in differential geometry exposed in [17].

3. Projective Cartan connections

For the paper to be self-contained, we recall here the most important facts about Cartan connections. We begin with a general definition and then we give more details about the projective Cartan connections and their links with projective structures. For more detailed information, the reader may refer to [15].

3.1. **Cartan connections.** Let G be a Lie group and H a closed subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras. Let $P \to M$ be a principal H-bundle over M, such that $\dim M = \dim G/H$. A Cartan connection on P is a $\mathfrak{g}$-valued one-form ω on P such that

- If R_a denotes the right action of $a \in H$ on P, then $R_a^*\omega = Ad(a^{-1})\omega$,
- If k^* is the vertical vector field associated to $k \in \mathfrak{h}$, then $\omega(k^*) = k$,
- $\forall u \in P, \omega_u : T_uP \to \mathfrak{g}$ is a linear bijection.

3.2. **Projective structures and Projective connections.** We consider the group $G = PGL(m+1, \mathbb{R})$ acting on $\mathbb{R}P^m$. We denote by H the stabilizer of the element $[e_{m+1}]$ in $\mathbb{R}P^m$. One has

$$H = \{\begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix} : A \in GL(m, \mathbb{R}), \xi \in \mathbb{R}^{m*}, a \neq 0\}/\mathbb{R}_0\mathrm{Id}, \tag{1}$$

and it follows that H is the semi-direct product $G_0 \rtimes G_1$, where G_0 is isomorphic to $GL(m, \mathbb{R})$ and G_1 is isomorphic to $\mathbb{R}^{m*}$. Then there is a projection

$$\pi : H \to GL(m, \mathbb{R}) : \left[\begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix}\right] \mapsto \frac{A}{a}$$

The Lie algebra of $PGL(m+1, \mathbb{R})$ is $gl(m+1, \mathbb{R})/\mathbb{R}Id$. It is thus isomorphic to $sl(m+1, \mathbb{R})$ and it decomposes as a direct sum of subalgebras

$$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \cong \mathbb{R}^m \oplus gl(m, \mathbb{R}) \oplus \mathbb{R}^{m*}.$$

The isomorphism is given by

$$\left[\begin{pmatrix} A & v \\ \xi & a \end{pmatrix}\right] \mapsto (v, A - a\,Id, \xi).$$

This correspondance induces a structure of Lie algebra on $\mathbb{R}^m \oplus gl(m, \mathbb{R}) \oplus \mathbb{R}^{m*}$. The Lie algebras corresponding to G_0, G_1 and H are respectively $\mathfrak{g}_0$, $\mathfrak{g}_1$, and $\mathfrak{g}_0 \oplus \mathfrak{g}_1$.

Let us denote by G_m^2 the group of 2-jets at the origin $0 \in \mathbb{R}^m$ of local diffeomorphisms defined on a neighborhood of 0 and that leave 0 fixed. The group H acts on $\mathbb{R}^m$ by linear fractional transformations that leave the origin fixed. This allows to view H as a subgroup of G_m^2 , namely,

$$\imath : H \to G_m^2 : \begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix} \mapsto (\frac{A_j^i}{a}, -\frac{A_j^i\xi_k + A_k^i\xi_j}{a^2}) \tag{2}$$

A *Projective structure on M* is then a reduction of the second order jet-bundle P^2M to the group H. The following result ([15, Prop 7.2 p.147]) is the starting point of our method :

Proposition 1 (Kobayashi-Nagano)**.** *There is a natural one to one correspondance between the projective equivalence classes of torsion-free linear connections on M and the projective structures on M.*

In general, if ω is a Cartan connection defined on a H-principal bundle P, then its curvature Ω is defined as usual by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{3}$$

The notion of *Normal* Cartan connection is defined by natural conditions imposed on the components of the curvature.

Now, the following result ([15, p. 135]) gives the relationship between projective structures and Cartan connections :

Proposition 2. *A unique normal Cartan connection with values in the algebra $sl(m+1,\mathbb{R})$ is associated to every projective structure P. This association is natural.*

The connection associated to a projective structure P is called the normal projective connection of the projective structure.

