A new look at the $[70, 1^-]$ baryon multiplet in the $1/N_c$ expansion

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So far, the masses of excited states of mixed orbital symmetry and in particular those of nonstrange $[70, 1^-]$ baryons derived in the $1/N_c$ expansion were based on the separation of a system of $N_c$ quarks into a symmetric core and an excited quark. Here we avoid this separation and show that an advantage of this new approach is to substantially reduce the number of linearly independent operators entering the mass formula. A novelty is that the isospin-isospin term becomes as dominant in $\Delta$ as the spin-spin term in $N$ resonances.

I. INTRODUCTION

In 1974 't Hooft $^1$ suggested a perturbative expansion of QCD in terms of the parameter $1/N_c$, where $N_c$ is the number of colors. This suggestion, together with the power counting rules of Witten $^2$ has led to the $1/N_c$ expansion method which allows to systematically analyze baryon properties. The current research status is described, for example, in Ref. $^3$. The success of the method stems from the discovery that the ground state baryons have an exact contracted SU(2) symmetry when $N_c \to \infty$ $^4,^5$, $N_f$ being the number of flavors. A considerable amount of work has been devoted to the ground state baryons $^6,^7,^8,^9,^10,^11$. For $N_c \to \infty$ the baryon masses are degenerate. For finite $N_c$ the mass splitting starts at order $1/N_c$. Operator reduction rules simplify the $1/N_c$ expansion $^6,^7$. It is customary to drop higher order corrections of order $1/N_c^2$.

It is thought that 't Hooft’s suggestion $^1$ would lead to an $1/N_c$ expansion to hold in all QCD regimes. Accordingly, the applicability of the approach to excited states is a subject of current investigation.

In the language of the constituent quark model the excited states can be grouped into excitation bands with $N = 1, 2, 3,$ etc. units of excitation energy. Among them, the $N = 1$ band, or equivalently the $[70, 1^-]$ multiplet, has been most extensively studied, either for $N_f = 2$ $^12$, $13, 14, 15, 16, 17, 18, 19$ or for $N_f = 3$ $^20$. In the latter case, first order corrections in SU(3) symmetry breaking were also included. In either case, the conclusion was that the splitting starts at order $N_c^0$.

The $N = 2$ band contains the $[56', 0^+], [56, 2^+]$, $[70, \ell^+]$ ($\ell = 0, 2$) and $[20, 1^+]$ multiplets. There are no physical resonances associated to $[20, 1^+]$. The few studies related to the $N = 2$ band concern the $[56', 0^+]$ for $N_f = 2$ $^21$, $[56, 2^+]$ for $N_f = 3$ $^22$ and $[70, \ell^+]$ for $N_f = 2$ $^23$, later extended to $N_f = 3$ $^24$. The method has also been applied to highly excited nonstrange and strange baryons belonging to $[56, 4^+]$ $^25$ which is the lowest of the 17 multiplets of the $N = 4$ band $^26$.

The mass operator $M$ is defined as a linear combination of independent operators $O_i$

$$M = \sum_i c_i O_i, \quad (1)$$

where the coefficients $c_i$ are reduced matrix elements that encode the QCD dynamics and are determined from a fit to the existing data. Here we are concerned with nonstrange baryons only. The building blocks of the operators $O_i$ are the SU(2) generators $S_i$, $T_a$ and $G_{ia}$ and the SO(3) generators $\ell_i$. Their general form is

$$O_i = \frac{1}{N_c^{n-1}} O^{(k)}_i \cdot O^{(k)}_{SF}, \quad (2)$$

where $O^{(k)}_i$ is a $k$-rank tensor in SO(3) and $O^{(k)}_{SF}$ a $k$-rank tensor in SU(2)-spin, but invariant in SU($N_f$). Thus $O_i$ are rotational invariant. For the ground state one has $k = 0$. The excited states also require $k = 1$ and $k = 2$ terms.

The spin-flavor (SF) operators $O^{(k)}_{SF}$ are combinations of SU($2N_f$) generators, the lower index $i$ in the left hand side of (2) representing a specific combination. Each $n$-body operator is multiplied by an explicit factor of $1/N_c^{n-1}$ resulting from the power counting rules. Some compensating $N_c$ factors may arise in the matrix elements when $O_i$ contains a coherent operator such as $G^{ia}$ or $T^a$.

The excited states belonging to $[56, \ell]$ multiplets are rather simple and can be studied by analogy with the ground state. In this case both the orbital and the spin-flavor parts of the wave function are symmetric. Naturally, it turned out that the splitting starts at order $1/N_c$ $^22, ^23, ^24$, as for the ground state.

The states belonging to $[70, \ell]$ multiplets are apparently more difficult. So far, the general practice was to decouple the baryon into an excited quark and a symmetric core. This means that each generator of SU(2$N_f$) must be written as a sum of two terms, one acting on the excited quark and the other on the core. As a consequence, the number of linearly independent operators $O_i$ increases tremendously and the number of coefficients $c_i$, to be determined, becomes much larger than the experimental data available. For example, for the $[70, 1^-]$
multiplet with $N_f = 2$ one has 13 linearly independent operators up to order $1/N_c$, included \[16\], instead of 7 (see below). We recall that there are only 7 nonstrange resonances belonging to this band. Consequently, selecting the most dominant operators is very difficult so that one risks to make an arbitrary choice \[16\].

In this practice the matrix elements of the excited quark are straightforward, as being described by single-particle operators. The matrix elements of the core operators $S^c_i, T^c$ are also simple to calculate, while those of $G^{at}_c$ are more involved. Analytic formulas for the matrix elements of all SU(4) generators have been derived in Ref. \[27\]. Every matrix element is factorized according to a generalized Wigner-Eckart theorem into a reduced matrix element and an SU(4) Clebsch-Gordan coefficient. These matrix elements have been used in nuclear physics, which is governed by the SU(4) symmetry. Recently we have extended the approach of Ref. \[27\] to SU(6) \[28\] and obtained matrix elements of all SU(6) generators between symmetric $[N_c]$ states.

