Affine bundles are affine spaces

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Abstract

We show that the category of affine bundles over a smooth manifold M is equivalent to the category of affine spaces modeled on locally free $\mathrm{C}^\infty(M)$-modules.

1. Differential Calculus in the language of Commutative Algebra

Differential Calculus on smooth manifolds (Classical Physics) can be developed in the language of commutative algebra (see [1]).
Algebraic translation allows to define in an *elegant and powerful* way Differential Calculus on more "exotic" spaces (manifolds with boundaries, etc.).
In practice, Differential Calculus is *embedded* into Commutative algebra according to the following *dictionary* :

Geometry :	➔	**Algebra :**
smooth manifold M; smooth map $f : M \to M'$		$\mathbb{R}$-algebra $\mathrm{C}^\infty(M)$; $\mathbb{R}$-homomorphism $f^* : \mathrm{C}^\infty(M') \to \mathrm{C}^\infty(M)$, $g \mapsto g \circ f$
vector bundle $\eta : E_\eta \to M$; morphism of vector bundles $\eta \to \eta'$		$\mathrm{C}^\infty(M)$-module of sections $\Gamma(\eta)$; $\mathrm{C}^\infty(M)$-linear map $\Gamma(\eta) \to \Gamma(\eta')$
affine bundle $\pi : Z_\pi \to M$ modeled on $\eta : E_\eta \to M$; morphism of affine bundles $\pi \to \pi'$		???; ???

2. Affine bundles over vector bundles

Definition 1
An affine bundle modeled on a vector $\eta : E_\eta \to M$ is a fibered manifold $\pi : Z_\pi \to M$ that looks locally like the product of M and an affine space. More precisely :

- there is an affine space A_π modeled on the typical fiber V_η ;
- for every $x \in M$, $\pi_x := \pi^{-1}(x)$ is an affine space modeled on η_x ;
- the base M is covered by fiberwise affine diffeomorphisms

$$\pi^{-1}(U) \xrightarrow{\sim} U \times A_\pi.$$

An M-morphism of affine bundles $\pi \to \pi'$ is a fiberwise affine smooth map $\alpha : Z_\pi \to Z_{\pi'}$, i.e. for any $x \in M$, $f|_{\pi_x} : \pi_x \to \pi'_x$ is an affine map.

Remark The choice of a global section $\sigma_0 \in \Gamma(\pi)$ determines an isomorphism of affine bundles $\pi \simeq \eta$, namely

$$T_{\sigma_0} : Z_\pi \to E_\eta$$
$$z \mapsto z - \sigma_0(x) \text{ if } z \in \pi_x$$

Thus, **any affine bundle is isomorphic to the vector bundle on which it is modeled, but not canonically !**

3. Affine spaces over modules

Definition 2
Let R be a commutative ring and P be an R-module. An R-affine space modeled on P is a set A together with a free and transitive group action

$$\mathrm{t} : P \times A \to A$$
$$(p, a) \mapsto \mathrm{t}_p(a).$$

We often write $a + p$ instead of $\mathrm{t}_p(a)$ and $a - a_0$ for the unique p "moving" a_0 to a.

A map $T : A \to A'$ is called an R-affine map if there is an R-linear map $\vec{T} : P \to P'$ such that

$$T(a + p) = T(a) + \vec{T}(p)$$

Remark The choice of an element $a_0 \in A$ determines an R-affine isomorphism $A \simeq P$, namely

$$T_{a_0} : A \to P$$
$$a \mapsto a - a_0.$$

Thus, **any R-affine space is isomorphic to the module on which it is modeled, but not canonically !**

4. Affine bundles are torsors

Affine bundles are "affine spaces over vector bundles" :

Proposition 1
A fibered manifold $\pi : Z_\pi \to M$ is an *affine bundle modeled on* η iff there exists a fibered smooth map

$$\mathrm{t} : E_\eta \times_M Z_\pi \to Z_\pi$$

such that for any $x \in M$, $\mathrm{t}|_{\eta_x \times \pi_x} : \eta_x \times \pi_x \to \pi_x$ makes π_x an affine space modeled on η_x.

5. Vector bundles are $\mathrm{C}^\infty(M)$-modules

The space of sections $\Gamma(\eta)$ is a $\mathrm{C}^\infty(M)$-module :

$$(s_1 + s_2)(x) := s_1(x) + s_2(x) \in \eta_x, \; g.s(x) := g(x)s(x) \in \eta_x.$$

Moreover, any M-morphism of vector bundles $\beta : \eta \to \eta'$ induces a $\mathrm{C}^\infty(M)$-linear map $\Gamma(\beta) : \Gamma(\eta) \to \Gamma(\eta')$, $s \mapsto \beta \circ s$

Theorem 1 (see [2])
The functor of sections $\Gamma : \mathrm{VB}(M) \to \mathrm{Mod}_{\text{locally free}}\mathrm{C}^\infty(M)$ is an equivalence of categories.

6. Affine bundles are affine spaces

The space of sections $\Gamma(\pi)$ is a $\mathrm{C}^\infty(M)$-affine space modeled on $\Gamma(\eta)$:

$$(\sigma + s)(x) := \sigma(x) + s(x) \in \pi_x$$

Moreover, any M-morphism of affine bundles $\alpha : \pi \to \pi'$ induces a $\mathrm{C}^\infty(M)$-affine map

$$\Gamma(\alpha) : \Gamma(\pi) \to \Gamma(\pi'), \; \sigma \mapsto \alpha \circ \sigma$$

Theorem 2
The functor of sections $\Gamma : \mathrm{AB}(M) \to \mathrm{AS}(\mathrm{Mod}_{\text{locally free}}\mathrm{C}^\infty(M))$ is an equivalence of categories.

Idea of the proof
Both concepts are torsors under group objects in equivalent categories (cf. Definition 2, Theorem 1 and Proposition 2).

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[1] Jet Nestruev. *Smooth manifolds and observables*, volume 220 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.

[2] D. Husemöller, M. Joachim, B. Jurčo, and M. Schottenloher. *Basic bundle theory and K-cohomology invariants*, volume 726 of *Lecture Notes in Physics*. Springer, Berlin, 2008.

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