AN EXPLICIT FORMULA FOR THE NATURAL AND CONFORMALLY INVARIANT QUANTIZATION

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ABSTRACT. In [10], P. Lecomte conjectured the existence of a natural and conformally invariant quantization. In [13], we gave a proof of this theorem thanks to the theory of Cartan connections. In this paper, we give an explicit formula for the natural and conformally invariant quantization of trace-free symbols thanks to the method used in [13] and to tools already used in [14] in the projective setting. This formula is extremely similar to the one giving the natural and projectively invariant quantization in [14].

Mathematics Subject Classification (2000) : 53B05, 53A30, 53D50, 53C10.
Key words : Conformal Cartan connections, differential operators, natural maps, quantization maps.

1. Introduction

A quantization can be defined as a linear bijection from the space $\mathcal{S}(M)$ of symmetric contravariant tensor fields on a manifold M (also called the space of *Symbols*) to the space $\mathcal{D}_{\frac{1}{2}}(M)$ of differential operators acting between half-densities.

It is known that there is no natural quantization procedure. In other words, the spaces of symbols and of differential operators are not isomorphic as representations of $\mathrm{Diff}(M)$.

The idea of equivariant quantization, introduced by P. Lecomte and V. Ovsienko in [11] is to reduce the group of local diffeomorphisms in the following way : if a Lie group G acts (locally) on a manifold M, the action can be lifted to tensor fields and to differential operators and symbols. A G-equivariant quantization is then a quantization that exchanges the actions of G on symbols and differential operators.

In [11] and [6], the authors considered the case of the projective group $PGL(m+1,\mathbb{R})$ acting on the manifold $M=\mathbb{R}^m$ by linear fractional transformations. This leads to the notion of projectively equivariant quantization or its infinitesimal counterpart, the $sl(m+1,\mathbb{R})$-equivariant quantization. In [4], the authors considered the group $SO(p+1,q+1)$ acting on the space

Date: August 26, 2009.

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$\mathbb{R}^{p+q}$ or on a manifold endowed with a flat pseudo-conformal structure of signature (p,q). In both situations (projective and conformal), the results were the existence and uniqueness of equivariant quantization in situations called *non-critical* situations. These results settled the problem of projectively- and conformally- equivariant quantizations in the framework of manifolds endowed with flat structures.

In [2], the author expressed the projectively equivariant quantization at the third order using a torsion-free connection in such a way that the quantization only depends on the *projective class* of the connection. In [5], the authors showed that the conformally equivariant quantization procedure can be expressed using the Levi-Civita connection associated to a pseudo-Riemannian metric in such a way that it only depends on the *conformal class* of the metric. They give in this reference an explicit formula at the second order for this conformal quantization.

The problem of the $sl(m+1,\mathbb{R})$ (resp. $so(p+1,q+1)$)-equivariant quantization on $\mathbb{R}^m$ has indeed a counterpart on an arbitrary manifold M. In [10], P. Lecomte conjectured the existence of a natural and projectively (resp. conformally) invariant quantization, i.e. a quantization procedure depending on a torsion-free connection (resp. pseudo-riemannian metric), that would be natural (in all arguments) and that would be left invariant by a projective (resp. conformal) change of connection (resp. metric).

In [1], M. Bordemann was the first one to prove the existence of an equivariant quantization on an arbitrary manifold by giving the proof of the existence of a natural and projectively equivariant quantization for differential operators acting between densities. We showed in [12] the existence of such a quantization for differential operators acting between arbitrary tensor fields using Cartan connections theory. Eventually, we proved in [13] the existence of a natural conformally equivariant quantization in a similar way.

The goal of this paper is to obtain an explicit formula on M for the natural and conformally invariant quantization of trace-free symbols in the case where the differential operators act beween densities. This task can be realized using tools exposed in [13] and [3].

The paper is organized as follows. In the first section, we recall briefly the notions exposed in [13] necessary to understand the article. In the second part, we calculate the explicit formula giving the natural and conformally invariant quantization for trace-free symbols on the Cartan fiber bundle using the method exposed in [13]. In the third section, we develop as in [14] this formula in terms of natural operators on the base manifold M, using tools explained in [3], in order to obtain the announced explicit formula. It constitutes the generalization to any weight of density of the formula given by M. Eastwood in [7] thanks to a completely different method. Moreover, Eastwood gives the formula under a different form. Remark that J. Kroeske gives too the formula thanks to a different method in a non-published thesis (see [9]).

2. Fundamental tools

Throughout this work, we let M be a smooth manifold of dimension $m \geq 3$.