Here we propose a method where no decoupling is necessary. All one needs to know are the matrix elements of the SU(2$N_f$) generators between mixed symmetric states $[N_c - 1, 1]$. For SU(4) they were obtained by Hecht and Pang \[27\]. They can be easily applied to a system of $N_c$ nonstrange quarks. To our knowledge such matrix elements are yet unknown for $N_f = 3$.

II. THE WAVE FUNCTION

We deal with a system of $N_c$ quarks having one unit of orbital excitation. Then the orbital wave function must have a mixed symmetry $[N_c - 1, 1]$. Its spin-flavor part must have the same symmetry in order to obtain a totally symmetric state in the orbital-spin-flavor space. The general form of such a wave function is \[29\]

$$|[N_c]⟩ = \frac{1}{\sqrt{d_{[N_c - 1, 1]}}} \sum_{Y} |[N_c - 1, 1]Y⟩_O |[N_c - 1, 1]Y⟩_{FS} \, (3)$$

where $d_{[N_c - 1, 1]} = N_c - 1$ is the dimension of the representation $[N_c - 1, 1]$ of the permutation group $S_{N_c}$ and $Y$ is a symbol for a Young tableau (Yamanouchi symbol). The sum is performed over all possible standard Young tableaux. In each term the first basis vector represents the orbital space $(O)$ and the second the spin-flavor space $(FS)$. In this sum there is only one $Y$ (the normal Young tableau) where the last particle is in the second row and $N_c - 2$ terms where the last particle is in the first row. All these terms were neglected in the procedure of decoupling the excited quark, which implies that the permutation symmetry $S_{N_c}$ was broken, i.e. the orbital-spin-flavor wave function was no more symmetric, as it should be. One can easily prove the above assertion by looking at the expression of the wave function, Eqs. (3.4)-(3.5) in the second paper of Ref. \[16\], for example. This definition contains the coefficients $c_{pq}$ which are defined as coefficients of an “orthogonal rotation”. In Ref. \[23\] we have shown that $c_{pq}$ are some specific isoscalar factors of the permutation group $S_{N_c}$. These are factors of the Clebsch-Gordan coefficients, factorized as isoscalar factors times Clebsch-Gordan coefficients of the group $S_{N_c - 1}$. In the case under concern the isoscalar factors incorporate the position of the $N_c$-th particle in a Young tableau. By identifying our expressions with those of Ref. \[16\] we found that they correspond to the term where the last particle is located in the second row of the Young tableau of the representation $[N_c - 1, 1]$. Thus the other $N_c - 2$ terms of the wave function, with the $N_c$-th particle in the first row, are missing. In Appendix A we show explicitly which are the missing terms for $N_c = 3$ in the sectors $^2S^2$, $^4S^2$ and $^2I^2$. In addition, as an example, the orbital basis vectors of configuration $s^3p$, containing one unit of orbital excitation, which span the invariant subspace of the irreducible representation of $S_5$ are given in Appendix B. The definition and the orthogonality properties together with examples of isoscalar factors can be found in Ref. \[20\]. In Sec. VI we discuss the validity of the approximate (asymmetric) wave function of Ref. \[16\].

If there is no decoupling, there is no need to specify $Y$, the matrix elements being identical for all $Y$’s, due to Weyl’s duality between a linear group and a symmetric group in a given tensor space. Then the explicit form of a wave function of total angular momentum $\vec{J} = \vec{l} + \vec{s}$ and isospin $I$ is

$$|ℓSJ; J_3⟩ = \sum_{m_ℓ, S_3} \left( \frac{ℓ}{m_ℓ}, S \right) J^J_3 \, (4)$$

$$\times |[N_c - 1, 1]ℓm_ℓ⟩ |[N_c - 1, 1]S S_3 H_3⟩,$$

where the $S$ is a symbol for a Young tableau. By identifying our expressions with those of Ref. \[16\] we found that they correspond to the term where the last particle is located in the second row of the Young tableau of the representation $[N_c - 1, 1]$. Thus the other $N_c - 2$ terms of the wave function, with the $N_c$-th particle in the first row, are missing. In Appendix A we show explicitly which are the missing terms for $N_c = 3$ in the sectors $^2S^2$, $^4S^2$ and $^2I^2$.

III. SU(4) GENERATORS AS TENSOR OPERATORS

The SU(4) generators $S_i, T_a$ and $G_{ia}$, globally denoted by $E_{ia}$ \[27\], are components of an irreducible tensor operator which transform according to the adjoint representation \[211\] of dimension 15 of SU(4). We recall that the SU(4) algebra is

$$[S_i, T_a] = 0, \quad [S_i, G_{ja}] = iε_{ijk}G_{ka}, \quad [T_a, G_{ib}] = iε_{abc}G_{ic},$$

$$[S_i, S_j] = iε_{ijk}S_k, \quad [T_a, T_b] = iε_{abc}T_c, \quad [G_{ia}, G_{jb}] = \frac{i}{4}δ_{ij}ε_{abc}T_c + \frac{i}{4}δ_{ab}ε_{ijk}S_k. \, (5)$$

