Matrix Elements of SU(6) Generators for Baryons at Arbitrary N_c

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Abstract

We present explicit formulas for the matrix elements of the SU(6) generators for totally symmetric spin-flavor states, relevant for baryon spectroscopy in large N_c QCD. We rely on the interplay between two different methods to calculate these matrix elements. As an outcome, general analytic formulas of the corresponding SU(6) isoscalar factors are derived for arbitrary N_c. These results can be used to study excited states of nonstrange and strange baryons.

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I. INTRODUCTION

The large N_c limit of QCD suggested by 't Hooft [1] and the power counting rules of Witten [2] lead to the powerful $1/N_c$ expansion method to study baryon spectroscopy. The method is based on the result that the SU($2N_f$) spin-flavor symmetry, where N_f is the number of flavors, is exact in the large N_c limit of QCD [3]. For $N_c \to \infty$ the baryon masses are degenerate. For large N_c the mass splitting starts at order $1/N_c$ for the ground state. The method has been applied with great success to the ground state baryons (N = 0 band), described by the symmetric representation **56** of SU(6) [3, 4, 5, 6, 7, 8, 9]. Although the SU(6) symmetry is broken for excited states, it has been realized that the $1/N_c$ expansion can still be applied.

The excited states belonging to the $[\mathbf{70}, 1^-]$ multiplet (N = 1 band) [10, 11, 12, 13, 14, 15, 16, 17, 18, 19] have been studied extensively in SU(4) ($N_f = 2$). The approach has been extended to $N_f = 3$ in Ref. [20] and it included first order in SU(3) symmetry breaking. There are also a few studies of the physically important multiplets belonging to the N = 2 band. These are related to $[\mathbf{56}', 0^+]$ [21] in SU(4), to $[\mathbf{56}, 2^+]$ [22] in SU(6) and to $[\mathbf{70}, \ell^+]$ [23] in SU(4). The method of Ref. [22] has been applied to highly excited nonstrange and strange baryons [24] belonging to the $[\mathbf{56}, 4^+]$ multiplet (N = 4 band). Configuration mixing have also been discussed [25].

In calculating the mass spectrum, the general procedure is to split the baryon into an excited quark and a core. The latter is in its ground state for the N = 1 band but generally carries some excitation for N > 1 (for example the $[\mathbf{70}, \ell^+]$ multiplet [23]). The excitation is implemented into the orbital part of the wave function. The spin-flavor part of the core wave function remains always symmetric.

The building blocks of the mass operator describing baryons with u, d and s quarks are the excited quark operators formed of the SO(3) generators ℓ^i_q and of the SU(6) generators s^i, t^a and g^{ia} and the corresponding core operators ℓ^i_c, S^i_c, T^a_c and G^{ia}_c. The matrix elements of the excited quark are straightforward, as being single-particle operators. The matrix elements of the core operators S^i_c, T^a_c are also simple to calculate, while those of G^{ia}_c are more intricate. The purpose of this work is to derive explicit formulas for the matrix elements of these SU(6) generators, for arbitrary N_c. So far, only the SU(4) case was solved analytically [12] for G^{ia}_c and it was used as such in studies of nonstrange baryons, as for example, in Ref. [14].

Recently, explicit formulas for SU(3) Clebsch-Gordan coefficients, relevant for couplings of mesons to baryons at large N_c, have also been derived [26].

There are several ways to calculate the matrix elements of the SU(6) generators. One is the standard group theory method. It is the way Hecht and Pang [27] derived matrix elements of the SU(4) generators and it can straightforwardly be generalized to SU(6). The difficulty in using this method is that it involves the knowledge of isoscalar factors of SU(6). So far, the literature provides a few examples of isoscalar factors: $\mathbf{56} \times \mathbf{35} \to \mathbf{56}$ [28, 29], $\mathbf{35} \times \mathbf{35} \to \mathbf{35}$ [29] or $\mathbf{35} \times \mathbf{70} = \mathbf{20} + \mathbf{56} + 2 \times \mathbf{70} + \mathbf{540} + \mathbf{560} + \mathbf{1134}$ [30] which can be applied to baryons composed of three quarks or to pentaquarks.

Here we propose an alternative method, based on the decomposition of an SU(6) state into a product of SU(3) and SU(2) states. It involves knowledge of isoscalar factors of the permutation group S_n, with $n = N_c - 1$, for the core of a baryon with an arbitrary number N_c of quarks. As we shall see, these isoscalar factors can easily be derived in various ways.

We recall that the group SU(6) has 35 generators S_i, T_a, G_{ia} with $i = 1, 2, 3$ and $a = 1, 2, ..., 8$, where S_i are the generators of the spin subgroup SU(2) and T_a the generators of the flavor subgroup SU(3). The group algebra is

$$\begin{aligned}
[S_i, S_j] &= i\varepsilon_{ijk} S_k, \quad [T_a, T_b] = i f_{abc} T_c, \quad [S_i, T_a] = 0, \\
[S_i, G_{ia}] &= i\varepsilon_{ijk} G_{ka}, \quad [T_a, G_{ib}] = i f_{abc} G_{ic}, \\
[G_{ia}, G_{jb}] &= \frac{i}{4} \delta_{ij} f_{abc} T_c + \frac{i}{2} \varepsilon_{ijk} \left(\frac{1}{3} \delta_{ab} S_k + d_{abc} G_{kc} \right),
\end{aligned} \tag{1}$$

by which the normalization of the generators is fixed.

The structure of the paper is as follows. In the next section we recall the standard group theory method to calculate the matrix elements of the generators of an unitary group. In Sec. 3 we introduce the isoscalar factors of S_n needed to decompose a symmetric spin-flavor wave function of N_c quarks into its spin and flavor parts. In Section 4 we derive matrix elements of the SU(6) generators between symmetric SU(6) states by using the decomposition introduced in Sec. 3. In Sec. 5 we compare the results of the methods of Secs. 2 and 4. This allows us to obtain explicit formulas for the isoscalar factors of the SU(6) generators as a function of arbitrary N_c and spin S. In Sec. 6, we return to the SU(4) case for completeness and consistency. In the before last section we introduce the baryon mass operator for which the above matrix elements are needed. A summary is given in the last section.

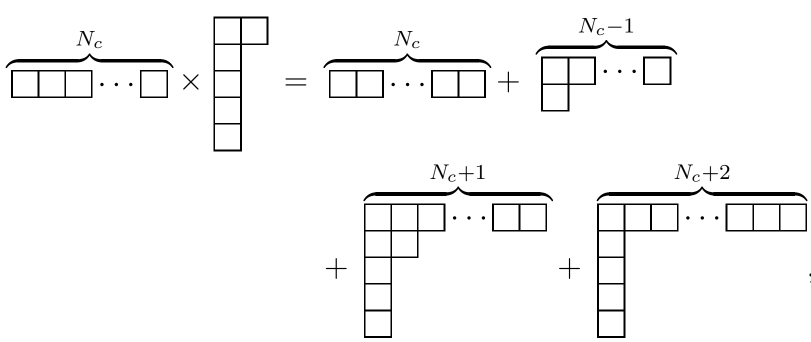
II. SU(6) GENERATORS AS TENSOR OPERATORS

The SU(6) generators are the components of an irreducible tensor operator which transform according to the adjoint representation $[21^4]$, equivalent to **35**, in dimensional notation. The matrix elements of any irreducible tensor can be expressed in terms of a generalized Wigner-Eckart theorem which is a factorization theorem, involving the product between a reduced matrix element and a Clebsch-Gordan (CG) coefficient. The case SU(4) $\supset$ SU(2) $\times$ SU(2) has been worked out by Hecht and Pang [27] and applied to nuclear physics.