3.3. **Invariant differentiation.** We will use the concept of invariant differentiation with respect to a Cartan connection developed in [8, 9]. Let P be a projective structure and let ω be the associated normal projective connection.

Definition 1. Let (V,ρ) be a representation of H. If $f \in C^\infty(P,V)$, then the invariant differential of f with respect to ω is the function $\nabla^\omega f \in C^\infty(P, \mathbb{R}^{m*} \otimes V)$ defined by

$$\nabla^\omega f(u)(X) = L_{\omega^{-1}(X)} f(u) \quad \forall u \in P, \quad \forall X \in \mathbb{R}^m.$$

We will also use an iterated and symmetrized version of the invariant differentiation

Definition 2. If $f \in C^\infty(P,V)$ then $(\nabla^\omega)^k f \in C^\infty(P, S^k\mathbb{R}^{m*} \otimes V)$ is defined by

$$(\nabla^\omega)^k f(u)(X_1,\dots,X_k) = \frac{1}{k!}\sum_\nu L_{\omega^{-1}(X_{\nu_1})} \circ \dots \circ L_{\omega^{-1}(X_{\nu_k})} f(u)$$

for $X_1,\dots,X_k \in \mathbb{R}^m$.

3.4. **Lift of equivariant functions.** In order to make use of the invariant differentiation, we need to know the relationship between equivariant functions on P^1M and equivariant functions on P. The following results were already quoted in [8, p. 47].

If (V, ρ) is a representation of $GL(m, \mathbb{R})$, then we define a representation (V, ρ') of H by

$$\rho' : H \to GL(V) : \left[\begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix}\right] \mapsto \rho \circ \pi(\left[\begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix}\right]) = \rho(\frac{A}{a})$$

for every $A \in GL(m, \mathbb{R}), \xi \in \mathbb{R}^{m*}, a \neq 0$.

Now, using the representation ρ', we can give the relationship between equivariant functions on P^1M and equivariant functions on P : If P is a projective structure on M, the natural projection $P^2M \to P^1M$ induces a projection $p : P \to P^1M$ and we have :

Proposition 3. *If (V, ρ) is a representation of $GL(m, \mathbb{R})$, then the map*

$$p^* : C^\infty(P^1M, V) \to C^\infty(P, V) : f \mapsto f \circ p$$

defines a bijection from $C^\infty(P^1M, V)_{\mathrm{GL}(m,\mathbb{R})}$ to $C^\infty(P, V)_H$.

This result is well-known and comes from the following facts

- (p, Id, π) is a morphism of principal bundles from P to P^1M
- the equivariant functions on P are constant on the orbits of the action of G_1 on P.

Now, since $\mathbb{R}^m$ and $\mathbb{R}^{m*}$ are natural representations of $GL(m, \mathbb{R})$, they become representations of H and we can state an important property of the invariant differentiation :

Proposition 4. *If f belongs to $C^\infty(P, V)_{G_0}$ then $\nabla^\omega f \in C^\infty(P, \mathbb{R}^{m*} \otimes V)_{G_0}$.*

Proof. The result is a direct consequence of the Ad-invariance of the Cartan connection ω. □

The main point that we will discuss in the next sections is that this result is not true in general for H-equivariant functions : for an H-equivariant function f, the function $\nabla^\omega f$ is in general not G_1-equivariant.

As we continue, we will use the representation ρ'_* of the Lie algebra of H on V. If we recall that this algebra is isomorphic to $gl(m, \mathbb{R}) \oplus \mathbb{R}^{m*}$ then we have

$$\rho'_*(A, \xi) = \rho_*(A), \quad \forall A \in gl(m, \mathbb{R}), \xi \in \mathbb{R}^{m*}. \tag{4}$$

In our computations, we will make use of the infinitesimal version of the equivariance relation : If $f \in C^\infty(P, V)_H$ then one has

$$L_{h^*} f(u) + \rho'_*(h) f(u) = 0, \quad \forall h \in gl(m, \mathbb{R}) \oplus \mathbb{R}^{m*} \subset sl(m+1, \mathbb{R}), \forall u \in P. \tag{5}$$

4. Construction of a projectively invariant quantization

The existence of a natural and projectively equivariant quantization is linked to the existence of an $sl(m+1,\mathbb{R})$-equivariant quantization in the sense of [18] in the flat situation. It is known that for some *critical values* of δ such a quantization does not exist. Let us recall the following definition of [19, Prop 2, p. 289] :

Definition 3. We define the numbers

$$\gamma_{2k-l} = \frac{(m+2k-l-(m+1)\delta)}{m+1}.$$

A value of δ is critical if there exists $k, l \in \mathbb{N}$ such that $1 \leq l \leq k$ and $\gamma_{2k-l} = 0$.