As one can see, the tensor operators $E_{ia}$ are of three types: $E_i$ ($i = 1, 2, 3$) which form the subalgebra of SU(2)-spin, $E_a$ ($a = 1, 2, 3$) which form the subalgebra of SU(2)-isospin and $E_{ia}$ which act both in the spin and
the isospin spaces. They are related to $S_i$, $T_a$ and $G_{ia}$ ($i = 1, 2, 3; a = 1, 2, 3$) by
\begin{equation}
E_i = \frac{S_i}{\sqrt{2}}; \quad E_a = \frac{T_a}{\sqrt{2}}; \quad E_{ia} = \sqrt{2}G_{ia}. \tag{6}
\end{equation}

The matrix elements of every $E_{ia}$ between states belonging to
the representation $[N_c - 1, 1]$ can be expressed as a
generalized Wigner-Eckart theorem which reads \[27\]
\begin{equation}
\langle [N_c - 1, 1]|I'\ell'S'\rho'\rangle_{E_{ia}}| [N_c - 1, 1]|I\ell S\rangle = \sqrt{C^{[N_c - 1, 1]}(SU(4))} \left( \begin{array}{ccc}
S & S' & S' \\
S_3 & S_3 & S_3 \\
I & I' & I'
\end{array} \right) \left( \begin{array}{ccc}
I_3 & I_3 & I_3 \\
I_3 & I_3 & I_3 \\
I_3 & I_3 & I_3
\end{array} \right)_{\rho = 1}, \tag{7}
\end{equation}
where $C^{[N_c - 1, 1]}(SU(4)) = N_c(3N_c + 4)/8$ is the
eigenvalue of the SU(4) Casimir operator for the representation
$[N_c - 1, 1]$. The other three factors are: an SU(2)-spin CG coefficient, an SU(2)-isospin CG coefficient and
an isoscalar factor of SU(4). Note that the isoscalar factor
carries a lower index $\rho = 1$. In general, this index
is necessary to distinguish between reducible representations,
whenever the multiplicity in the inner product
$[N_c - 1, 1] \times [211] \rightarrow [N_c - 1, 1]$ is larger than one. In
that case, the matrix elements of the SU(4) generators
in a fixed reducible representation $[f]$ are defined such as
the reduced matrix elements take the following values [27]
\begin{equation}
\langle [f]|E|[f]\rangle = \left\{ \begin{array}{ll}
\sqrt{C^{[N_c - 1, 1]}(SU(4))} & \text{for } \rho = 1 \\
0 & \text{for } \rho \neq 1
\end{array} \right.. \tag{8}
\end{equation}

Thus the knowledge of the matrix elements of SU(4) generators
amounts to the knowledge of the corresponding
SU(4) isoscalar factors. In Ref. \[27\] a variety of isoscalar
factors were obtained. We need those for $|f]\rangle = [N_c - 1, 1]$. They are reproduced in Table II in terms of our notation
and typographical errors corrected. They contain the
phase factor introduced in Eq. (35) of Ref. \[27\]. As compared
to the symmetric $[N_c]$ representation, where $I = S$ always, here one has $I = S$ (13 cases) but also $I \neq S$ (10 cases). Some of the properties of these isoscalar factors are
given in Appendix C.

One can easily identify the matrix elements associated to
the generators of SU(4). One has $S_2I_2 = 10$ for $S_i$, $S_2I_2 = 01$ for $T_a$ and $S_2I_2 = 11$ for $G_{ia}$, where 1 or 0 is
the rank of the SU(2)-spin or SU(2)-isospin tensor contained
in the generator. The generalized Wigner-Eckart theorem [17] is used to calculate the matrix elements of $O_i$ needed for the
mass operator, as described below.

**IV. THE MASS OPERATOR**

As specified in the introduction, here we are concerned with
nonstrange baryons only. Table III contains the
seven independent operators up to order $1/N_c$ appearing
in the mass operator Eq. (1). As already mentioned,
the building blocks of $O_i$ are $S^i$, $T^a$, $G^{ia}$ and $\ell^i$. We also
need the rank $\ell = 2$ tensor operator
\begin{equation}
(\ell^{(2)i}) = \frac{1}{2} \{\ell^i, \ell^j\} - \frac{1}{3} \delta_{i,j} \ell^i \cdot \ell^j, \tag{9}
\end{equation}
which, like $\ell^i$, acts on the orbital wave function $|\ell m\rangle$
of the whole system of $N_c$ quarks (see Ref. \[23\] for the
normalization of $\ell^{(2)i}$).

In Table III the first nontrivial operator is the spin-orbit operator $O_2$. In the spirit of the Hartree picture [2], generally adopted for the description of baryons, we identify the spin-orbit operator with the single-particle operator
\begin{equation}
\ell \cdot s = \sum_{i=1}^{N_c} \ell(i) \cdot s(i), \tag{10}
\end{equation}
Accordingly, its matrix elements are of order $N_c^0$. For
simplicity we ignore the two-body part of the spin-orbit operator,
denoted by $1/N_c (\ell \cdot S_i)$ in Ref. \[16\], as being of a lower order (the lower case indicates operators acting
on the excited quark and the subscript $c$ is attached to
those acting on the core).

The operators $O_3$ and $O_4$ are two-body and linearly
independent. However, in the decoupling procedure the
appropriate isospin-isospin operator $\ell T_c/N_c$ has always
been avoided in the numerical analysis \[16, 20\].

To be consistent with Ref. \[10\] we assume that the
operators $O_5$ and $O_6$ are dominantly two-body, which
means that they carry a factor $1/N_c$. Moreover, as $G^{ia}$
sums coherently, it introduces an extra factor $N_c$ and
makes the matrix elements of $O_5$ and $O_6$ of order $N_c^0$ as
well (what it matters in the mass operator are the
products $c_5 O_5$ and $c_6 O_6$ and it will turn out that their
contribution is small in any case).