Let us consider that the tensor operator $[21^4]$ acts on an SU(6) state of symmetry $[f]$. The symmetry of the final state, denoted by $[f']$, labels one of the irreducible representations (irreps) appearing in the Clebsch-Gordan series

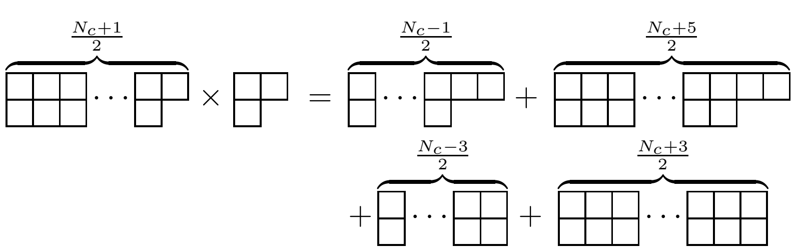
$$[f] \times [21^4] = \sum_{[f']} m_{[f']} [f'], \tag{2}$$

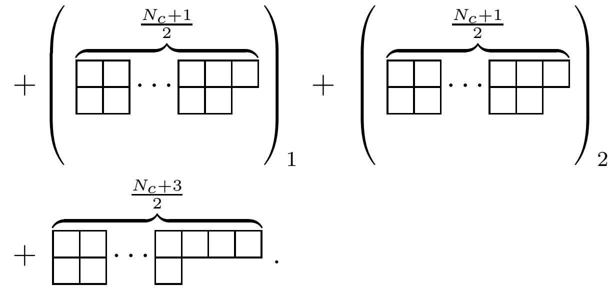
where $m_{[f']}$ denotes the multiplicity of the irrep $[f']$. The multiplicity problem arises if $[f'] = [f]$. An extra label ρ is then necessary. It is not the case here in connection with SU(6). Indeed, if $N_c = 3$, for the symmetric state **56**, one has $\mathbf{56}\times\mathbf{35} \to \mathbf{56}$ with multiplicity 1. For arbitrary N_c and $[f] = [N_c]$ the reduction (2) in terms of Young diagrams reads



(3)

which gives $m_{[f']} = 1$ for all terms, including the case $[f'] = [f]$. But the multiplicity problem arises at the level of the subgroup SU(3). Introducing the SU(3) $\times$ SU(2) content of the **56** and **35** irreps into their direct product one finds that the product $\mathbf{8} \times \mathbf{8}$ appears twice. For arbitrary N_c, this product is given by





(4)

In the following, ρ is used to distinguish between various $\mathbf{8} \times \mathbf{8}$ products. Then this label is carried over by the isoscalar factors of SU(6) (see below). In particle physics one uses the label s for the symmetric and a for the antisymmetric product (see *e.g.* [28]). Here we shall use the notation ρ of Hecht [31]. For $N_c = 3$ the relation to other labels is: $\rho = 1$ corresponds to $(8 \times 8)_a$ or to $(8 \times 8)_2$ of De Swart [32]; $\rho = 2$ corresponds to $(8 \times 8)_s$ or to $(8 \times 8)_1$ of De Swart. Throughout the paper, we shall use the SU(3) notations and phase conventions of Hecht [31]. Accordingly an irreducible representation of SU(3) carries the label $(\lambda\mu)$, introduced by Elliott [33], who applied SU(3) for the first time in physics, to describe rotational bands of deformed nuclei [34]. In particle physics the corresponding notation is (p, q). By analogy to SU(4) [27] one can write the matrix elements of the SU(6) generators E_{ia} as

$$\langle [N_c](\lambda'\mu')Y'I'I'_3S'S'_3|E_{ia}|[N_c](\lambda\mu)YII_3SS_3\rangle = \sqrt{C(\mathrm{SU}(6))} \left(\begin{array}{cc|c} S & S^i & S' \\ S_3 & S^i_3 & S'_3 \end{array}\right) \left(\begin{array}{cc|c} I & I^a & I' \\ I_3 & I^a_3 & I'_3 \end{array}\right)$$
$$\times \sum_{\rho=1,2} \left(\begin{array}{cc||c} (\lambda\mu) & (\lambda^a\mu^a) & (\lambda'\mu') \\ YI & Y^aI^a & Y'I' \end{array}\right)_\rho \left(\begin{array}{cc||c} [N_c] & [21^4] & [N_c] \\ (\lambda\mu)S & (\lambda^a\mu^a)S^i & (\lambda'\mu')S' \end{array}\right)_\rho, \quad (5)$$

where $C(\mathrm{SU}(6)) = 5[N_c(N_c + 6)]/12$ is the Casimir operator of SU(6), followed by Clebsch-Gordan coefficients of SU(2)-spin and SU(2)-isospin. The sum over ρ is over terms containing products of isoscalar factors of SU(3) and SU(6) respectively. We introduce T_a as an SU(3) irreducible tensor operator of components $T^{(11)}_{Y^aI^a}$. It is a scalar in SU(2) so that the index i is no more necessary. The generators S_i form a rank 1 tensor in SU(2) which is a scalar in SU(3), so the index i suffices. Although we use the same symbol for the operator S_i and its quantum numbers we hope that no confusion is created. The relation with the algebra (1) is

$$E_i = \frac{S_i}{\sqrt{3}}; \quad E_a = \frac{T_a}{\sqrt{2}}; \quad E_{ia} = \sqrt{2}G_{ia}. \quad (6)$$

Thus, for the generators S_i and T_a, which are elements of the $su(2)$ and $su(3)$ subalgebras of (1), the above expression simplifies considerably. In particular, as S_i acts only on the spin part of the wave function, we apply the usual Wigner-Eckart theorem for SU(2) to get

$$\langle [N_c](\lambda'\mu')Y'I'I'_3; S'S'_3|S_i|[N_c](\lambda\mu)YII_3; SS_3\rangle = \delta_{SS'}\delta_{\lambda\lambda'}\delta_{\mu\mu'}\delta_{YY'}\delta_{II'}\delta_{I_3I'_3}$$
$$\times\sqrt{C(\mathrm{SU}(2))}\left(\begin{array}{cc|c} S & 1 & S' \\ S_3 & i & S'_3 \end{array}\right), \tag{7}$$

with $C(\mathrm{SU}(2)) = S(S+1)$. Similarly, we use the Wigner-Eckart theorem for T_a which is a generator of the subgroup SU(3)

$$\langle [N_c](\lambda'\mu')Y'I'I'_3; S'S'_3|T_a|[N_c](\lambda\mu)YII_3; SS_3\rangle = \delta_{SS'}\delta_{S_3S'_3}\delta_{\lambda\lambda'}\delta_{\mu\mu'}$$
$$\times \sum_{\rho=1,2}\langle(\lambda'\mu')||T^{(11)}||(\lambda\mu)\rangle_\rho \left(\begin{array}{cc|c} (\lambda\mu) & (11) & (\lambda'\mu') \\ YII_3 & Y^aI^aI^a_3 & Y'I'I'_3 \end{array}\right)_\rho, \tag{8}$$

where the reduced matrix element is defined as [31]

$$\langle(\lambda\mu)||T^{(11)}||(\lambda\mu)\rangle_\rho = \begin{cases} \sqrt{C(\mathrm{SU}(3))} & \text{for } \rho = 1 \\ 0 & \text{for } \rho = 2 \end{cases}, \tag{9}$$

in terms of the eigenvalue of the Casimir operator $C(\mathrm{SU}(3)) = \frac{1}{3}g_{\lambda\mu}$ where

$$g_{\lambda\mu} = \lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu. \tag{10}$$