2.1. **Tensor densities.** Denote by $\Delta^\lambda(\mathbb{R}^m)$ the one dimensional representation of $GL(m,\mathbb{R})$ given by

$$\rho(A)e = |detA|^{-\lambda}e, \quad \forall A \in GL(m,\mathbb{R}),\ \forall e \in \Delta^\lambda(\mathbb{R}^m).$$

The vector bundle of λ-densities is then defined by

$$P^1M \times_\rho \Delta^\lambda(\mathbb{R}^m) \to M,$$

where P^1M is the linear frame bundle of M.

Recall that the space $\mathcal{F}_\lambda(M)$ of smooth sections of this bundle, the space of λ-densities, can be identified with the space $C^\infty(P^1M, \Delta^\lambda(\mathbb{R}^m))_{GL(m,\mathbb{R})}$ of functions f such that

$$f(uA) = \rho(A^{-1})f(u) \quad \forall u \in P^1M,\ \forall A \in GL(m,\mathbb{R}).$$

2.2. **Differential operators and symbols.** As usual, we denote by $\mathcal{D}_{\lambda,\mu}(M)$ the space of differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$.

The space $\mathcal{D}_{\lambda,\mu}$ is filtered by the order of differential operators. The space of *symbols* is then the associated graded space of $\mathcal{D}_{\lambda,\mu}$. It is also known that the principal operators σ_l ($l \in \mathbb{N}$) allow to identify the space of symbols with the space of contravariant symmetric tensor fields with coefficients in δ-densities where $\delta = \mu - \lambda$ is the shift value.

More precisely, we denote by $S^l_\delta(\mathbb{R}^m)$ or simply S^l_δ the space $S^l\mathbb{R}^m \otimes \Delta^\delta(\mathbb{R}^m)$ endowed with the natural representation ρ of $GL(m,\mathbb{R})$. Then the vector bundle of symbols of degree l is

$$P^1M \times_\rho S^l_\delta(\mathbb{R}^m) \to M.$$

The space $\mathcal{S}^l_\delta(M)$ of symbols of degree l is then the space of smooth sections of this bundle, which can be identified with $C^\infty(P^1M, S^l_\delta(\mathbb{R}^m))_{GL(m,\mathbb{R})}$. Finally, the whole space of symbols is

$$\mathcal{S}_\delta(M) = \bigoplus_{l=0}^{\infty} \mathcal{S}^l_\delta(M),$$

endowed with the classical actions of diffeomorphisms and of vector fields.

2.3. **Natural and invariant quantizations.** A *quantization on M* is a linear bijection Q_M from the space of symbols $\mathcal{S}_\delta(M)$ to the space of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ such that

$$\sigma_l(Q_M(S)) = S,\ \forall S \in \mathcal{S}^l_\delta(M),\ \forall l \in \mathbb{N},$$

where σ_l is the principal symbol operator on the space of operators of order less or equal to l.

In the conformal sense, a *natural quantization* is a collection of quantizations Q_M depending on a pseudo-Riemannian metric such that

- For all pseudo-Riemannian metric g on M, $Q_M(g)$ is a quantization,
- If ϕ is a local diffeomorphism from M to N, then one has

$$Q_M(\phi^* g)(\phi^* S) = \phi^*(Q_N(g)(S)),$$

for all pseudo-Riemannian metrics g on N, and all $S \in \mathcal{S}_\delta(N)$.

Recall now that two pseudo-Riemannian metrics g and g' on a manifold M are conformally equivalent if and only if there exists a positive function f such that $g' = fg$.

A quantization Q_M is then *conformally invariant* if one has $Q_M(g) = Q_M(g')$ whenever g and g' are conformally equivalent.

2.4. **Conformal group and conformal algebra.** These tools were presented in details in [13, Section 3]. We give here the most important ones for this paper to be self-contained.

Given p and q such that $p + q = m$, we consider the conformal group $G = SO(p+1, q+1)$ and its following subgroup H:

$$H = \{ \begin{pmatrix} a^{-1} & 0 & 0 \\ a^{-1}A\xi^\sharp & A & 0 \\ \frac{1}{2a}|\xi|^2 & \xi & a \end{pmatrix} : A \in O(p,q), a \in \mathbb{R}_0, \xi \in \mathbb{R}^{m*} \}/\{\pm I_{m+2}\}.$$

The subgroup H is a semi-direct product $G_0 \rtimes G_1$. Here G_0 is isomorphic to $CO(p,q)$ and G_1 is isomorphic to $\mathbb{R}^{m*}$.