We have also included in the fit the following operator
\begin{equation}
O_7 = \frac{3}{N_c} S^i T^{ia} G^{ia}, \tag{11}
\end{equation}
an SU(4) invariant built from products of all generators of
SU(4), $S_i$, $T_a$, and $G_{ia}$. In the core plus excited quark
procedure its counterpart was listed in Table I of the
second paper of Ref. \[16\] as $O_{16} = g S_c T_c/N_c^2$ but completely
ignoroted in the numerical fit, one reason being that
the number of operators in the mass formula was much
too large as compared to the data. The operator $O_{16}$ is
only a part of $O_7$, as it can be easily seen. As shown
below, its matrix elements are of order $1/N_c$, like those of
the pure spin $O_3$ or pure isospin $O_4$ operators. Therefore
there is no a priori reason to ignore it.

Naturally, one should also include the operator
\begin{equation}
O_8 = \frac{1}{N_c} (\ell^{(2)}) S \cdot S, \tag{12}
\end{equation}
also of order $1/N_c$. However, in our basis we found a
proportionality relation between expectation values of two
different operators
\begin{equation}
(\ell^{(2)} i S^i S^j) = 12 (\ell^{(2)} i G^{ja} G^{ia}), \tag{13}
\end{equation}
| $S_1$ | $I_1$ | $S_2 I_2$ | $SI$ | \[
\begin{pmatrix}
[S_{11} - 1, 1] & [211] & [S_{11} - 1, 1] \\
S_{21} & S_{22} & SI
\end{pmatrix}_{\rho \equiv 1}
\] |
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>$S + 1$</td>
<td>$S + 1$</td>
<td>11</td>
<td>$SS$</td>
<td>$\sqrt{\frac{S(S + 2)(2S + 3)(N_c - 2 - 2S)(N_c + 2 + 2S)}{(2S + 1)(S + 1)^2 N_c(3N_c + 4)}}$</td>
</tr>
<tr>
<td>$S + 1$</td>
<td>$S$</td>
<td>11</td>
<td>$SS$</td>
<td>$-\sqrt{\frac{(2S + 3)(N_c + 2 + 2S)}{(2S + 1)(3N_c + 4)}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S + 1$</td>
<td>11</td>
<td>$SS$</td>
<td>$-\sqrt{\frac{S(S + 2)(N_c - 2 - 2S)(N_c + 2 + 2S)}{(2S + 1)(3N_c + 4)}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>11</td>
<td>$SS$</td>
<td>$\sqrt{\frac{(2S - 1)(N_c - 2S)}{(2S + 1)(3N_c + 4)}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>11</td>
<td>$SS$</td>
<td>$\sqrt{\frac{(S - 1)(S + 1)(2S - 1)(N_c + 2S)(N_c - 2S)}{(2S + 1)N_c(3N_c + 4)}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>01</td>
<td>$SS$</td>
<td>0</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>01</td>
<td>$SS$</td>
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</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>01</td>
<td>$SS$</td>
<td>$\sqrt{\frac{4S(S + 1)}{N_c(3N_c + 4)}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>11</td>
<td>$SS - 1$</td>
<td>$\sqrt{\frac{(2S + 3)(N_c + 2 + 2S)(N_c - 2S)}{(2S + 1)(3N_c + 4)}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>11</td>
<td>$SS - 1$</td>
<td>$\sqrt{\frac{S(S + 2)}{3N_c + 4}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>11</td>
<td>$SS - 1$</td>
<td>$\sqrt{\frac{(S - 1)(S + 1)N_c}{3N_c + 4}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>11</td>
<td>$SS - 1$</td>
<td>$\sqrt{\frac{N_c + 4S^2}{S(2S - 1)(2S + 1)N_c(3N_c + 4)}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>11</td>
<td>$SS - 1$</td>
<td>$\sqrt{\frac{(2S - 3)(N_c - 2 - 2S)(N_c + 2S)}{(2S - 1)N_c(3N_c + 4)}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>10</td>
<td>$SS - 1$</td>
<td>$\sqrt{\frac{4S(S + 1)}{N_c(3N_c + 4)}}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
<td>10</td>
<td>$SS - 1$</td>
<td>0</td>
</tr>
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<td>$S$</td>
<td>$S$</td>
<td>10</td>
<td>$SS - 1$</td>
<td>0</td>
</tr>
<tr>
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<td>$S$</td>
<td>01</td>
<td>$SS - 1$</td>
<td>$\sqrt{\frac{4(S - 1)S}{N_c(3N_c + 4)}}$</td>
</tr>
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</table>
TABLE II: List of operators and the coefficients resulting from numerical fits. The values of $c_i$ are indicated under the headings Fit $n$, in each case.

<table>
<thead>
<tr>
<th>Operator</th>
<th>Fit 1 (MeV)</th>
<th>Fit 2 (MeV)</th>
<th>Fit 3 (MeV)</th>
<th>Fit 4 (MeV)</th>
<th>Fit 5 (MeV)</th>
<th>Fit 6 (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1 = N_c \mathbb{I}$</td>
<td>481 ± 5</td>
<td>482 ± 5</td>
<td>484 ± 4</td>
<td>484 ± 4</td>
<td>498 ± 3</td>
<td>495 ± 3</td>
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<tr>
<td>$O_2 = \ell^a \ell^a$</td>
<td>-31 ± 26</td>
<td>-20 ± 23</td>
<td>-12 ± 20</td>
<td>3 ± 15</td>
<td>38 ± 34</td>
<td>-30 ± 25</td>
</tr>
<tr>
<td>$O_3 = \frac{1}{N_c} S^a S^a$</td>
<td>161 ± 16</td>
<td>149 ± 11</td>
<td>163 ± 16</td>
<td>150 ± 11</td>
<td>156 ± 16</td>
<td></td>
</tr>
<tr>
<td>$O_4 = \frac{1}{N_c} T^a T^a$</td>
<td>169 ± 36</td>
<td>170 ± 36</td>
<td>141 ± 27</td>
<td>139 ± 27</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$O_5 = \frac{1}{N_c} \ell^a (G^a)^a G^a$</td>
<td>-29 ± 31</td>
<td>-34 ± 30</td>
<td>-34 ± 31</td>
<td>-32 ± 29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$O_6 = \frac{3}{N_c} \ell^a (G^a)^a$</td>
<td>32 ± 26</td>
<td>35 ± 26</td>
<td>-67 ± 30</td>
<td>28 ± 20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$O_7 = \frac{3}{N_c^2} S^a T^a G^a$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>649 ± 61</td>
<td></td>
</tr>
<tr>
<td>$\chi^2_{\text{obs}}$</td>
<td>0.43</td>
<td>0.68</td>
<td>0.94</td>
<td>1.04</td>
<td>11.5</td>
<td>0.24</td>
</tr>
</tbody>
</table>