The SU(3) CG coefficient factorizes into an SU(2)-isospin CG coefficient and an SU(3) isoscalar factor [32]

$$\left(\begin{array}{cc|c} (\lambda\mu) & (11) & (\lambda'\mu') \\ YII_3 & Y^aI^aI^a_3 & Y'I'I'_3 \end{array}\right)_\rho = \left(\begin{array}{cc|c} I & I^a & I' \\ I_3 & I^a_3 & I'_3 \end{array}\right)\left(\begin{array}{cc||c} (\lambda\mu) & (11) & (\lambda'\mu') \\ YI & Y^aI^a & Y'I' \end{array}\right)_\rho. \tag{11}$$

The ρ dependence is consistent with Eq. (5) and reflects the multiplicity problem appearing in Eq. (19) below. We shall return to this point in Sec. V.

III. SU(6) SYMMETRIC WAVE FUNCTIONS

Here we consider a wave function which is symmetric in the spin-flavor space. To write its decomposition into its SU(2)-spin and SU(3)-flavor parts, one can use the Kronecker or inner product of the permutation group S_n. The advantage is that one can treat the permutation symmetry separately in each degree of freedom [34]. A basis vector $|[f]Y\rangle$ of an irreducible representation of S_n is completely defined by the partition $[f]$, and by a Young tableau Y or its equivalent, an Yamanouchi symbol. In the following we do not need to specify the full Young tableau, we only need to know the position p of the last particle in each tableau. In this short-hand notation a symmetric state of N_c quarks is $|[N_c]1]\rangle$, because $p = 1$. A symmetric spin-flavor wave function can be obtained from the product $[f'] \times [f'']$ of spin and flavor states of symmetries $[f']$ and $[f'']$ respectively, provided $[f'] = [f'']$.

Let us consider a system of N_c quarks having a total spin S. The group SU(2) allows only partitions with maximum two rows, in this case with $N_c/2 + S$ boxes in the first row and $N_c/2 - S$ in the second row. So, one has

$$[f'] = [\frac{N_c}{2} + S, \frac{N_c}{2} - S]. \tag{12}$$

By using the Clebsch-Gordan coefficients of S_n and their factorization property, described in the Appendix A, one can write a symmetric state of N_c particles with spin S as the linear combination

$$|[N_c]1\rangle \; = \; c_{11}^{[N_c]}(S)|[f']1\rangle|[f']1\rangle + c_{22}^{[N_c]}(S)|[f']2\rangle|[f']2\rangle, \tag{13}$$

where the coefficients $c_{pp}^{[\mathrm{N_c}]}(p = 1, 2)$ in the right-hand side are isoscalar factors of the permutation group. Their meaning is the following. The square of the first (second) coefficient is the fraction of Young tableaux of symmetry $[f']$ having the last particle of both states $|[f']p\rangle$ in the first (second) row. That is why they carry the double index 11 and 22 respectively, one index for each state. Examples of such isoscalar factors can be found in Ref. [35]. In the following, the first index refers to the spin part of the wave function and the second index to the flavor part. The total number of Young tableaux gives the dimension of the irrep $[f']$, so that the sum of squares of the two isoscalar factors is equal to one.

In the context of SU(6) $\supset$ SU(2) $\times$ SU(3) there are two alternative forms of each $c_{pp}^{[N_c]}$. They are derived in Appendix B. The first form is

$$\begin{aligned} c_{11}^{[N_c]}(S) \; &= \; \sqrt{\frac{S[N_c + 2(S+1)]}{N_c(2S+1)}}, \\ c_{22}^{[N_c]}(S) \; &= \; \sqrt{\frac{(S+1)(N_c - 2S)}{N_c(2S+1)}}. \end{aligned} \tag{14}$$

These expressions were obtained by acting with S_i on the spin part of the total wave function and by calculating the matrix elements of S_i in two different ways, one involving the Wigner-Eckart theorem and the other the linear combination (13). The coefficients (14) are precisely the so called "elements of orthogonal basis rotation" of Refs. [14] with the identification $c_{11}^{[N_c]} = c_{0-}^{\mathrm{SYM}}$ and $c_{22}^{[N_c]} = c_{0+}^{\mathrm{SYM}}$. The other form of the same coefficients, obtained by acting with T_a on the flavor part of the total wave function is

$$\begin{aligned} c_{11}^{[N_c]}(\lambda\mu) &= \sqrt{\frac{2g_{\lambda\mu} - N_c(\mu - \lambda + 3)}{3N_c(\lambda+1)}}, \\ c_{22}^{[N_c]}(\lambda\mu) &= \sqrt{\frac{N_c(6 + 2\lambda + \mu) - 2g_{\lambda\mu}}{3N_c(\lambda+1)}}, \end{aligned} \tag{15}$$

with $g_{\lambda\mu}$ given by Eq. (10). One can use either form, (14) or (15), depending on the SU(2) or the SU(3) context of the quantity to calculate. The best is to use the version which leads to simplifications. One can easily see that the expressions (14) and (15) are equivalent to each other. By making the replacement $\lambda = 2S$ and $\mu = N_c/2 - S$ in (15) one obtains (14).

The coefficients c_{0+}^{MS} and c_{0-}^{MS} of Refs. [14], are also isoscalar factors of the permutation group. They are needed to construct a mixed symmetric state from the inner product of S_n which generated the symmetric state as well. As shown in Appendix A, they can be obtained from orthogonality relations. The identification is $c_{11}^{[N_c-1,1]} = c_{0-}^{\mathrm{MS}}$ and $c_{22}^{[N_c-1,1]} = c_{0+}^{\mathrm{MS}}$. There are also the coefficients $c_{12}^{[N_c-1,1]} = c_{++}^{\mathrm{MS}}$ and $c_{21}^{[N_c-1,1]} = c_{--}^{\mathrm{MS}}$.

IV. MATRIX ELEMENTS OF THE SU(6) GENERATORS

Besides the standard group theory method of Sec. 2, another method to calculate the matrix elements of the SU(6) generators is based on the decoupling of the last particle from the rest, in each part of the wave function. This is easily done inasmuch as the row p of the last particle in a Young tableau is specified.

Let us first consider the spin part. The decoupling is

$$|S_1, 1/2; SS_3; p\rangle = \sum_{m_1,m_2} \left(\begin{array}{cc|c} S_1 & 1/2 & S \\ m_1 & m_2 & S_3 \end{array} \right) |S_1, m_1\rangle |1/2, m_2\rangle, \tag{16}$$

in terms of an SU(2)-spin CG coefficient with $S_1 = S - 1/2$ for $p = 1$ and $S_1 = S + 1/2$ for $p = 2$.

As for the flavor part, a wave function of symmetry $(\lambda\mu)$ with the last particle in the row p decouples to

$$|(\lambda_1\mu_1)(10); (\lambda\mu)YII_3; p\rangle =$$

$$\sum_{Y_1,I_1,I_{1_3},Y_2,I_2,I_{2_3}} \left(\begin{array}{cc|c} (\lambda_1\mu_1) & (10) & (\lambda\mu) \\ Y_1 I_1 I_{1_3} & Y_2 I_2 I_{2_3} & Y I I_3 \end{array} \right) |(\lambda_1\mu_1) Y_1 I_1 I_{1_3}\rangle |(10) Y_2 I_2 I_{2_3}\rangle, \tag{17}$$

where $(\lambda_1\mu_1) = (\lambda-1,\mu)$ for $p=1$ and $(\lambda_1\mu_1) = (\lambda+1,\mu-1)$ for $p=2$.