One of the results of [19] is then the following

Theorem 5. *If δ is not critical, there exist an unique $sl(m+1,\mathbb{R})$-equivariant quantization.*

A link between the natural quantization and the $sl(m+1,\mathbb{R})$-equivariant quantization is then given in [20] by

Theorem 6. *If Q_M is a natural projectively equivariant quantization and if we denote by ∇_0 the flat connection on $\mathbb{R}^m$, then $Q_{\mathbb{R}^m}(\nabla_0)$ is $sl(m+1,\mathbb{R})$-equivariant.*

Now, let us introduce the divergence operator associated to a Cartan connection. This operator will be the main tool of our construction.

4.1. The Divergence operator.

We fix a basis $(e_1,\ldots,e_m)$ of $\mathbb{R}^m$ and we denote by $(\epsilon^1,\ldots,\epsilon^m)$ the dual basis in $\mathbb{R}^{m*}$.

The *Divergence operator* with respect to the Cartan connection ω is then defined by

$$div^\omega : C^\infty(P, S^k_\delta(\mathbb{R}^m)) \to C^\infty(P, S^{k-1}_\delta(\mathbb{R}^m)) : S \mapsto \sum_{j=1}^m i(\epsilon^j)\nabla^\omega_{e_j} S,$$

where i denotes the inner product.

This operator can be seen as a curved generalization of the divergence operator used in [18]. The following propositions shows its most important properties.

Lemma 7. *If $S \in C^\infty(P, S^k_\delta(\mathbb{R}^m))_{G_0}$ then $div^\omega S \in C^\infty(P, S^{k-1}_\delta(\mathbb{R}^m))_{G_0}$.*

Proof. This can be checked directly from the definition. One can also remark that $div^\omega S$ is the contraction of the invariant function $\nabla^\omega S$ (see proposition 4) and of the constant and invariant function

$$ID : P \to \mathbb{R}^m \otimes \mathbb{R}^{m*}, u \mapsto \sum_{j=1}^m \epsilon^j \otimes e_j.$$

□

The purpose of the next results is to measure the failure of G_1- equivariance of the operators defined so far. At the infinitesimal level, in view of equations (4) and (5), this leads to the computation of the commutator of these operators with the Lie derivative L_{h^*}, for $h \in \mathfrak{g}_1$. We begin with the divergence operator :

Lemma 8. *For every $S \in C^\infty(P, S^k_\delta(\mathbb{R}^m))_{G_0}$ we have*

$$L_{h^*} div^\omega S - div^\omega L_{h^*} S = (m+1)\gamma_{2k-1} i(h) S,$$

for every $h \in \mathbb{R}^{m*} \cong \mathfrak{g}_1$.

Proof. First we remark that the Lie derivative with respect to a vector field commutes with the evaluation : If $\eta^1, \ldots, \eta^{k-1} \in \mathbb{R}^{m*}$, we have

$$\begin{array}{rcl}(L_{h^*} div^\omega S)(\eta^1, \ldots, \eta^{k-1}) & = & L_{h^*}(div^\omega S(\eta^1, \ldots, \eta^{k-1})) \\ & = & \sum_{j=1}^m (L_{h^*} L_{\omega^{-1}(e_j)} S(\epsilon^j, \eta^1, \ldots, \eta^{k-1}))\end{array}$$

Now, the definition of a projective Cartan connection implies the relation

$$[h^*, \omega^{-1}(X)] = \omega^{-1}([h, X]), \quad \forall h \in gl(m, \mathbb{R}) \oplus \mathbb{R}^{m*}, X \in \mathbb{R}^m,$$

where the bracket on the right is the one of $sl(m+1, \mathbb{R})$. It follows that the expression we have to compute is equal to

$$\sum_{j=1}^m (L_{\omega^{-1}(e_j)} L_{h^*} S(\epsilon^j, \eta^1, \ldots, \eta^{k-1}) + (L_{[h,e_j]^*} S)(\epsilon^j, \eta^1, \ldots, \eta^{k-1})).$$

