

Abstract

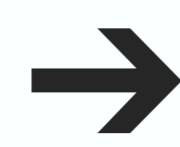
We show that the category of affine bundles over a smooth manifold M is equivalent to the category of affine spaces modeled on locally free $C^\infty(M)$ -modules.

1. Differential Calculus in the language of Commutative Algebra

Differential Calculus on smooth manifolds (Classical Physics) can be developed in the language of commutative algebra (see [1]). Algebraic translation allows to define in an *elegant and powerful* way Differential Calculus on more “exotic” spaces (manifolds with boundaries, etc.). In practice, Differential Calculus is *embedded* into Commutative algebra according to the following *dictionary* :

Geometry :

- { smooth manifold M
- { smooth map $f : M \rightarrow M'$
- { vector bundle $\eta : E_\eta \rightarrow M$
- { morphism of vector bundles $\eta \rightarrow \eta'$
- { **affine bundle** $\pi : Z_\pi \rightarrow M$ modeled on $\eta : E_\eta \rightarrow M$
- { **morphism of affine bundles** $\pi \rightarrow \pi'$



Algebra :

- { \mathbb{R} -algebra $C^\infty(M)$
- { \mathbb{R} -homomorphism $f^* : C^\infty(M') \rightarrow C^\infty(M), g \mapsto g \circ f$
- { $C^\infty(M)$ -module of sections $\Gamma(\eta)$
- { $C^\infty(M)$ -linear map $\Gamma(\eta) \rightarrow \Gamma(\eta')$
- { ???
- { ???

2. Affine bundles over vector bundles

Definition 1

An **affine bundle** modeled on a vector $\eta : E_\eta \rightarrow M$ is a fibered manifold $\pi : Z_\pi \rightarrow M$ that looks locally like the product of M and an affine space. More precisely :

- ▶ there is an affine space A_π modeled on the typical fiber V_η ;
- ▶ for every $x \in M$, $\pi_x := \pi^{-1}(x)$ is an affine space modeled on η_x ;
- ▶ the base M is covered by fiberwise affine diffeomorphisms

$$\pi^{-1}(U) \xrightarrow{\sim} U \times A_\pi.$$

An **M -morphism of affine bundles** $\pi \rightarrow \pi'$ is a fiberwise affine smooth map $\alpha : Z_\pi \rightarrow Z_{\pi'}$, i.e. for any $x \in M$, $f|_{\pi_x} : \pi_x \rightarrow \pi'_x$ is an affine map.

Remark The choice of a global section $\sigma_0 \in \Gamma(\pi)$ determines an isomorphism of affine bundles $\pi \simeq \eta$, namely

$$T_{\sigma_0} : Z_\pi \rightarrow E_\eta$$

$$z \mapsto z - \sigma_0(x) \text{ if } z \in \pi_x$$

Thus, **any affine bundle is isomorphic to the vector bundle on which it is modeled, but not canonically !**

4. Affine bundles are torsors

Affine bundles are “affine spaces over vector bundles” :

Proposition 1

A fibered manifold $\pi : Z_\pi \rightarrow M$ is an *affine bundle modeled on η* iff there exists a fibered smooth map

$$t : E_\eta \times_M Z_\pi \rightarrow Z_\pi$$

such that for any $x \in M$, $t|_{\eta_x \times \pi_x} : \eta_x \times \pi_x \rightarrow \pi_x$ makes π_x an affine space modeled on η_x .

6. Affine bundles are affine spaces

The space of sections $\Gamma(\pi)$ is a $C^\infty(M)$ -affine space modeled on $\Gamma(\eta)$:

$$(\sigma + s)(x) := \sigma(x) + s(x) \in \pi_x$$

Moreover, any M -morphism of affine bundles $\alpha : \pi \rightarrow \pi'$ induces a $C^\infty(M)$ -affine map

$$\Gamma(\alpha) : \Gamma(\pi) \rightarrow \Gamma(\pi'), \sigma \mapsto \alpha \circ \sigma$$

3. Affine spaces over modules

Definition 2

Let R be a commutative ring and P be an R -module. An **R -affine space** modeled on P is a set A together with a free and transitive group action

$$t : P \times A \rightarrow A$$

$$(p, a) \mapsto t_p(a).$$

We often write $a + p$ instead of $t_p(a)$ and $a - a_0$ for the unique p “moving” a_0 to a .

A map $T : A \rightarrow A'$ is called an **R -affine map** if there is an R -linear map $\vec{T} : P \rightarrow P'$ such that

$$T(a + p) = T(a) + \vec{T}(p)$$

Remark The choice of an element $a_0 \in A$ determines an R -affine isomorphism $A \simeq P$, namely

$$T_{a_0} : A \rightarrow P$$

$$a \mapsto a - a_0.$$

Thus, **any R -affine space is isomorphic to the module on which it is modeled, but not canonically !**

5. Vector bundles are $C^\infty(M)$ -modules

The space of sections $\Gamma(\eta)$ is a $C^\infty(M)$ -module :

$$(s_1 + s_2)(x) := s_1(x) + s_2(x) \in \eta_x, \quad g \cdot s(x) := g(x)s(x) \in \eta_x.$$

Moreover, any M -morphism of vector bundles $\beta : \eta \rightarrow \eta'$ induces a $C^\infty(M)$ -linear map $\Gamma(\beta) : \Gamma(\eta) \rightarrow \Gamma(\eta')$, $s \mapsto \beta \circ s$

Theorem 1 (see [2])

The functor of sections $\Gamma : \text{VB}(M) \rightarrow \text{Mod}_{\text{locally free } C^\infty(M)}$ is an equivalence of categories.

Theorem 2

The functor of sections $\Gamma : \text{AB}(M) \rightarrow \text{AS}(\text{Mod}_{\text{locally free } C^\infty(M)})$ is an equivalence of categories.

Idea of the proof

Both concepts are torsors under group objects in equivalent categories (cf. Definition 2, Theorem 1 and Proposition 2).

[1] Jet Nestruev. *Smooth manifolds and observables*, volume 220 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.

[2] D. Husemöller, M. Joachim, B. Jurčo, and M. Schottenloher. *Basic bundle theory and K-cohomology invariants*, volume 726 of *Lecture Notes in Physics*. Springer, Berlin, 2008.