Now we use the fact that S_i, T_a and G_{ia} are one-body operators, *i.e.* their general form is

$$O = \sum_{i=1}^{N_c} O(i).$$

The expectation value of O between symmetric states is equal to N_c times the expectation value of any $O(i)$. Taking $i = N_c$ one has

$$\langle O \rangle = N_c \langle O(N_c) \rangle. \tag{18}$$

This means that one can reduce the calculation of $\langle O \rangle$ to the calculation of $\langle O(N_c) \rangle$,

To proceed, we recall that the flavor part of G_{ia} is a $T^{(11)}$ tensor in SU(3). To find out its matrix elements we have to consider the direct product

$$\begin{aligned} (\lambda\mu) \times (11) = &(\lambda+1,\mu+1) + (\lambda+2,\mu-1) + (\lambda\mu)_1 + (\lambda\mu)_2 \\ &+ (\lambda-1,\mu+2) + (\lambda-2,\mu+1) + (\lambda+1,\mu-2) + (\lambda-1,\mu-1). \end{aligned} \tag{19}$$

From the right-hand side, the only terms which give non-vanishing matrix elements of G_{ia} are $(\lambda'\mu') = (\lambda\mu), (\lambda+2,\mu-1)$ and $(\lambda-2,\mu+1)$, *i.e.* those with the same number of boxes, equal to $\lambda+2\mu$, as on the left-hand side. In addition, as G_{ia} is a rank 1 tensor in SU(2) it has non-vanishing matrix elements only for $S' = S, S\pm 1$, *i.e.* again only three distinct possibilities. The proper combinations of flavor and spin parts to get $|[N_c]1\rangle$ will be seen in the following subsections.

A. Diagonal matrix elements of G_{ia}

The diagonal matrix element have $(\lambda'\mu') = (\lambda\mu)$ and $S' = S$. The first step is to use the relation (18) and the factorization (13) of the spin-flavor wave function into its spin and flavor parts. This gives

$$\begin{aligned} &\langle [N_c](\lambda\mu) Y' I' I'_3 S S'_3 | G_{ia} | [N_c](\lambda\mu) Y I I_3 S S_3 \rangle = N_c \\ &\times \sum_{p=1,2} \left(c^{[N_c]}_{pp}(S) \right)^2 \langle S_1 1/2; S S'_3; p \,| s_i(N_c) |\, S_1 1/2; S S_3; p \rangle \\ &\times \langle (\lambda_1\mu_1)(10); (\lambda\mu) Y' I' I'_3; p \,| t_a(N_c) |\, (\lambda_1\mu_1)(10); (\lambda\mu) Y I I_3; p \rangle, \end{aligned} \tag{20}$$

where s_i and t_a are the SU(2) and SU(3) generators of the N_c-th particle respectively. The matrix elements of $s_i(N_c)$ between the states (16) are

$$\begin{aligned}
&\langle S_1 1/2; SS'_3; p|s_i(N_c)|S_1 1/2; SS_3; p\rangle = \\
&\sqrt{\frac{3}{4}} \sum_{m_1 m_2 m'_2} \left(\begin{array}{cc|c} S_1 & 1/2 & S \\ m_1 & m_2 & S_3 \end{array}\right) \left(\begin{array}{cc|c} S_1 & 1/2 & S \\ m_1 & m'_2 & S'_3 \end{array}\right) \left(\begin{array}{cc|c} 1/2 & 1 & 1/2 \\ m_2 & i & m'_2 \end{array}\right) \\
&= (-)^{S+S_1-1/2} \sqrt{\frac{3}{2}\,(2S+1)} \left(\begin{array}{cc|c} S & 1 & S \\ S_3 & i & S'_3 \end{array}\right) \left\{\begin{array}{ccc} 1 & S & S \\ S_1 & 1/2 & 1/2 \end{array}\right\}.
\end{aligned} \tag{21}$$

The matrix elements of the single particle operator t_a between the states (17) are

$$\begin{aligned}
&\langle (\lambda_1\mu_1)(10); (\lambda\mu)Y'I'I'_3; p|t_a(N_c)|(\lambda_1\mu_1)(10); (\lambda\mu)YII_3; p\rangle = \\
&\sqrt{\frac{4}{3}} \sum_{Y_1 I_1 I_{1_3} Y_2 I_2 I_{2_3} Y'_2 I'_2 I'_{2_3}} \left(\begin{array}{cc|c} (\lambda_1\mu_1) & (10) & (\lambda\mu) \\ Y_1 I_1 I_{1_3} & Y_2 I_2 I_{2_3} & Y I I_3 \end{array}\right) \left(\begin{array}{cc|c} (\lambda_1\mu_1) & (10) & (\lambda\mu) \\ Y_1 I_1 I_{1_3} & Y'_2 I'_2 I'_{2_3} & Y' I' I'_3 \end{array}\right) \\
&\times \left(\begin{array}{cc|c} (10) & (11) & (10) \\ Y_2 I_2 I_{2_3} & Y^a I^a I^a_3 & Y'_2 I'_2 I'_{2_3} \end{array}\right) \\
&= \sqrt{\frac{4}{3}} \left(\begin{array}{cc|c} I & I^a & I' \\ I_3 & I^a_3 & I'_3 \end{array}\right) \sum_{\rho=1,2} \left(\begin{array}{cc||c} (\lambda\mu) & (11) & (\lambda\mu) \\ YI & Y^a I^a & Y'I' \end{array}\right)_\rho \\
&\times U((\lambda_1,\mu_1)(10)(\lambda\mu)(11); (\lambda\mu)(10))_\rho\,,
\end{aligned} \tag{22}$$

where U are SU(3) Racah coefficients [39]. Note that the sum over ρ is consistent with Eq. (5) and expresses the fact that the direct product $(\lambda\mu) \times (11) \rightarrow (\lambda\mu)$ has multiplicity 2 in the reduction from SU(6) to SU(3), as discussed in Sec. 2.

Introducing (21) and (22) into (20) one obtains

$$\begin{aligned}
&\langle [N_c](\lambda\mu)Y'I'I'_3; SS'_3|G_{ia}|[N_c](\lambda\mu)YII_3; SS_3\rangle = (-)^{2S} N_c \sqrt{2(2S+1)} \\
&\times \left(\begin{array}{cc|c} S & 1 & S \\ S_3 & i & S'_3 \end{array}\right) \left(\begin{array}{cc|c} I & I^a & I' \\ I_3 & I^a_3 & I'_3 \end{array}\right) \sum_{\rho=1,2} \left(\begin{array}{cc||c} (\lambda\mu) & (11) & (\lambda\mu) \\ YI & Y^a I^a & Y'I' \end{array}\right)_\rho \\
&\times \left[\left(c^{[N_c]}_{22}(S)\right)^2 \left\{\begin{array}{ccc} S+1/2 & 1/2 & S \\ 1 & S & 1/2 \end{array}\right\} U((\lambda+1,\mu-1)(10)(\lambda\mu)(11); (\lambda\mu)(10))_\rho \right. \\
&\left. - \left(c^{[N_c]}_{11}(S)\right)^2 \left\{\begin{array}{ccc} S-1/2 & 1/2 & S \\ 1 & S & 1/2 \end{array}\right\} U((\lambda-1,\mu)(10)(\lambda\mu)(11); (\lambda\mu)(10))_\rho \right],
\end{aligned} \tag{23}$$