Finally, using the relation (5), we obtain

$$\begin{array}{rl} & div^\omega(L_{h^*} S) - (\rho'_*([h, e_j])S)(\epsilon^j, \eta^1, \ldots, \eta^{k-1}) \\ = & div^\omega(L_{h^*} S) - (\rho_*(h \otimes e_j + \langle h, e_j \rangle Id)S)(\epsilon^j, \eta^1, \ldots, \eta^{k-1}).\end{array}$$

The result then easily follows from the definition of ρ on $S^k_\delta(\mathbb{R}^m)$. □

Eventually we obtain

Proposition 9. *For every $S \in C^\infty(P, S^k_\delta(\mathbb{R}^m))_{G_0}$, we have*

$$L_{h^*}(div^\omega)^l S - (div^\omega)^l L_{h^*} S = (m+1) l \gamma_{2k-l} i(h) (div^\omega)^{l-1} S,$$

for every $h \in \mathbb{R}^{m*} \cong \mathfrak{g}_1$.

Proof. For $l = 1$, this is just lemma 8. Then the result follows by induction, using lemmas 7 and 8. □

Next, we analyse the failure of invariance of the iterated invariant differentiation :

Proposition 10. *If $f \in \mathcal{C}^\infty(P, \Delta^\lambda \mathbb{R}^m)_{G_0}$, then*

$$L_{h^*}(\nabla^\omega)^k f - (\nabla^\omega)^k L_{h^*} f = -k((m+1)\lambda + k - 1)((\nabla^\omega)^{k-1} f \vee h),$$

for every $h \in \mathbb{R}^{m*} \cong \mathfrak{g}_1$.

Proof. If $k = 0$, then the formula is obviously true. Then we proceed by induction. In view of the symmetry of the expressions that we have to compare, it is sufficient to check that they coincide when evaluated on the k-tuple $(X, \dots, X)$ for every $X \in \mathbb{R}^m$. The proof is similar to the one of lemma 8 : first the evaluation and the Lie derivative commute :

$$(L_{h^*}(\nabla^\omega)^k f)(X, \dots, X) = L_{h^*}((\nabla^\omega)^k f)(X, \dots, X)).$$

Next, we use the definition of the iterated invariant differential and we let the operators L_{h^*} and $L_{\omega^{-1}(X)}$ commute so that the latter expression becomes

$$L_{\omega^{-1}(X)} L_{h^*}((\nabla^\omega)^{k-1} f)(X, \dots, X) + (L_{[h,X]^*}((\nabla^\omega)^{k-1} f))(X, \dots, X).$$

By the induction, the first term is equal to

$$(\nabla^\omega)^k L_{h^*} f(X, \dots, X) - (k-1)((m+1)\lambda + k - 2)((\nabla^\omega)^{k-1} f \vee h)(X, \dots, X).$$

For the second term, we use proposition 4 and relation (5) and we obtain

$$(\rho_*((h \otimes X) + \langle h, X\rangle Id)((\nabla^\omega)^{k-1} f))(X, \dots, X).$$

The result follows by the definition of ρ_*. □

4.2. **The main result.** In this section, we give an explicit formula for the natural and projectively equivariant quantization, using the properties of the iterated invariant differentiation and of the divergence operator.