The Lie algebra of G is $\mathfrak{g} = so(p+1, q+1)$. It decomposes as a direct sum of subalgebras :

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{1}$$

where $\mathfrak{g}_{-1} \cong \mathbb{R}^m$, $\mathfrak{g}_0 \cong co(p,q)$, and $\mathfrak{g}_1 \cong \mathbb{R}^{m*}$.

This correspondence induces a structure of Lie algebra on $\mathbb{R}^m \oplus co(p,q) \oplus \mathbb{R}^{m*}$. It is easy to see that the adjoint actions of G_0 and of $co(p,q)$ on $\mathfrak{g}_{-1} = \mathbb{R}^m$ and on $\mathfrak{g}_1 = \mathbb{R}^{m*}$ coincides with the natural actions of $CO(p,q)$ and of $co(p,q)$. It is interesting for the sequel to note that :

$$[v, \xi] = v \otimes \xi + \xi(v) I_m - \xi^\sharp \otimes v^\flat,$$

if $v \in \mathfrak{g}_{-1}$, $\xi \in \mathfrak{g}_1$ and if I_m denotes the identity matrix of dimension m. The applications $\flat$ and $\sharp$ represent the classical isomorphisms between $\mathbb{R}^m$ and $\mathbb{R}^{m*}$ detailed in [13].

The Lie algebras corresponding to G_0, G_1 and H are respectively $\mathfrak{g}_0$, $\mathfrak{g}_1$, and $\mathfrak{g}_0 \oplus \mathfrak{g}_1$.

2.5. **Cartan fiber bundles.** It is well-known that there is a bijective and natural correspondence between the conformal structures on M and the reductions of P^1M to the structure group $G_0 \cong CO(p,q)$. The representations (V, ρ) of $GL(m, \mathbb{R})$ defined so far can be restricted to the group $CO(p,q)$. Therefore, once a conformal structure is given, i.e. a reduction P_0 of P^1M to G_0, we can identify tensors fields of type V as G_0-equivariant functions on P_0.

In [8], one shows that it is possible to associate at each G_0-structure P_0 a principal H-bundle P on M, this association being natural and obviously conformally invariant. Since H can be considered as a subgroup of G^2_m, this H-bundle can be considered as a reduction of P^2M. The relationship between conformal structures and reductions of P^2M to H is given by the following proposition.

Proposition 1. *There is a natural one-to-one correspondence between the conformal equivalence classes of pseudo-Riemannian metrics on M and the reductions of P^2M to H.*

Throughout this work, we will freely identify conformal structures and reductions of P^2M to H.

2.6. **Cartan connections.** Let L be a Lie group and L_0 a closed subgroup. Denote by $\mathfrak{l}$ and $\mathfrak{l}_0$ the corresponding Lie algebras. Let $N \to M$ be a principal L_0-bundle over M, such that $\dim M = \dim L/L_0$. A Cartan connection on N is an $\mathfrak{l}$-valued one-form ω on N such that

(1) If R_a denotes the right action of $a \in L_0$ on N, then $R_a^*\omega = Ad(a^{-1})\omega$,
(2) If k^* is the vertical vector field associated to $k \in \mathfrak{l}_0$, then $\omega(k^*) = k$,
(3) $\forall u \in N$, $\omega_u : T_uN \mapsto \mathfrak{l}$ is a linear bijection.

When considering in this definition a principal H-bundle P, and taking as group L the group G and for L_0 the group H, we obtain the definition of Cartan conformal connections.

If ω is a Cartan connection defined on an H-principal bundle P, then its curvature Ω is defined by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{2}$$

The notion of *Normal* Cartan connection is defined by natural conditions imposed on the components of the curvature.

Now, the following result ([8, p. 135]) gives the relationship between conformal structures and Cartan connections :

Proposition 2. *A unique normal Cartan conformal connection is associated to every conformal structure P. This association is natural.*

The connection associated to a conformal structure P is called the normal conformal connection of the conformal structure.

2.7. **Lift of equivariant functions.** In a previous subsection, we recalled how to associate an H-principal bundle P to a conformal structure P_0. We now recall how the densities and the symbols can be regarded as equivariant functions on P.

If (V, ρ) is a representation of G_0, then we may extend it to a representation (V, ρ') of H (see [13]). Now, using the representation ρ', we can recall the relationship between equivariant functions on P_0 and equivariant functions on P (see [3]): if we denote by p the projection $P \to P_0$, we have

Proposition 3. *If (V,ρ) is a representation of G_0, then the map*
$$p^* : C^\infty(P_0,V) \mapsto C^\infty(P,V) : f \mapsto f \circ p$$
defines a bijection from $C^\infty(P_0,V)_{G_0}$ to $C^\infty(P,V)_H$.