where $c^{[N_c]}_{pp}$ are given by Eqs. (14) or by the equivalent form (15). Using the definition of U given in Ref. [31], Table 4 of the same reference and Table 1 of Ref. [36] we have obtained the following expressions

$$U((\lambda+1,\mu-1)(10)(\lambda\mu)(11);(\lambda\mu)(10))_{\rho=1} = \frac{\mu-\lambda+3}{4\sqrt{g_{\lambda\mu}}}, \tag{24}$$

$$U((\lambda-1,\mu)(10)(\lambda\mu)(11);(\lambda\mu)(10))_{\rho=1} = \frac{\mu+2\lambda+6}{4\sqrt{g_{\lambda\mu}}}, \tag{25}$$

$$U((\lambda+1,\mu-1)(10)(\lambda\mu)(11);(\lambda\mu)(10))_{\rho=2} = \frac{1}{4}\sqrt{\frac{3\lambda(\mu+2)(\lambda+\mu+1)(\lambda+\mu+3)}{\mu(\lambda+2)g_{\lambda\mu}}}, \tag{26}$$

$$U((\lambda-1,\mu)(10)(\lambda\mu)(11);(\lambda\mu)(10))_{\rho=2} = -\frac{1}{4}\sqrt{\frac{3(\lambda+2)\mu(\mu+2)(\lambda+\mu+3)}{\lambda(\lambda+\mu+1)g_{\lambda\mu}}}, \tag{27}$$

where $g_{\lambda\mu}$ is given by the relation (10).

B. Off-diagonal matrix elements of G_{ia}

We have applied the procedure of the previous subsection to obtain the off-diagonal matrix elements of G_{ia} as well. As mentioned above, there are only two types of non-vanishing matrix elements, those with $(\lambda'\mu') = (\lambda+2,\mu-1)$, $S' = S+1$ and those with $(\lambda'\mu') = (\lambda-2,\mu+1)$, $S' = S-1$. We found that they are given by

$$\begin{aligned}
&\langle [N_c](\lambda+2,\mu-1)Y'I'I'_3; S+1, S'_3|G_{ia}|[N_c](\lambda\mu)YII_3; SS_3\rangle = (-)^{2S+1}N_c\sqrt{2(2S+1)} \\
&\times \left(\begin{array}{cc|c} S & 1 & S+1 \\ S_3 & i & S'_3 \end{array}\right) \left(\begin{array}{cc|c} I & I^a & I' \\ I_3 & I^a_3 & I'_3 \end{array}\right) \left(\begin{array}{cc||c} (\lambda\mu) & (11) & (\lambda+2,\mu-1) \\ YI & Y^aI^a & Y'I' \end{array}\right) c^{[N_c]}_{11}(S+1)c^{[N_c]}_{22}(S) \\
&\times \left\{\begin{array}{ccc} S+1/2 & 1/2 & S \\ 1 & S+1 & 1/2 \end{array}\right\} U((\lambda+1,\mu-1)(10)(\lambda+2,\mu-1)(11);(\lambda\mu)(10)),
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\langle [N_c](\lambda-2,\mu+1)Y'I'I'_3; S-1, S'_3|G_{ia}|[N_c](\lambda\mu)YII_3; SS_3\rangle = (-)^{2S}N_c\sqrt{2(2S+1)} \\
&\times \left(\begin{array}{cc|c} S & 1 & S-1 \\ S_3 & i & S'_3 \end{array}\right) \left(\begin{array}{cc|c} I & I^a & I' \\ I_3 & I^a_3 & I'_3 \end{array}\right) \left(\begin{array}{cc||c} (\lambda\mu) & (11) & (\lambda-2,\mu+1) \\ YI & Y^aI^a & Y'I' \end{array}\right) c^{[N_c]}_{22}(S-1)c^{[N_c]}_{11}(S) \\
&\times \left\{\begin{array}{ccc} S-1/2 & 1/2 & S \\ 1 & S-1 & 1/2 \end{array}\right\} U((\lambda-1,\mu)(10)(\lambda-2,\mu+1)(11);(\lambda\mu)(10)).
\end{aligned} \tag{29}$$

Note that the off-diagonal matrix elements do not contain a summation over ρ because in the right-hand side of the SU(3) product (19) the terms $(\lambda+2,\mu-1)$ and $(\lambda-2,\mu+1)$ appear with multiplicity 1.

The above expression require one of the following U coefficients

$$U((\lambda+1,\mu-1)(10)(\lambda+2,\mu-1)(11);(\lambda\mu)(10)) = \frac{1}{2}\sqrt{\frac{3(\lambda+\mu+\lambda\mu+1)}{2\mu(\lambda+2)}}, \qquad (30)$$

and

$$U((\lambda-1,\mu)(10)(\lambda-2,\mu+1)(11);(\lambda\mu)(10)) = -\frac{1}{2}\sqrt{\frac{3(\lambda+1)(\lambda+\mu+2)}{2\lambda(\lambda+\mu+1)}}. \qquad (31)$$

As a practical application for $N_c = 3$, the off-diagonal matrix element are needed to couple $^4 8$ and $^2 8$ baryon states, for example.

V. ISOSCALAR FACTORS OF SU(6) GENERATORS FOR ARBITRARY N_c

Here we derive analytic formulas for isoscalar factors related to matrix elements of SU(6) generators between symmetric states by comparing the definition (5) with results from Sec. 4. In doing this, we have to replace E_{ia} by the corresponding generators S_i, T_a or G_{ia} according to the relations (6). Then from the result (23) we obtain the isoscalar factor of G_{ia} for $(\lambda'\mu') = (\lambda\mu)$, $S' = S$ as

$$\left(\begin{array}{cc||c} [N_c] & [21^4] & [N_c] \\ (\lambda\mu)S & (11)1 & (\lambda\mu)S \end{array}\right)_\rho = N_c(-1)^{2S}\sqrt{\frac{4(2S+1)}{C(SU(6))}}$$
$$\times \left[\left(c^{[N_c]}_{22}(S)\right)^2 \begin{Bmatrix} S+1/2 & 1/2 & S \\ 1 & S & 1/2 \end{Bmatrix} U((\lambda+1,\mu-1)(10)(\lambda\mu)(11);(\lambda\mu)(10))_\rho \right.$$
$$\left. - \left(c^{[N_c]}_{11}(S)\right)^2 \begin{Bmatrix} S-1/2 & 1/2 & S \\ 1 & S & 1/2 \end{Bmatrix} U((\lambda-1,\mu)(10)(\lambda\mu)(11);(\lambda\mu)(10))_\rho \right]. \qquad (32)$$