TABLE III: Matrix elements of $O_i$ for all states belonging to the $[70, 1^-]$ multiplet.

<table>
<thead>
<tr>
<th>$O_1$</th>
<th>$O_2$</th>
<th>$O_3$</th>
<th>$O_4$</th>
<th>$O_5$</th>
<th>$O_6$</th>
<th>$O_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^2N_\frac{1}{2}$</td>
<td>$^4N_\frac{1}{2}$</td>
<td>$^2N_\frac{3}{2}$</td>
<td>$^4N_\frac{3}{2}$</td>
<td>$^4N_\frac{1}{2}$</td>
<td>$^2\Delta_\frac{1}{2}$</td>
<td>$^4\Delta_\frac{1}{2}$</td>
</tr>
<tr>
<td>$N_c - \frac{2N_c - 3}{3N_c} \frac{3}{4N_c}$</td>
<td>$N_c - \frac{5}{6} \frac{15}{4N_c} \frac{3}{4N_c}$</td>
<td>$N_c - \frac{2N_c - 3}{6N_c} \frac{3}{4N_c} \frac{3}{4N_c}$</td>
<td>$N_c - \frac{1}{3} \frac{15}{4N_c} \frac{3}{4N_c}$</td>
<td>$N_c - \frac{1}{6} \frac{15}{4N_c} \frac{3}{4N_c}$</td>
<td>$N_c - \frac{1}{5} \frac{15}{4N_c} \frac{3}{4N_c}$</td>
<td>$N_c - \frac{1}{5} \frac{15}{4N_c} \frac{3}{4N_c}$</td>
</tr>
</tbody>
</table>

TABLE IV: The partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion using Fit 1. The last two columns give the empirically known masses, and name-status.

<table>
<thead>
<tr>
<th>Part. contrib. (MeV)</th>
<th>Total (MeV)</th>
<th>Exp. (MeV)</th>
<th>Name, status</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1 O_1$</td>
<td>$c_2 O_2$</td>
<td>$c_3 O_3$</td>
<td>$c_4 O_4$</td>
</tr>
<tr>
<td>$^2N_\frac{1}{2}$</td>
<td>1444</td>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>$^4N_\frac{1}{2}$</td>
<td>1444</td>
<td>26</td>
<td>201</td>
</tr>
<tr>
<td>$^2N_\frac{3}{2}$</td>
<td>1444</td>
<td>-5</td>
<td>40</td>
</tr>
<tr>
<td>$^4N_\frac{3}{2}$</td>
<td>1444</td>
<td>10</td>
<td>201</td>
</tr>
<tr>
<td>$^4N_\frac{1}{2}$</td>
<td>1444</td>
<td>-16</td>
<td>201</td>
</tr>
<tr>
<td>$^2\Delta_\frac{1}{2}$</td>
<td>1444</td>
<td>-10</td>
<td>40</td>
</tr>
<tr>
<td>$^2\Delta_\frac{3}{2}$</td>
<td>1444</td>
<td>5</td>
<td>40</td>
</tr>
</tbody>
</table>
for all states belonging to the \([70, 1^-]\) multiplet. This implies that we cannot include \(O_8\) independently in the fit to the experimental spectrum, because its expectation values are proportional to those of \(O_5\).

The operators \(O_3, O_6\) and \(O_7\) are normalized to allow their coefficients \(c_i\) to have a natural size \(20\) \cite{31}. The normalization factors follow from the matrix elements of \(O_i\) presented in Table \(\text{III}\). These matrix elements have been calculated for all available states of the multiplet \([70, 1^-]\) starting from the wave function \(\text{I}\) and using the isoscalar factors of Table \(\text{I}\). The general analytic expressions of \(O_3, O_6\) and \(O_7\), up to an obvious factor, are given in Appendix D. For completeness, in Table \(\text{III}\) we also indicate the off-diagonal matrix elements of \(O_5\) and \(O_6\).

V. RESULTS

We have implemented the matrix elements of Table III into the mass formula \(\text{I}\) and have performed several distinct fits of the theoretical masses to the experiment \[32\]. Each of the six fits corresponds to a selection of operators \(O_i\) used in Eq. \(\text{I}\), such as to cover the most relevant possibilities, in our view.

In this way we have obtained sets of values for the dynamical coefficients \(c_i\) presented in Table \(\text{II}\). In Tables \(\text{IV}\) and \(\text{V}\) we present the masses of the nonstrange resonances belonging to the \([70, 1^-]\) multiplet obtained from the coefficients resulting from the Fit 1 and the Fit 6, which correspond to the lowest values of \(\chi^2_{\text{dof}}\).