As we continue, we will use the representation ρ'_* of the Lie algebra of H on V. If we recall that this algebra is isomorphic to $\mathfrak{g}_0 \oplus \mathfrak{g}_1$, then we have
$$\rho'_*(A,\xi) = \rho_*(A), \quad \forall A \in \mathfrak{g}_0, \xi \in \mathfrak{g}_1. \tag{3}$$

2.8. **The application Q_ω.** The construction of the application Q_ω is based on the concept of invariant differentiation developed in [3]. Let us recall the definition :

Definition 1. If $f \in C^\infty(P,V)$ then $(\nabla^\omega)^k f \in C^\infty(P, \otimes^k \mathbb{R}^{m*} \otimes V)$ is defined by
$$(\nabla^\omega)^k f(u)(X_1,\ldots,X_k) = L_{\omega^{-1}(X_1)} \circ \ldots \circ L_{\omega^{-1}(X_k)} f(u)$$
for $X_1,\ldots,X_k \in \mathbb{R}^m$.

Definition 2. The map Q_ω is defined by its restrictions to $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$, $(k \in \mathbb{N})$: we set
$$Q_\omega(T)(f) = \langle T, (\nabla^\omega)^k f \rangle, \tag{4}$$
for all $T \in C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$ and $f \in C^\infty(P, \Delta^\lambda(\mathbb{R}^m))$.

Explicitly, when the symbol T writes $tA \otimes h_1 \otimes \cdots \otimes h_k$ for $t \in C^\infty(P)$, $A \in \Delta^\delta(\mathbb{R}^m)$ and $h_1,\cdots,h_k \in \mathbb{R}^m \cong \mathfrak{g}_{-1}$ then one has
$$Q_\omega(T)f = tA \circ L_{\omega^{-1}(h_1)} \circ \cdots \circ L_{\omega^{-1}(h_k)} f,$$
where t is considered as a multiplication operator.

2.9. **The map γ.**

Definition 3. We define γ on $\otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m)$ by
$$\begin{aligned}\gamma(h)(x_1 \otimes \cdots \otimes x_k \otimes A) = {} & \lambda \textstyle\sum_{i=1}^k tr([h,x_i]) x_1 \otimes \cdots (i) \cdots \otimes x_k \otimes A \\ & + \textstyle\sum_{i=1}^k \sum_{j>i} x_1 \otimes \cdots (i) \cdots \otimes \underbrace{[[h,x_i],x_j]}_{(j)} \otimes \cdots \otimes x_k \otimes A.\end{aligned}$$
for every $x_1,\cdots,x_k \in \mathfrak{g}_{-1}$, $A \in \Delta^\delta(\mathbb{R}^m)$ and $h \in \mathfrak{g}_1$. Then we extend it to $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$ by $C^\infty(P)$-linearity.

Definition 4. A trace-free symbol S is a symbol such that $i(g)S = 0$ if g is a metric belonging to the conformal structure P.

If S is an equivariant function representing a trace-free symbol, $i(g_0)S = 0$ if g_0 represents the canonical metric on $\mathbb{R}^m$ corresponding to the conformal structure P (see [13], section 3). It is then easy to show that

Proposition 4. *If S is a trace-free symbol of degree k,*
$\gamma(h)S = -k(\lambda m + k - 1) i(h) S$. In particular, $\gamma(h)S$ is trace-free.

2.10. **Casimir-like operators.** Recall that we can define an operator called the Casimir operator $C^\flat$ on $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$ (see [13]). This operator $C^\flat$ is semi-simple. The vector space $\otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m)$ can be decomposed as an $o(p,q)$-representation into irreducible components (since $o(p,q)$ is semi-simple):

$$\otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m) = \oplus_{s=1}^{n_k} I_{k,s}.$$

The restriction of $C^\flat$ to $C^\infty(P, I_{k,s})$ is then a scalar multiple of the identity.

We defined in [13] two other operators. If we denote respectively by $(e_1, \ldots, e_m)$ and $(\epsilon^1, \ldots, \epsilon^m)$ a basis of $\mathfrak{g}_{-1}$ and a basis of $\mathfrak{g}_1$ which are dual with respect to the Killing form of $so(p+1, q+1)$, then

Definition 5. The operator N^ω is defined on $C^\infty(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$ by

$$N^\omega = -2 \sum_{i=1}^m \gamma(\varepsilon^i) L_{\omega^{-1}}(e_i),$$

and we set

$$C^\omega := C^\flat + N^\omega.$$

2.11. **Construction of the quantization.** Recall that $\otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m)$ is decomposed as a representation of $o(p,q)$ as the direct sum of irreducible components $I_{k,s}$ with $0 \leq s \leq \frac{k}{2}$ (see [4]). Remark that if S is a trace-free symbol of degree k, then $S \in I_{k,0}$. Denote by $E_{k,s}$ the space $C^\infty(P, I_{k,s})$ and by $\alpha_{k,s}$ the eigenvalue of $C^\flat$ restricted to $E_{k,s}$.