Similarly, but using the formula (29) we obtain the isoscalar factors for $(\lambda'\mu') \neq (\lambda\mu)$ $S' \neq S$. These are

$$\left(\begin{array}{cc||c} [N_c] & [21^4] & [N_c] \\ (\lambda\mu)S & (11)1 & (\lambda+2,\mu-1)S+1 \end{array}\right) = N_c(-1)^{2S+1}\sqrt{\frac{4(2S+1)}{C(SU(6))}}$$
$$\times\, c^{[N_c]}_{11}(S+1)\; c^{[N_c]}_{22}(S) \begin{Bmatrix} S+1/2 & 1/2 & S \\ 1 & S+1 & 1/2 \end{Bmatrix}$$
$$\times\, U((\lambda+1,\mu-1)(10)(\lambda+2,\mu-1)(11);(\lambda\mu)(10)), \qquad (33)$$

and

$$\left(\begin{array}{cc\|c} [N_c] & [21^4] & [N_c] \\ (\lambda\mu)S & (11)1 & (\lambda-2,\mu+1)S-1 \end{array}\right) = N_c(-1)^{2S}\sqrt{\frac{4(2S+1)}{C(SU(6))}}$$
$$\times\, c_{11}^{[N_c]}(S)c_{22}^{[N_c]}(S-1)\left\{\begin{array}{ccc} S-1/2 & 1/2 & S \\ 1 & S-1 & 1/2 \end{array}\right\}$$
$$\times\, U((\lambda-1,\mu)(10)(\lambda-2,\mu+1)(11);(\lambda\mu)(10)). \tag{34}$$

Replacing $c_{pp}^{[N_c]}$ by definitions (14) and the U coefficients by their expressions we have obtained the isoscalar factors for arbitrary N_c listed in the first four rows of Table 1.

For completeness, we now return to the generators S_i and T_a which have only diagonal matrix elements. For S_i we use the equivalence between Eq. (5) and the Wigner-Eckart theorem (7). This leads to row 5 of Table 1. For T_a we use the equivalence between Eq. (5) and the Wigner-Eckart theorem (8). The calculation of the isoscalar factors for $\rho = 1$ and $\rho = 2$ gives the results shown in rows 6 and 7 of Table 1.

One can alternatively express the SU(6) isoscalar factors of Table 1 in terms of λ and μ by using the identities $\lambda = 2S$, $\mu = N_c/2 - S$ and $g_{\lambda\mu} = [N_c(N_c+6) + 12S(S+1)]/4$.

Before ending this section let us calculate, as an example, the diagonal matrix element of G_{i8} by using Eq. (5). We consider a system of N_c quarks with spin S, isospin I and strangeness $\mathcal{S}$ defined by $Y = N_c/3 + \mathcal{S}$. In this case Eq. (5) becomes

$$\langle [N_c](\lambda\mu)Y'I'I'_3SS'_3|G_{i8}|[N_c](\lambda\mu)YII_3SS_3\rangle = \delta_{YY'}\delta_{II'}\delta_{I_3I'_3}\sqrt{\frac{C(\mathrm{SU(6)})}{2}}\left(\begin{array}{cc|c} S & 1 & S \\ S_3 & i & S'_3 \end{array}\right)$$
$$\times \left(\begin{array}{cc|c} I & 0 & I' \\ I_3 & 0 & I'_3 \end{array}\right)\sum_{\rho=1,2}\left(\begin{array}{cc\|c} (\lambda\mu) & (11) & (\lambda\mu) \\ YI & 00 & Y'I' \end{array}\right)_\rho \left(\begin{array}{cc\|c} [N_c] & [21^4] & [N_c] \\ (\lambda\mu)S & (11)1 & (\lambda\mu)S \end{array}\right)_\rho, \tag{35}$$

Using Table 4 of Ref. [31] and our Table 1 for the isoscalar factors of SU(3) and SU(6) respectively we have obtained

$$\langle [N_c](\lambda\mu)Y'I'I'_3SS'_3|G_{i8}|[N_c](\lambda\mu)YII_3SS_3\rangle = \frac{\delta_{YY'}\delta_{II'}\delta_{I_3I'_3}}{4\sqrt{3S(S+1)}}\left(\begin{array}{cc|c} S & 1 & S \\ S_3 & i & S'_3 \end{array}\right)$$
$$\times\,[3I(I+1) - S(S+1) - \frac{3}{4}\mathcal{S}(\mathcal{S}-2)]. \tag{36}$$

With $\mathcal{S} = -N_s$, where N_s is the number of strange quarks, we can recover the relation

$$S_iG_{i8} = \frac{1}{4\sqrt{3}}[3I(I+1) - S(S+1) - \frac{3}{4}N_s(N_s+2)], \tag{37}$$

used in our previous work [24]. (Ref. [24] contains a typographic error. In the denominator of Eq. (13) one should read $\sqrt{3}$ instead of $\sqrt{2}$.)

VI. BACK TO ISOSCALAR FACTORS OF SU(4) GENERATORS

In the case of SU(4) $\supset$ SU(2) $\times$ SU(2) the analogue of Eq. (5) is [27]

$$\langle [N_c] I' I'_3 S' S'_3 | E_{ia} | [N_c] I I_3 S S_3 \rangle = \sqrt{C(\mathrm{SU}(4))}$$
$$\times \left(\begin{array}{cc||c} [N_c] & [21^2] & [N_c] \\ IS & I^a S^i & I'S' \end{array} \right) \left(\begin{array}{cc|c} S & S^i & S' \\ S_3 & S_3^i & S'_3 \end{array} \right) \left(\begin{array}{cc|c} I & I^a & I' \\ I_3 & I_3^a & I'_3 \end{array} \right), \tag{38}$$

where $C(\mathrm{SU}(4)) = [3N_c(N_c+4)]/8$ is the eigenvalue of the SU(4) Casimir operator. Note that for a symmetric state one has $I = S$. We recall that the SU(4) algebra is

$$\begin{aligned} [S_i, S_j] &= i\varepsilon_{ijk} S_k, \quad [T_a, T_b] = i\varepsilon_{abc} T_c, \quad [S_i, T_a] = 0, \\ [S_i, G_{ia}] &= i\varepsilon_{ijk} G_{ka}, \quad [T_a, G_{ib}] = i\varepsilon_{abc} G_{ic}, \\ [G_{ia}, G_{jb}] &= \frac{i}{4}\delta_{ij}\varepsilon_{abc} T_c + \frac{i}{2}\delta_{ab}\varepsilon_{ijk} S_k. \end{aligned} \tag{39}$$

The tensor operators E_{ia} are related to S_i, T_a and G_{ia} ($i = 1,2,3;\ a = 1,2,3$) by

$$E_i = \frac{S_i}{\sqrt{2}}; \quad E_a = \frac{T_a}{\sqrt{2}}; \quad E_{ia} = \sqrt{2} G_{ia}. \tag{40}$$

In Eq. (38) they are identified by $I^a S^i = 01, 10$ and 11 respectively. Now we want to obtain the SU(4) isoscalar factors as particular cases of the SU(6) results with $Y^a = 0$. In SU(4) the hypercharge of a system of N_c quarks takes the value $Y = N_c/3$. By comparing (5) and (38) we obtained the relation

$$\left(\begin{array}{cc||c} [N_c] & [21^2] & [N_c] \\ IS & I^a S^i & I'S' \end{array} \right) = r^{I^a S^i} \sqrt{\frac{C(\mathrm{SU}(6))}{C(\mathrm{SU}(4))}}$$
$$\times \sum_{\rho=1,2} \left(\begin{array}{cc||c} (\lambda\mu) & (\lambda^a\mu^a) & (\lambda'\mu') \\ \frac{N_c}{3} I & 0 I^a & \frac{N_c}{3} I' \end{array} \right)_\rho \left(\begin{array}{cc||c} [N_c] & [21^4] & [N_c] \\ (\lambda\mu) S & (\lambda^a\mu^a) S^i & (\lambda'\mu') S' \end{array} \right)_\rho, \tag{41}$$

where

$$r^{I^a S^i} = \begin{cases} \sqrt{\frac{3}{2}} & \text{for } I^a S^i = 01 \\ 1 & \text{for } I^a S^i = 10 \\ 1 & \text{for } I^a S^i = 11 \end{cases}, \tag{42}$$

due to (6), (38) and (40). In Eq. (41) we have to make the replacement

$$\lambda = 2I, \ \mu = \frac{N_c}{2} - I; \ \lambda' = 2I', \ \mu' = \frac{N_c}{2} - I', \tag{43}$$

and take

$$(\lambda^a \mu^a) = \begin{cases} (00) \ \text{ for } I^a = 0 \\ (11) \ \text{ for } I^a = 1 \end{cases}. \tag{44}$$

In this way we recovered the SU(4) isoscalar factors presented in Table A4.2 of Ref. [27] up to a phase factor. By introducing these isoscalar factors into the matrix elements (38) we obtained the expressions given in Eqs. (A1-A3) of Ref. [14].