In Tables \(\text{IV}\) and \(\text{V}\) we have also indicated the partial contribution (without error bars) of each term present in the total mass. These are obtained from the values of \(c_i\) of Table \(\text{II}\) and the values of \(\langle O_i \rangle\) of Table \(\text{III}\). The Fit 1, containing all operators but \(O_7\), is indeed excellent, giving \(\chi^2_{\text{dof}} \approx 0.43\). From Table \(\text{II}\) one can see that the values of the coefficients \(c_3\) and \(c_4\) are closed to each other, which shows the importance of including \(O_4\), besides the usual \(O_3\). In addition, one can see that \(O_3\) is dominant for the \(4N_J\) resonances while \(O_4\) is dominant for the \(2\Delta_J\) resonances, the contribution being of about 200 MeV in both cases. This brings a new aspect into the description of excited states studied so far, where the dominant term was always the spin-spin term \(\text{II}\), the isospin term being absent in the numerical analysis. To get a better idea about the role of the operator \(O_4\) we have also made a fit by removing it from the definition of the mass operator \(\text{I}\). The result is shown in Table \(\text{II}\) column Fit 5. The \(\chi^2_{\text{dof}}\) deteriorates considerably, becoming 11.5 instead of 0.43. This clearly shows that \(O_4\) is crucial in the fit.

The coefficient \(c_2\) of the spin-orbit term is small and its magnitude and sign remains comparable to that of Ref. \[24\] obtained in the analysis of the \([70, \ell^+]\) multiplet. The value of \(c_2\) implies a small spin-orbit contribution to the total mass, in agreement with the general pattern observed for the excited states \[25\] and in agreement with constituent quark models.

The error bars of \(c_5\) and \(c_6\) are comparable to their central values. However, the removal of \(O_3\) and/or \(O_6\) from the mass operator does not deteriorate the fit too badly, as shown in Table \(\text{III}\) Fits 2–4, the \(\chi^2_{\text{dof}}\) becoming at most 1.04. The contribution of \(O_5\) or \(O_6\) is comparable to that of the spin-orbit operator. Note that the structure of \(O_6\) is related to that of the spin-orbit term, which makes its small contribution entirely plausible. Thus the contribution of all operators containing angular momentum is small, which may be a dynamic effect.

Table \(\text{V}\) shows explicitly the role of the operator \(O_7\), never included so far in numerical fits. One can see that this operator plays a dominant role in \(4N_J\) and \(2\Delta_J\), where it contributes with about 200 MeV to the mass, value comparable to that of \(O_3\) or \(O_4\) in the Fit 1. Including \(O_3, O_4\) and \(O_7\) together, their contributions remains equally large but \(c_7\) changes sign and \(\chi^2_{\text{dof}}\) increases to 0.24 to about 2. This suggests that \(O_7\) somehow compensates for the pure spin \(S \cdot S\) or pure isospin \(T \cdot T\) operators, or in other words, plays a kind of common role with \(O_3\) and \(O_4\). We consider that more theoretical work is needed to better understand the algebraic relations between various \(O_i\) operators, in particular to find new operator identities for mixed symmetric states.

VI. VALIDITY OF THE APPROXIMATE WAVE FUNCTION

In Sec. II it was mentioned that all previous studies of the \([70, 1^-]\) multiplet were performed with the asymmetric wave function \(3.4\) of the second paper of Ref. \[10\]. Here we discuss the validity of this approximation by comparing matrix elements of the same operators calculated both with the exact (symmetric) and the approximate (asymmetric) wave function.

First we consider the operator \(O_3\), common to previous and present calculations. It is a one-body operator, defined by Eq. \(\text{III}\). Its matrix elements can be written as

\[
\langle \ell \cdot s \rangle = N_c \langle \ell(N_c) \cdot s(N_c)\rangle,
\]

because the orbital-spin-flavor wave function is symmetric. Thus it is enough to know the matrix element of a single quark operator, say \(N_c\). Let us illustrate the case \(N_c = 5\), for which the components of the orbital wave function are given in Table \(\text{VII}\). One can see that only the first basis vector, associated to the normal Young tableau, gives a nonvanishing contribution to \(\langle \ell(N_c) \cdot s(N_c)\rangle\) and this comes only from the term \(ssssp\), because it is the only one where the particle 5 is in a \(p\) state. The generalization of this argument to an arbitrary \(N_c\) is obvious and equally good. Thus in the case of a single excited quark it is equally well to calculate \(\langle \ell \cdot s \rangle\) with the exact or with the approximate wave function of
TABLE V: The partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion using Fit 6. The last two columns give the empirically known masses, name and status.

<table>
<thead>
<tr>
<th>Part. contrib. (MeV)</th>
<th>Total (MeV)</th>
<th>Exp. (MeV)</th>
<th>Name, status</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1 O_1$</td>
<td>$c_2 O_2$</td>
<td>$c_3 O_5$</td>
<td>$c_6 O_6$</td>
</tr>
<tr>
<td>$^2N_{1/2}$</td>
<td>1486</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$^4N_{1/2}$</td>
<td>1486</td>
<td>25</td>
<td>-33</td>
</tr>
<tr>
<td>$^2N_{3/2}$</td>
<td>1486</td>
<td>-5</td>
<td>0</td>
</tr>
<tr>
<td>$^4N_{3/2}$</td>
<td>1486</td>
<td>10</td>
<td>26</td>
</tr>
<tr>
<td>$^4N_{5/2}$</td>
<td>1486</td>
<td>-15</td>
<td>7</td>
</tr>
<tr>
<td>$^2\Delta_{1/2}$</td>
<td>1486</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>$^2\Delta_{3/2}$</td>
<td>1486</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

TABLE VI: Matrix elements of operators from the decoupling scheme [16] corresponding to the $[70, 1^-]$ multiplet. The columns asym reproduce results obtained with the asymmetric wave function of Ref. [16] and the columns sym show results obtained with the exact wave function [9], detailed in Appendix A.