Theorem 11. *If δ is not critical, then the collection of maps $Q_M : \mathcal{C}_M \times \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M)$ defined by*

$$Q_M(\nabla, S)(f) = p^{*^{-1}}(\sum_{l=0}^{k} C_{k,l} \langle Div^{\omega^l} p^* S, \nabla_s^{\omega^{k-l}} p^* f\rangle), \forall S \in \mathcal{S}^k_\delta(M) \qquad (6)$$

defines a projectively invariant natural quantization if

$$C_{k,l} = \frac{(\lambda + \frac{k-1}{m+1}) \cdots (\lambda + \frac{k-l}{m+1})}{\gamma_{2k-1} \cdots \gamma_{2k-l}} \begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1.$$

Proof. First, we have to check that the formula makes sense : the function

$$\sum_{l=0}^{k} C_{k,l} \langle (Div^\omega)^l p^* S, (\nabla^\omega)^{k-l} p^* f\rangle \qquad (7)$$

has to be H-equivariant. It is obviously G_0-equivariant by proposition 4 and lemma 7. It is then sufficient to check that it is $\mathfrak{g}_1$-equivariant. This follows directly from propositions 9 and 10 and from the relation

$$C_{k,l} l(m + 2k - l - (m+1)\delta) = C_{k,l-1}(k - l + 1)((m+1)\lambda + k - l). \qquad (8)$$

Next we see, using the results of [8, p.47] that the principal symbol of $Q_M(\nabla, S)$ is exactly S, and formula (6) defines a quantization, that is projectively invariant, by the definition of ω. Next, the naturality of the quantization defined in this way is easy to understand : it follows from the naturality of the association of a projective structure $P \to M$ endowed with a normal Cartan connection ω to a class of projectively equivalent torsion-free connections on M and from the naturality of the lift of the equivariant functions on P^1M to equivariant functions on P. $\square$

Remarks :

- Theorems 5 and 6 directly imply that, when M is taken to be $\mathbb{R}^m$ and ∇ is the flat connection, formula 6 must coincide with the ones of [18] (formulas 4.14 and 4.15) and [12] (Formula 2.4), at least when δ is not critical. What is more surprising is that our coefficients $C_{k,l}$ coincide with the ones of [12] (formulas 2.5 and 3.6), up to a combinatorial coefficient, which is due to a slightly different definition of the divergence operator. In particular, our formula can be expressed, as the one of [12], in terms of hypergeometric functions.
- A long computation, involving the explicit form of the normal Cartan connection in coordinates allows to show that our formula coincides for the case of third order differential operators with the formula provided by S. Bouarroudj in [5, 6].
- It was shown in [18] how an $sl(m+1)$-equivariant quantization (in the flat case) induced a projectively invariant star product on the space of symbols. It was shown in [7] that the bilinear operators appearing in the deformation were not bidifferential operators. We conjecture that our formula 6 will induce a deformation of the algebra of symbols (depending on a connection) and that this deformation will not be local and therefore will not be a star-product in the sense of [1].

Now, the proof of the previous theorem also allows to analyse the existence problem when δ is a critical value : assume that there exist $k \in \mathbb{N}$ and $r \in \mathbb{N}$ such that $1 \leq r \leq k$ and $\gamma_{2k-r} = 0$. Then if there exists $i \in \{1, \dots, r\}$ such that $\lambda = -\frac{k-i}{m+1}$, then one can replace the coefficients $C_{k,i}, \dots, C_{k,k}$ by zero and the function (7) is still H-equivariant. Then the collection Q_M still defines a projectively equivariant and natural quantization. If λ does not belong to the set $\{-\frac{k-1}{m+1}, \dots, -\frac{k-r}{m+1}\}$, then there is no solution since the $sl(m+1, \mathbb{R})$- equivariant quantization in the sense of [18, 19] does not exist. To sum up, we have shown the following

Theorem 12. *There exists a natural and projectively equivariant quantization if and only if there exists an $sl(m+1, \mathbb{R})$-equivariant quantization in the sense of* [18] *over $M = \mathbb{R}^m$.*

Acknowledgements

It is a pleasure to thank S. Hansoul and P. Lecomte for numerous fruitful discussions and for their interest in our work. We thank D. Sternheimer for his comments and suggestions. We are grateful to C. Roberts, D.J. Saunders, M. Crampin and W. Bertram for their help and the references they gave us about Thomas-Whitehead and Cartan connections.

F. Radoux thanks the Belgian FRIA for his Research Fellowship.

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Quantum mechanics as a deformation of classical mechanics. *Lett. Math. Phys.*, 1(6):521–530, 1975/77.

[2] F. Boniver, S. Hansoul, P. Mathonet, and N. Poncin. Equivariant symbol calculus for differential operators acting on forms. *Lett. Math. Phys.*, 62(3):219–232, 2002.