VII. THE MASS OPERATOR

As mentioned in the introduction, in calculating the mass spectrum of baryons belonging to [70]-plets, the general procedure is to decompose the generators of a system of N_c quarks representing a large N_c baryon, into sums of core and excited quark generators. In order to apply these separate parts one also has to decouple the excited quark from the core. As an essential ingredient, the spin-flavor part of the core wave function is totally symmetric. Then the matrix elements of the core operators can be calculated by using Table I where one has to replace N_c by $N_c - 1$.

When the SU(3) symmetry is exact, the mass operator of an excited state can be written as the linear combination

$$M = \sum_i c_i O_i, \tag{45}$$

where c_i are unknown coefficients which parametrize the QCD dynamics and the operators O_i are of type

$$O_i = \frac{1}{N_c^{n-1}} O_\ell^{(k)} \cdot O_{SF}^{(k)} \tag{46}$$

where $O_\ell^{(k)}$ is a k-rank tensor in SO(3) and $O_{SF}^{(k)}$ a k-rank tensor in SU(2), but scalar in SU(3)-flavor. This implies that O_i is a combination of SO(3) generators ℓ_i and of SU(6) generators. Additional operators are needed when SU(3) is broken. The values of the coefficients c_i are found by a numerical fit to data.

An essential step is to find all linearly independent operators contributing to a given order $\mathcal{O}(1/N_c)$. This problem was extensively discussed for example in Ref. [14] for excited baryons belonging to the $[\mathbf{70}, 1^-]$ multiplet. For $[\mathbf{70}, \ell^+]$ the problem is more complicated because the core can also be excited. However the practice on the $[\mathbf{70}, 1^-]$ multiplet showed that some operators are dominant [14, 20], which is expected to remain valid for the $[\mathbf{70}, \ell^+]$ multiplet as well [23]. These are $O_1 = N_c \mathbb{1}$, $O_2 = \ell_q^i s^i$, $O_3 = \frac{3}{N_c} \ell_q^{(2)ij} g^{ia} G_c^{ja}$, $O_4 = \frac{4}{N_c + 1} \ell^i t^a G_c^{ia}$ and $O_5 = \frac{1}{N_c}(S_c^i S_c^i + s^i S_c^i)$. Among them, the operators O_3 and O_4 contain the core generator G_c^{ia}. Table I provides the necesary matrix elements for this generator when the study is extended to strange baryons. Applications to baryons belonging to the $[\mathbf{70}, \ell^+]$ multiplet are presented elsewhere [37] where problems related to the corresponding description of the wave functions are also discussed.

VIII. SUMMARY

We have derived general explicit formulas for specific matrix elements of the SU(6) generators as a function of N_c. They refer to spin-flavor symmetric states $|[N_c]1\rangle$. They can be applied, for example, to the study of baryon excited states belonging to the $[\mathbf{70}, \ell]$ multiplet. In that case the system of N_c quarks describing the baryon is divided into an excited quark and a core, which is the remaining of a baryon state after an excited quark has been removed. The core is always described by a symmetric spin-flavor state. In using Table I for the core operators one must replace N_c by $N_c - 1$. The results of Table I can equally be applied to the study of pentaquarks [38]

Putting $N_c = 3$ in Table I we could, for example, reproduce the values of Refs. [28] and [29] up to a phase. To further check the validity of our results we have calculated the matrix elements of the operator O_3 [20], which contains G_{ia}. We have recovered the expressions of O_3 needed to calculate masses of the strange and nonstrange baryons belonging to the $[\mathbf{70}, 1^-]$ multiplet.

APPENDIX A

Here we shortly recall the definition of isoscalar factors of the permutation group S_n. Let us denote a basis vector in the invariant subspace of the irrep $[f]$ of S_n by $|[f]Y\rangle$, where Y is the corresponding Young tableau or Yamanouchi symbol. A basis vector obtained from the inner product of two irreps $[f']$ and $[f'']$ is defined by the sum over products of basis vectors of $[f']$ and $[f'']$ as

$$|[f]Y\rangle = \sum_{Y'Y''} S([f']Y'[f'']Y''|[f]Y)|[f']Y'\rangle|[f'']Y''\rangle, \tag{A1}$$

where $S([f']Y'[f'']Y''|[f]Y)$ are Clebsch-Gordan (CG) coefficients of S_n. Any CG coefficient can be factorized into an isoscalar factor, here called K matrix [34], and a CG coefficient of S_{n-1}. To apply the factorization property it is necessary to specify the row p of the n-th particle and the row q of the $(n-1)$-th particle. The remaining particles are distributed in a Young tableau denoted by y. Then the isoscalar factor K associated to a given CG of S_n is defined as

$$S([f']p'q'y'[f'']p''q''y''|[f]pqy) = K([f']p'[f'']p''|[f]p)S([f'_{p'}]q'y'[f''_{p''}]q''y''|[f_p]qy), \tag{A2}$$

where the right-hand side contains a CG coefficient of S_{n-1} containing $[f_p]$, $[f'_{p'}]$ and $[f''_{p''}]$ which are the partitions obtained from $[f]$ after the removal of the n-th particle. The K matrix obeys the following orthogonality relations

$$\begin{aligned} \sum_{p'p''} K([f']p'[f'']p''|[f]p)K([f']p'[f'']p''|[f_1]p_1) &= \delta_{ff_1}\delta pp_1, && \text{(A3)} \\ \sum_{fp} K([f']p'[f'']p''|[f]p)K([f']p'_1[f'']p''_1|[f]p) &= \delta_{p'p'_1}\delta_{p''p''_1}. && \text{(A4)} \end{aligned}$$

The isoscalar factors used to construct the spin-flavor symmetric state (13) are

$$\begin{aligned} c_{11}^{[N_c]} &= K([f']1[f']1|[N_c]1), \\ c_{22}^{[N_c]} &= K([f']2[f']2|[N_c]1), \end{aligned} \tag{A5}$$

with $[f'] = [N_c/2 + S, N_c/2 - S]$. The isoscalar factors needed to construct the state of mixed symmetry $[N_c - 1, 1]$ from the same inner product are

$$\begin{aligned} c_{11}^{[N_c-1,1]} &= K([f']1[f']1|[N_c-1,1]2), \\ c_{22}^{[N_c-1,1]} &= K([f']2[f']2|[N_c-1,1]2). \end{aligned} \tag{A6}$$

The above coefficients and the orthogonality relation (A4) give

$$\begin{aligned} c_{11}^{[N_c-1,1]} &= -c_{22}^{[N_c]}, \\ c_{22}^{[N_c-1,1]} &= c_{11}^{[N_c]}. \end{aligned} \tag{A7}$$

When the last particle is located in different rows in the flavor and spin parts the needed coefficients are

$$\begin{aligned} c_{12}^{[N_c-1,1]} &= K([f']1[f']2|[N_c-1,1]2) = 1, \\ c_{21}^{[N_c-1,1]} &= K([f']2[f']1|[N_c-1,1]2) = 1, \end{aligned} \tag{A8}$$

which are identical because of the symmetry properties of K. For $N_c \leq 6$ one can check the above identification with the isoscalar factors given in Ref. [35]. The identification of the so called "elements of orthogonal basis rotation" of Ref. [14] with the above isoscalar factors of S_n is given in Sec. III. The expressions of these coefficients are derived in the following appendix for arbitrary N_c.