The tree-like susbspace $\mathcal{T}_\gamma(I_{k,s})$ associated to $I_{k,s}$ is defined by

$$\mathcal{T}_\gamma(I_{k,s}) = \bigoplus_{l \in \mathbb{N}} \mathcal{T}_\gamma^l(I_{k,s}),$$

where $\mathcal{T}_\gamma^0(I_{k,s}) = I_{k,s}$ and $\mathcal{T}_\gamma^{l+1}(I_{k,s}) = \gamma(\mathfrak{g}_1)(\mathcal{T}_\gamma^l(I_{k,s}))$, for all $l \in \mathbb{N}$. The space $\mathcal{T}_\gamma^l(E_{k,s})$ is then defined in the same way. Since γ is $C^\infty(P)$-linear, this space is equal to $C^\infty(P, \mathcal{T}_\gamma^l(I_{k,s}))$.

Definition 6. A value of δ is *critical* if there exists k, s such that the eigenvalue $\alpha_{k,s}$ corresponding to an irreducible component $I_{k,s}$ of $\otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m)$ belongs to the spectrum of the restriction of $C^\flat$ to $\bigoplus_{l \geq 1} \mathcal{T}_\gamma^l(E_{k,s})$.

Recall now the following result :

Theorem 5. *If δ is not critical, for every T in $C^\infty(P, I_{k,s})$, (where $I_{k,s}$ is an irreducible component of $\otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m)$) there exists a unique function $\hat{T}$ in $C^\infty(P, \mathcal{T}_\gamma(I_{k,s}))$ such that*

$$\left\{ \begin{array}{lll} \hat{T} & = & T_k + \cdots + T_0, \quad T_k = T \\ C^\omega(\hat{T}) & = & \alpha_{k,s} \hat{T}. \end{array} \right. \tag{5}$$

This result allows to define the main ingredient in order to define the quantization : The "modification map", acting on symbols.

Definition 7. Suppose that δ is not critical. Then the map

$$R : \oplus_{k=0}^{\infty} C^{\infty}(P, S^k_\delta) \to \oplus_{k=0}^{\infty} C^{\infty}(P, \otimes^k \mathfrak{g}_{-1} \otimes \Delta^\delta(\mathbb{R}^m))$$

is the linear extension of the association $T \mapsto \hat{T}$.

And finally, the main result :

Theorem 6. *If δ is not critical, then the formula*

$$Q_M : (g, T) \mapsto Q_M(g, T)(f) = (p^*)^{-1}[Q_\omega(R(p^*T))(p^*f)],$$

(where Q_ω is given by (4)) defines a natural and conformally invariant quantization.

3. The first explicit formula

Define now the numbers γ_{2k-l} :

$$\gamma_{2k-l} = \frac{m + 2k - l - m\delta}{m}.$$

We will say that a value of δ is *critical* if there are $k, l \in \mathbb{N}$ such that $2 \leq l \leq k+1$ and $\gamma_{2k-l} = 0$.

We can then give the formula giving the natural and conformally invariant quantization in terms of the normal Cartan connection for the trace-free symbols (see [14] for the definitions of ∇^ω_s and Div^ω) :

Theorem 7. *If δ is not critical, then the collection of maps* $Q_M : S^2T^*M \times \mathcal{S}^k_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M)$ *defined by*

$$Q_M(g, S)(f) = {p^*}^{-1}\Big(\sum_{l=0}^{k} C_{k,l} \langle {Div^{\omega}}^{l} p^*S, {\nabla^{\omega}_s}^{k-l} p^*f\rangle\Big) \tag{6}$$

defines a conformally invariant natural quantization for the trace-free symbols if

$$C_{k,l} = \frac{(\lambda + \frac{k-1}{m}) \cdots (\lambda + \frac{k-l}{m})}{\gamma_{2k-2} \cdots \gamma_{2k-l-1}} \begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1.$$

Proof. Thanks to Theorem 5, to the definition of N^ω and to Proposition 4, one has

$$S_l = \frac{2\sum_{i=1}^m \gamma(\varepsilon^i) L_{\omega^{-1}}(e_i) S_{l+1}}{\alpha_{l,0} - \alpha_{k,0}}, \quad 0 \leq l \leq k-1.$$

