Affine bundles are affine spaces

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Abstract

We show that the category of affine bundles over a smooth manifold $M$ is equivalent to the category of affine spaces modeled on locally free $C^\infty(M)$-modules.

1. Differential Calculus in the language of Commutative Algebra

Differential Calculus on smooth manifolds (Classical Physics) can be developed in the language of commutative algebra (see [1]). Algebraic translation allows to define in an elegant and powerful way Differential Calculus on more “exotic” spaces (manifolds with boundaries, etc.). In practice, Differential Calculus is embedded into Commutative algebra according to the following dictionary:

Algebra:  
- $\mathbb{R}$-algebra $C^\infty(M)$  
- $\mathbb{R}$-homomorphism $f^* : C^\infty(M') \to C^\infty(M)$, $g \mapsto g \circ f$  

Geometry:  
- smooth manifold $M$  
- smooth map $f : M \to M'$  
- vector bundle $\eta : E \to M$  
- morphism of vector bundles $\eta \to \eta'$  
- affine bundle $\pi : Z \to M$ modeled on $\eta : E \to M$  
- morphism of affine bundles $\pi \to \pi'$

2. Affine bundles over vector bundles

Definition 1

An affine bundle modeled on a vector $\eta : E \to M$ is a fibered manifold $\pi : Z \to M$ that looks locally like the product of $M$ and an affine space. More precisely:

- there is an affine space $A_x$, modeled on the typical fiber $V_{\eta_x}$;  
- for every $x \in M$, $\pi_x := \pi^{-1}(x)$ is an affine space modeled on $\eta_x$;  
- the base $M$ is covered by fiberwise affine diffeomorphisms $ \pi^{-1}(U) \to U \times A$. 

An $M$-morphism of affine bundles $\pi \to \pi'$ is a fiberwise affine smooth map $\alpha : Z \to Z'$, i.e. for any $x \in M$, $f|_{\pi_x} : \pi_x \to \pi'_x$ is an affine map.

Remark

The choice of a global section $s_0 \in \Gamma(\pi)$ determines an isomorphism of affine bundles $\pi \cong \eta$, namely

$$T_{s_0} : Z \to E, \quad z \mapsto z - s_0(x) \iff z \in \pi_x$$

Thus, any affine bundle is isomorphic to the vector bundle on which it is modeled, but not canonically!

4. Affine bundles are torsors

Affine bundles are “affine spaces over vector bundles”:

Proposition 1

A fibered manifold $\pi : Z \to M$ is an affine bundle modeled on $\eta$ if there exists a fibered smooth map

$$t : E \times M \to Z, \quad (E \times M, t) \to \pi_x$$

such that for any $x \in M$, $t|_{\pi_{x} \times M} : \pi_x \times M \to \pi_x$ makes $\pi_x$ an affine space modeled on $\eta_x$.

5. Vector bundles are $C^\infty(M)$-modules

The space of sections $\Gamma(\eta)$ is a $C^\infty(M)$-module:

$$(s_1 + s_2)(x) := s_1(x) + s_2(x) \in \eta_x, \quad g(s)(x) := g(x)s(x) \in \eta_x$$

Moreover, any $M$-morphism of vector bundles $\beta : \eta \to \eta'$ induces a $C^\infty(M)$-linear map $\Gamma(\beta) : \Gamma(\eta) \to \Gamma(\eta')$, $s \mapsto \beta \circ s$

Theorem 1 (see [2])

The functor of sections $\Gamma : VB(M) \to \text{Mod}_{\text{locally free}} C^\infty(M)$ is an equivalence of categories.

6. Affine bundles are affine spaces

The space of sections $\Gamma(\pi)$ is a $C^\infty(M)$-affine space modeled on $\Gamma(\eta)$:

$$(\sigma + s)(x) := \sigma(x) + s(x) \in \pi_x$$

Moreover, any $M$-morphism of affine bundles $\alpha : \pi \to \pi'$ induces a $C^\infty(M)$-affine map

$$\Gamma(\alpha) : \Gamma(\pi) \to \Gamma(\pi'), \quad \sigma \mapsto \alpha \circ \sigma$$

Theorem 2

The functor of sections $\Gamma : VB(M) \to \text{AS}(\text{Mod}_{\text{locally free}} C^\infty(M))$ is an equivalence of categories.

Idea of the proof

Both concepts are torsors under group objects in equivalent categories (cf. Definition 2, Theorem 1 and Proposition 2).