APPENDIX B

Here we derive the expressions of the coefficients $\left(c_{pp}^{[N_c]}\right)^2$ $(p = 1, 2)$, defined in the context of SU(6) $\supset$ SU(2) $\times$ SU(3) as isoscalar factors of S_n. To get (14) we write the matrix elements of the generators S_i in two different ways. One is to use the Wigner-Eckart theorem (7). The other is to calculate the matrix elements of S_i by using (13), (18) and (21). By comparing the two expressions we obtain the equality

$$\begin{aligned} \sqrt{S(S+1)} &= (-)^{2S} N_c \sqrt{\frac{3}{2}} \sqrt{2S+1} \left[\left(c_{22}^{[N_c]}\right)^2 \begin{Bmatrix} 1 & S & S \\ S+1/2 & 1/2 & 1/2 \end{Bmatrix} \right. \\ &\left. - \left(c_{11}^{[N_c]}\right)^2 \begin{Bmatrix} 1 & S & S \\ S-1/2 & 1/2 & 1/2 \end{Bmatrix} \right], \end{aligned} \tag{B1}$$

which is an equation for the unknown quantities The other equation is the normalization relation (A3)

$$\left(c_{11}^{[N_c]}\right)^2 + \left(c_{22}^{[N_c]}\right)^2 = 1. \tag{B2}$$

We found

$$\begin{aligned} c_{11}^{[N_c]}(S) &= \sqrt{\frac{S[N_c + 2(S+1)]}{N_c(2S+1)}}, \\ c_{22}^{[N_c]}(S) &= \sqrt{\frac{(S+1)(N_c - 2S)}{N_c(2S+1)}}. \end{aligned} \tag{B3}$$

which are the relations (14). This phase convention is consistent with Ref. [35]. Similarly, to get (15) we calculate the matrix elements of the generators T_a from (13), (18) and (22) and compare to the Wigner-Eckart theorem (8). This leads to the equation

$$\begin{aligned}\sqrt{\frac{g_{\lambda\mu}}{3}} &= \frac{2}{\sqrt{3}}N_c\left[\left(c_{22}^{[N_c]}\right)^2 U((\lambda+1,\mu-1)(10)(\lambda\mu)(11);(\lambda\mu)(10))_{\rho=1}\right. \\ &\left.+\left(c_{11}^{[N_c]}\right)^2 U((\lambda-1,\mu)(10)(\lambda\mu)(11);(\lambda\mu)(10))_{\rho=1}\right], \end{aligned} \tag{B4}$$

which together with the normalization condition (B2) give

$$\begin{aligned} c_{11}^{[N_c]}(\lambda\mu) &= \sqrt{\frac{2g_{\lambda\mu}-N_c(\mu-\lambda+3)}{3N_c(\lambda+1)}}, \\ c_{22}^{[N_c]}(\lambda\mu) &= \sqrt{\frac{N_c(6+2\lambda+\mu)-2g_{\lambda\mu}}{3N_c(\lambda+1)}}. \end{aligned} \tag{B5}$$

i.e. the relations (15). In addition, we found that the following identity holds for $\rho=2$

$$\begin{aligned} 0 &= \frac{2}{\sqrt{3}}N_c\left[\left(c_{22}^{[N_c]}\right)^2 U((\lambda+1,\mu-1)(10)(\lambda\mu)(11);(\lambda\mu)(10))_{\rho=2}\right. \\ &\left.+\left(c_{11}^{[N_c]}\right)^2 U((\lambda-1,\mu)(10)(\lambda\mu)(11);(\lambda\mu)(10))_{\rho=2}\right]. \end{aligned} \tag{B6}$$

This cancellation is consistent with the definition of the matrix elements of the SU(3) generators Eqs. (8), (9) and it is an important check of our results.

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[39] To obtain the last equality in (22) we used the identity

$$\sum_{\rho=1,2} \langle(\lambda\mu)YI;(11)Y^aI^a||(\lambda\mu)Y'I'\rangle_\rho U((\lambda_1,\mu_1)(10)(\lambda\mu)(11);(\lambda\mu)(10))_\rho =$$
$$\sum_{Y_1I_1Y_2I_2Y_2'I_2'} \langle(\lambda_1\mu_1)Y_1I_1;(10)Y_2I_2||(\lambda\mu)YI\rangle\langle(10)Y_2I_2;(11)Y^aI^a||(10)Y_2'I_2'\rangle$$
$$\langle(\lambda_1\mu_1)Y_1I_1;(10)Y_2'I_2'||(\lambda\mu)Y'I'\rangle U(I_1I_2I'I^a;II_2'),$$

similar to the relation (12) of Ref. [31].

$(\lambda_1\mu_1)S_1$	$(\lambda_2\mu_2)S_2$	ρ	$\left(\begin{array}{cc\|c} [N_c] & [21^4] & [N_c] \\ (\lambda_1\mu_1)S_1 & (\lambda_2\mu_2)S_2 & (\lambda\mu)S \end{array}\right)_\rho$
$(\lambda+2,\mu-1)S+1$	$(11)1$	/	$-\sqrt{\frac{3}{2}}\sqrt{\frac{2S+3}{2S+1}}\sqrt{\frac{(N_c-2S)(N_c+2S+6)}{5N_c(N_c+6)}}$
$(\lambda\mu)S$	$(11)1$	1	$4(N_c+3)\sqrt{\frac{2S(S+1)}{5N_c(N_c+6)[N_c(N_c+6)+12S(S+1)]}}$
$(\lambda\mu)S$	$(11)1$	2	$-\sqrt{\frac{3}{2}}\sqrt{\frac{(N_c-2S)(N_c+4-2S)(N_c+2+2S)(N_c+6+2S)}{5N_c(N_c+6)[N_c(N_c+6)+12S(S+1)]}}$
$(\lambda-2,\mu+1)S-1$	$(11)1$	/	$-\sqrt{\frac{3}{2}}\sqrt{\frac{2S-1}{2S+1}}\sqrt{\frac{(N_c+4-2S)(N_c+2+2S)}{5N_c(N_c+6)}}$
$(\lambda\mu)S$	$(00)1$	/	$\sqrt{\frac{4S(S+1)}{5N_c(N_c+6)}}$
$(\lambda\mu)S$	$(11)0$	1	$\sqrt{\frac{N_c(N_c+6)+12S(S+1)}{10N_c(N_c+6)}}$
$(\lambda\mu)S$	$(11)0$	2	0

TABLE I: Isoscalar factors SU(6) for $[N_c]\times[21^4]\rightarrow[N_c]$ defined by Eq. (5).