One concludes using Proposition 4 and the fact that (see [4]) :

$$\alpha_{k,0} = 2k(1 - k + m(\delta - 1)) - m^2\delta(\delta - 1).$$

Indeed, if $(e_1, \ldots, e_m)$ and $(\epsilon^1, \ldots, \epsilon^m)$ denote respectively the canonical bases of $\mathbb{R}^m$ and $\mathbb{R}^{m*}$, $(e_1, \ldots, e_m)$ and $(-\epsilon^1, \ldots, -\epsilon^m)$ are Killing-dual with respect to the Killing form given in [4]. One applies eventually Theorem 6. □

4. The second explicit formula

In order to obtain an explicit formula for the quantization, we need to know the developments of the operators ∇^{ω^l} and Div^{ω^l} in terms of operators on M.

Let γ be a connection on P_0 corresponding to a covariant derivative ∇ and belonging to the underlying structure of a conformal structure P. Recall that γ is the Levi-Civita connection of a metric belonging to P. We denote by τ the corresponding function on P with values in $\mathfrak{g}_1$, by Γ the corresponding deformation tensor (see [3]) and by ω the normal Cartan connection on P.

Let (V, ρ) be a representation of G_0 inducing a representation (V, ρ_*) of $\mathfrak{g}_0$. If we denote by $\rho_*^{(l)}$ the canonical representation on $\otimes^l \mathfrak{g}_{-1}^* \otimes V$ and if $s \in C^\infty(P_0, V)_{G_0}$, then the development of $\nabla^{\omega^l}(p^*s)(X_1, \ldots, X_l)$ is obtained inductively as follows (see [3], [14]) :

$$\begin{aligned}\nabla^{\omega^l}(p^*s)(X_1, \ldots, X_l) = \rho_*^{(l-1)}&([X_l, \tau])(\nabla^{\omega^{l-1}}(p^*s))(X_1, \ldots, X_{l-1})\\ &+S_\tau(\nabla^{\omega^{l-1}}(p^*s))(X_1, \ldots, X_{l-1})\\ &+S_\nabla(\nabla^{\omega^{l-1}}(p^*s))(X_1, \ldots, X_{l-1})\\ &+S_\Gamma(\nabla^{\omega^{l-1}}(p^*s))(X_1, \ldots, X_{l-1}).\end{aligned}$$

Recall that S_τ replaces successively each τ by $-\frac{1}{2}[\tau, [\tau, X_l]]$, that S_∇ adds successively a covariant derivative on the covariant derivatives of Γ and s and that S_Γ replaces successively each τ by $\Gamma.X_l$.

Recall too that Γ is equal in the conformal case to (see [3]) :

$$\frac{-1}{m-2}(\mathrm{Ric} - \frac{g_0 \mathrm{R}}{2(m-1)}),$$

where Ric and R denote the equivariant functions on P representing respectively the Ricci tensor and the scalar curvature of the connection γ.

Proposition 8. *If $f \in C^\infty(P_0, \Delta^\lambda(\mathbb{R}^m))_{G_0}$, then $\nabla^{\omega^l}(p^*f)(X, \ldots, X)$ is equal to $g_0(X, X)T(X, \ldots, X)$, where $T \in C^\infty(P, \otimes^{l-2}\mathbb{R}^{m*} \otimes \Delta^\lambda(\mathbb{R}^m))$, plus a linear combination of terms of the form*

$$(\otimes^{n_{-1}}\tau \otimes p^*(\otimes^{n_{l-2}}\nabla^{l-2}\Gamma \otimes \ldots \otimes \otimes^{n_0}\Gamma \otimes \nabla^q f))(X, \ldots, X).$$

If we denote by $T(n_{-1}, \ldots, n_{l-2}, q)$ such a term, then $\nabla^{\omega^{l+1}}(p^*f)(X, \ldots, X)$ is equal to the corresponding linear combination of the following sums

$$\begin{aligned}(-\lambda m - 2l + n_{-1})T(n_{-1} + 1, \ldots, n_{l-2}, q) + T(n_{-1}, \ldots, n_{l-2}, q + 1)\\ + \sum_{j=-1}^{l-2} n_j T(n_{-1}, \ldots, n_j - 1, n_{j+1} + 1, \ldots, n_{l-2}, q)\end{aligned}$$

plus $g_0(X, X)T'(X, \ldots, X)$, where $T' \in C^\infty(P, \otimes^{l-1}\mathbb{R}^{m*} \otimes \Delta^\lambda(\mathbb{R}^m))$.