[3] F. Boniver and P. Mathonet. IFFT-equivariant quantizations. *J. Geom. Phys.*, 56(4):712–730, 2006.

[4] M. Bordemann. Sur l'existence d'une prescription d'ordre naturelle projectivement invariante. *Submitted for publication, math.DG/0208171.*

[5] Sofiane Bouarroudj. Projectively equivariant quantization map. *Lett. Math. Phys.*, 51(4):265–274, 2000.

[6] Sofiane Bouarroudj. Formula for the projectively invariant quantization on degree three. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(4):343–346, 2001.

[7] Ranee Brylinski. Nonlocality of equivariant star products on $T^*(\mathbf{R}\mathrm{P}^n)$. *Lett. Math. Phys.*, 58(1):21–28, 2001.

[8] A. Čap, J. Slovák, and V. Souček. Invariant operators on manifolds with almost Hermitian symmetric structures. I. Invariant differentiation. *Acta Math. Univ. Comenian. (N.S.)*, 66(1):33–69, 1997.

[9] A. Čap, J. Slovák, and V. Souček. Invariant operators on manifolds with almost Hermitian symmetric structures. II. Normal Cartan connections. *Acta Math. Univ. Comenian. (N.S.)*, 66(2):203–220, 1997.

[10] Elie Cartan. Sur les variétés à connexion projective. *Bull. Soc. math. France*, 52:205–241, 1924.

[11] C. Duval, P. Lecomte, and V. Ovsienko. Conformally equivariant quantization: existence and uniqueness. *Ann. Inst. Fourier (Grenoble)*, 49(6):1999–2029, 1999.

[12] C. Duval and V. Ovsienko. Projectively equivariant quantization and symbol calculus: noncommutative hypergeometric functions. *Lett. Math. Phys.*, 57(1):61–67, 2001.

[13] Sarah Hansoul. Projectively equivariant quantization for differential operators acting on forms. *Lett. Math. Phys.*, 70(2):141–153, 2004.

[14] James Hebda and Craig Roberts. Examples of Thomas-Whitehead projective connections. *Differential Geom. Appl.*, 8(1):87–104, 1998.

[15] Shoshichi Kobayashi. *Transformation groups in differential geometry.* Springer-Verlag, New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70.

[16] Shoshichi Kobayashi and Tadashi Nagano. On projective connections. *J. Math. Mech.*, 13:215–235, 1964.

[17] Ivan Kolář, Peter W. Michor, and Jan Slovák. *Natural operations in differential geometry.* Springer-Verlag, Berlin, 1993.

[18] P. B. A. Lecomte and V. Yu. Ovsienko. Projectively equivariant symbol calculus. *Lett. Math. Phys.*, 49(3):173–196, 1999.

[19] Pierre B. A. Lecomte. Classification projective des espaces d'opérateurs différentiels agissant sur les densités. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(4):287–290, 1999.

[20] Pierre B. A. Lecomte. Towards projectively equivariant quantization. *Progr. Theoret. Phys. Suppl.*, (144):125–132, 2001. Noncommutative geometry and string theory (Yokohama, 2001).

[21] Craig Roberts. The projective connections of T. Y. Thomas and J. H. C. Whitehead applied to invariant connections. *Differential Geom. Appl.*, 5(3):237–255, 1995.

[22] Craig Roberts. Relating Thomas-Whitehead projective connections by a gauge transformation. *Math. Phys. Anal. Geom.*, 7(1):1–8, 2004.

[23] Tracy Yerkes Thomas. A projective theory of affinely connected manifolds. *Math. Z.*, 25:723–733, 1926.

[24] Oswald Veblen and Tracy Yerkes Thomas. The geometry of paths. *Trans. Amer. Math. Soc.*, 25(4):551–608, 1923.

[25] H. Weyl. Zur infinitesimalgeometrie;einordnung der projektiven und der konformen auffassung. *Göttingen Nachr.*, pages 99–122, 1921.

[26] J. H. C. Whitehead. The representation of projective spaces. *Ann. of Math. (2)*, 32(2):327–360, 1931.

[27] N. M. J. Woodhouse. *Geometric quantization*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1992. Oxford Science Publications.