<table>
<thead>
<tr>
<th></th>
<th>$\langle s \cdot S_c \rangle$</th>
<th>$\langle S_c^2 \rangle$</th>
<th>$\langle t \cdot T_c \rangle$</th>
<th>$\langle T_c^2 \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>asym</td>
<td>sym</td>
<td>asym</td>
<td>sym</td>
<td>asym</td>
</tr>
<tr>
<td>$^2{}^8$</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$^4{}^8$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$^2{}^{10}$</td>
<td>$-1$</td>
<td>$-\frac{1}{2}$</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
Ref. [16], because the missing terms in the latter function do not contribute. Note however that the spin-orbit operator has a negligible contribution to the mass in all previous and present calculations and in practice it can be neglected.

Next we consider a few of the two-body operators of Ref. [16] $s \cdot S_c$, $S_c^2$, $t \cdot T_c$ and $T_c^2$ and restrict the discussion to the case of physical interest, $N_c = 3$, which is enough for our purpose. Appendix A gives the approximate wave functions [16] and the exact wave functions for the submultiplets $^8_4S_8$, $^{14}_8S_8$ and $^{20}_8S_8$. The calculated matrix elements are shown in Table [17] One can see that for every operator there is a case where the approximation fails. This failure is related to those missing parts of the wave function, where the core has $I_c \neq S_c$. Moreover, the approximate matrix elements of the operators $s \cdot S_c$ and $S_c^2$ turn out to be isospin dependent and the approximate matrix elements of the operators $t \cdot T_c$ and $T_c^2$ are spin dependent. Using the exact wave function, this anomaly disappears.

For an arbitrary $N_c$ we expect that the exact wave function would generally give a dependence on $N_c$ for the matrix elements of operators from previous works, entirely different from that of Table II and III of Ref. [16]. As a by-product, one can also see that the operator $T_c^2$, always ignored previously, has matrix elements comparable to those of $S_c^2$. This is consistent with our result that the isospin-isospin becomes as dominant in $\Delta$ as the spin-spin in $N$ resonances.

Also we found that the operator $O_2$, never included previous fits and containing products of all generators of SU(4), Eqs. (5), plays by itself a dominant role in $^4N$ and $^2\Delta$, states where the spin and isospin are different. By contrast, all operators containing the $O(3)$ generators $\ell_i$ bring negligible contributions to the mass.

A comment is in order regarding Refs. [18, 19] where a submultiplet structure (distinct towers of states) has been found, in the procedure of decoupling the system into a core plus an excited quark. The present analysis would give similar results. The reason is that the existence of three towers of states in the $[70, 1^-]$ multiplet is due to the the presence of three operators when working up to order $N_c^0$: $1$ (of order $N_c$) and $\ell \cdot s$ and $(\ell \cdot s)G/G/N_c$ (of order $N_c^0$). The meson-baryon scattering analysis of Ref. [19] proves the compatibility between the three towers and three resonance poles in the scattering amplitude with quantum numbers corresponding to the states in the $[70, 1^-]$ multiplet.

It would be interesting to reconsider the study of higher excited baryons, for example those belonging to $[70, \ell^+]$ multiplets, in the spirit of the present approach.

In practical terms, the extension to three flavors would involve a considerable amount of work on isoscalar factors of SU(6) generators for mixed symmetric representations.

**APPENDIX A**

We consider the particular case of $N_c = 3$ to prove that the wave function given by Eq. (3.4) of the first paper of Ref. 16 breaks $S_3$ symmetry.

The basis vectors which span the invariant subspace of the mixed symmetric representation correspond to the following Young tableaux

$$X^\lambda \rightarrow \frac{1}{3}, \frac{2}{2}, X^\rho \rightarrow \frac{1}{3}, \frac{2}{2}, \frac{3}{2} \rho,$$

where $X = R, S, F$ and $FS$ are the orbital, spin, flavor and flavor-spin wave functions respectively.

In the spin space one can construct $|S^\lambda\rangle$ and $|S^\rho\rangle$ by first coupling the spin of quarks 1 and 2 to $S_c$ followed by the coupling of $S_c$ to the spin of the third quark. We explicitly have

$$|S^\lambda\rangle = \sum_{m_1, m_2} \left( \begin{array}{c} 1 \ 1/2 \ 1/2 \ S_3 \end{array} \right) \left| S_c = 1; m_1 \right| \left( \begin{array}{c} 1 \ 2 \ m_2 \end{array} \right),$$

and

$$|S^\rho\rangle = |S_c = 0; m_1 = 0 \rangle \left( \begin{array}{c} 1 \ 2 \ m_2 \ S_3 \end{array} \right),$$

and equivalently in the isospin space

$$|F^\lambda\rangle = \sum_{\alpha_1, \alpha_2} \left( \begin{array}{c} 1 \ 1/2 \ 1/2 \ J_3 \end{array} \right) \left| I^c = 1; \alpha_1 \right| \left( \begin{array}{c} 1 \ 2 \ \alpha_2 \end{array} \right),$$
where \( F^S \) and \( S^S \) denote symmetric states in isospin and spin respectively. In this notation the orbital-flavor-spin wave function of a baryon, which must be symmetric under \( S_3 \), is a particular case of Eq. (3) and can be written as

\[
|\psi \rangle = \frac{1}{\sqrt{2}} [R^\lambda (FS)^\lambda + R^\rho (FS)^\rho].
\]

We wish to rewrite the flavor-spin part of the wave function (3.4) of the first paper of Ref. [16], denoted by \(|II_3; SS_3\) in the above notation.