Proof. The proof is similar to the proof of Proposition 7 in [14]. □

One deduces easily from Proposition 8 the following corollary (see [14] for the definition of ∇_s) :

Proposition 9. *If $f \in C^\infty(P_0, \Delta^\lambda(\mathbb{R}^m))_{G_0}$, then $\nabla_s^{\omega^l}(p^*f)$ is equal to $g_0 \vee T$, where $T \in C^\infty(P, S^{l-2}\mathbb{R}^{m*} \otimes \Delta^\lambda(\mathbb{R}^m))$, plus a linear combination of terms of the form*

$$(\tau^{n_{-1}} \vee p^*((\nabla^{l-2}\Gamma)^{n_{l-2}} \vee \ldots \vee \Gamma^{n_0} \vee \nabla_s^q f)).$$

If we denote by $T(n_{-1}, \ldots, n_{l-2}, q)$ such a term, then $\nabla_s^{\omega^{l+1}}(p^*f)$ is equal to the corresponding linear combination of the following sums

$$(-\lambda m - 2l + n_{-1})T(n_{-1}+1, \ldots, n_{l-2}, q) + T(n_{-1}, \ldots, n_{l-2}, q+1)$$
$$+ \sum_{j=-1}^{l-2} n_j T(n_{-1}, \ldots, n_j - 1, n_{j+1}+1, \ldots, n_{l-2}, q)$$

plus $g_0 \vee T'$, where $T' \in C^\infty(P, S^{l-1}\mathbb{R}^{m*} \otimes \Delta^\lambda(\mathbb{R}^m))$.

Proof. The proof is similar to the proof of Proposition 8 in [14]. □

Remark that the action of the algorithm on the generic term of the part of the development of $\nabla_s^{\omega^l}(p^*f)$ that does not contain factors g_0 can be summarized. Indeed, this action gives first

$$(-\lambda m - 2l + n_{-1})T(n_{-1}+1, \ldots, n_{l-2}, q).$$

It gives next

$$n_{-1}T(n_{-1}-1, n_0+1, \ldots, n_{l-2}, q).$$

Finally, it makes act the covariant derivative ∇_s on

$$(\nabla_s^{l-2}\Gamma)^{n_{l-2}} \vee \ldots \vee \Gamma^{n_0} \vee \nabla_s^q f.$$

From now, we will denote by r the following multiple of the tensor Ric (recall that Ric is symmetric for a metric connection) :

$$r := \frac{1}{(2-m)}\mathrm{Ric}.$$

In the following proposition, *Div* denotes the divergence operator :

Proposition 10. *If $S \in C^\infty(P_0, \Delta^\delta\mathbb{R}^m \otimes S^k\mathbb{R}^m)_{G_0}$ is trace-free, then $Div^{\omega^l}(p^*S)$ is a linear combination of terms of the form*

$$\langle \tau^{n_{-1}} \vee p^*((\nabla_s^{k-2}r)^{n_{k-2}} \vee \ldots \vee r^{n_0}), p^*(Div^q S)\rangle.$$

If we denote by $T(n_{-1}, \ldots, n_{l-2}, q)$ such a term, then $Div^\omega T(n_{-1}, \ldots, n_{l-2}, q)$ is equal to

$$(\gamma_{2(k-l)-2}m + n_{-1})T(n_{-1}+1, \ldots, n_{l-2}, q) + T(n_{-1}, \ldots, n_{l-2}, q+1)$$
$$+ \sum_{j=-1}^{l-2} n_j T(n_{-1}, \ldots, n_j - 1, n_{j+1}+1, \ldots, n_{l-2}, q).$$

Proof. The proof is exactly similar to the one of Proposition 9 in [14], using the fact that S and its divergences are trace-free. □

Remark that the action of the algorithm on the generic term of the development of $Div^{\omega^l}(p^*S)$ can be summarized. Indeed, this action gives first

$$(\gamma_{2(k-l)-2}m + n_{-1})T(n_{-1}+1, \ldots, n_{l-2}, q).$$

It gives next

$$n_{-1}T(n_{-1}-1, n_0+1, \ldots, n_{l-2}, q).$$

Finally, it makes act the divergence Div on

$$\langle (\nabla_s^{k-2} r)^{n_{k-2}} \vee \ldots \vee r^{n_0}, Div^q S\rangle.$$

Because of the previous propositions, the quantization can be written as a linear combination of terms of the form

$$\langle\langle \tau^{n_{-1}} \vee p^*((\nabla_s^{k-2} r)^{n_{k-2}} \vee \ldots \vee r^{n_0}), p^*(Div^q S)\rangle, p^*(\nabla_s^l f)\rangle.$$

In this expression, recall that it suffices to consider the terms for which $n_{-1} = 0$ (see [14]).

In the sequel, we will need two operators that we will call T_1 and T_2.

If T is a tensor of type $\begin{pmatrix} 0 \\ j \end{pmatrix}$ with values in the λ-densities, then

$$T_1 T = (-\lambda m - j)(j+1)\Gamma \vee T.$$

If S is a trace-free symbol of degree j, then

$$T_2 S = (m\gamma_{2k-2} - k + j)(k - j + 1)i(r)S.$$

The following results give the explicit developments of $\nabla_s^{\omega^l}(p^*f)$ and of $Div^{\omega^l}(p^*S)$:

Proposition 11. *The term of degree t in τ in the part of the development of $\nabla_s^{\omega^l}(p^*f)$ that does not contain factors g_0 is equal to*

$$\begin{pmatrix} l \\ t \end{pmatrix} \prod_{j=1}^{t}(-\lambda m - l + j)p^*(\pi_{l-t}(\sum_{j=0}^{l-t}(\nabla_s + T_1)^j)f),$$

where π_{l-t} denotes the projection on the operators of degree $l-t$ (the degree of ∇_s is 1 whereas the degree of T_1 is 2). We set $\prod_{j=1}^{t}(-\lambda m - l + j)$ equal to 1 if $t = 0$.

Proof. The proof is exactly similar to the one of Proposition 10 in [14]. □

Proposition 12. *If S is trace-free, the term of degree t in τ in the development of $Div^{\omega^l}(p^*S)$ is equal to*

$$\begin{pmatrix} l \\ t \end{pmatrix} \prod_{j=1}^{t}(\gamma_{2k-2}m - l + j)p^*(\pi_{t-l}(\sum_{j=0}^{l-t}(Div + T_2)^j)S),$$

where π_{t-l} denotes the projection on the operators of degree $t-l$ (the degree of Div is -1 whereas the degree of T_2 is -2). We set the product $\prod_{j=1}^{t}(\gamma_{2k-2}m - l + j)$ equal to 1 if $t = 0$.

Proof. The proof is completely similar to the one of Proposition 11 in [14]. □

We can now write the explicit formula giving the natural and conformally invariant quantization for the trace-free symbols :

Theorem 13. *The quantization Q_M for the trace-free symbols is given by the following formula :*

$$Q_M(g,S)(f)=\sum_{l=0}^{k}C_{k,l}\langle\pi_l(\sum_{j=0}^{l}(Div+T_2)^j)S,\pi_{k-l}(\sum_{j=0}^{k-l}(\nabla_s+T_1)^j)f\rangle.$$

Remark that as S and its divergences are trace-free, one can replace in the definition of the operators T_1 the deformation tensor Γ by r. One can

easily derive from this formula the formula at the third order. Indeed, if we denote by D, T, ∂T the operators ∇_s, $r\vee$ and $(\nabla_s r)\vee$ (resp. Div, $i(r)$ and $i(\nabla_s r)$) and if we denote by β the number $-\lambda m$ (resp. $\gamma_4 m$), one obtains :

$$\pi_1(\sum_{j=0}^{1}(D+T)^j)=D,\quad \pi_2(\sum_{j=0}^{2}(D+T)^j)=D^2+\beta T,$$

$$\pi_3(\sum_{j=0}^{3}(D+T)^j)=D^3+\beta DT+2(\beta-1)TD=D^3+(3\beta-2)TD+\beta(\partial T).$$

We can then write the formula at the third order :

$$\begin{gathered}\langle S,(\nabla_s^3-(3m\lambda+2)r\vee\nabla_s-\lambda m(\nabla_s r))f\rangle\\ +C_{3,1}\langle DivS,(\nabla_s^2-m\lambda r)f\rangle+C_{3,2}\langle(Div^2+m\gamma_4 i(r))S,\nabla_s f\rangle\\ +C_{3,3}\langle(Div^3+(3\gamma_4 m-2)i(r)Div+m\gamma_4 i(\nabla_s r))S,f\rangle.\end{gathered}$$

At the second order, the formula is simply :

$$\langle S,(\nabla_s^2-m\lambda r)f\rangle+C_{2,1}\langle DivS,\nabla_s f\rangle+C_{2,2}\langle(Div^2+m\gamma_2 i(r))S,f\rangle.$$

5. Acknowledgements

It is a pleasure to thank P. Mathonet and V. Ovsienko for numerous fruitful discussions and for their interest in our work. We thank the Belgian FNRS for his Research Fellowship.

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