Let us first consider the case \( I = 3/2, S = 1/2 \). One has

\[
|3/2 I_3; 1/2 S_3 \rangle = \sum_{m_1, m_2, \alpha_1, \alpha_2} \left( \frac{S_e}{m_1} \frac{1/2}{m_2} \right) \left( \frac{I_e}{\alpha_1} \frac{1/2}{\alpha_2} \right) \frac{3/2}{I_3} c_{\alpha_1 \alpha_2}^{MS} |S^c = I^c = 1; m_1 \alpha_1 \rangle |1/2, m_2; 1/2, \alpha_2 \rangle,
\]

where \( c_{\alpha_1 \alpha_2}^{MS} = 1 \). The spin-flavor states are factorisable into spin and isospin, so that due to (A2) this state is identical to (A6). For the case \( I = 1/2, S = 3/2 \), one has

\[
|1/2 I_3; 3/2 S_3 \rangle = \sum_{m_1, m_2, \alpha_1, \alpha_2} \left( \frac{S_e}{m_1} \frac{1/2}{m_2} \right) \left( \frac{I_e}{\alpha_1} \frac{1/2}{\alpha_2} \right) \frac{1/2}{I_3} c_{\alpha_1 \alpha_2}^{MS} |S^c = I^c = 1; m_1 \alpha_1 \rangle |1/2, m_2; 1/2, \alpha_2 \rangle,
\]

where \( c_{\alpha_1 \alpha_2}^{MS} = 1 \). Due to (A4) this state is identical to (A8).

Next we consider the case \( I = 1/2, S = 1/2 \),

\[
|1/2 I_3; 1/2 S_3 \rangle = \sum_{m_1, \alpha_1, \eta} \left( \frac{S_e}{m_1} \frac{1/2}{S_3} \right) \left( \frac{I_e}{\alpha_1} \frac{1/2}{I_3} \right) c_{\alpha_1 \eta}^{MS} |S^c = I^c = 1 + \frac{\eta}{2}; m_1 \alpha_1 \rangle |1/2, m_2; 1/2, \alpha_2 \rangle
\]

\[
= \sqrt{\frac{1}{2}} \left\{ \sum_{m_1, \alpha_1} \left( \frac{1}{m_1} \frac{1/2}{S_3} \right) \left( \frac{1}{\alpha_1} \frac{1/2}{I_3} \right) |S^c = I^c = 1; m_1 \alpha_1 \rangle |1/2, m_2; 1/2, \alpha_2 \rangle
\]

\[
- |S^c = I^c = 0; m_1 = \alpha_1 = 0 \rangle |1/2; m_2 = \alpha_2 = 1/2 \rangle \right\},
\]

where we have introduced \( c_{\alpha_1 \eta}^{MS} = \sqrt{\frac{1}{2}} \) and \( c_{\alpha_1 \eta}^{MS} = -\sqrt{\frac{1}{2}} \) after the second equality sign. Due to (A2)-(A5) this
state is identical to (A10). This proves that in (A13), (A14) and (A15) the second term of Eq. (A12) is missing. Thus the wave function of Ref. [16] is truncated. It contains only one term instead of two as required by the \( S_3 \) symmetry. In Sec. VI we show that the missing terms (A7), (A9) and (A11) have a considerable contribution to the matrix elements of some operators used in the \( 1/N_c \) expansion mass formula.

**APPENDIX B**

As an example, in this Appendix we present the orbital basis vectors which span the invariant subspace of the representation [41] of \( S_3 \).

An exact orbital-spin-flavor wave function of five fermions (for which the color part is totally antisymmetric) having the configuration \( s^p \), i.e. a single quark excited to the \( p \) shell, has to be built from 4 independent basis vectors, each having a distinct Young tableau, both in the orbital and spin-flavor spaces. The basis vectors in the orbital space are shown in Table \( \text{VII} \). Note that every term in each state implies the normal order of particles: 1, 2, 3, 4, 5. One can see that the first basis vector, with the 5-th particle in the second row contains the configuration \( ssssp \), i.e. it is the only part of all these basis vectors which has the first four quarks in the ground state and the 5-th in a \( p \) state. One can see that in fact any quark can be excited to the \( p \) shell in a properly symmetrized state. Thus the wave function used in previous literature should contain only this \( ssssp \) term [13, 14, 15, 16, 18, 20] if the core was unexcited. The truncation of the spin-flavor part was discussed in Section II.

**APPENDIX C**

The grouping in Table II is justified by the observation that the isoscalar factors obey the following orthogonality relation

\[
\sum_{S_1I_1S_2I_2} \left( \begin{array}{c|c} [N_c - 1, 1] & [211] \\ \hline S_1I_1 & S_2I_2 \end{array} \right) \left( \begin{array}{c|c} [N_c - 1, 1] & [S-1] \\ \hline SI & S' \end{array} \right)_\rho = \delta_{SS'}\delta_{II'}, \tag{C1}
\]

For completeness also note that the isoscalar factors obey the following symmetry property

\[
\left( \begin{array}{c|c} [N_c - 1, 1] & [211] \\ \hline I_1S_1 & I_2S_2 \end{array} \right) \left( S-1 \right)_\rho = \left( \begin{array}{c|c} [N_c - 1, 1] & [211] \\ \hline S_1I_1 & S_2I_2 \end{array} \right) \left( S(S-1) \right)_\rho, \tag{C2}
\]

**APPENDIX D**

Here we present the analytic form of the matrix elements of operators proportional to \( O_5 \) and \( O_6 \). They have been obtained following the approach described in Sec. III. In that notation we have

\[
\langle \ell', \ell''; \ell'' | [\ell(2)i; ] G^{s\alpha} G^{s\alpha} | \ell SJJ; II \rangle = \delta_{\ell', \ell} \delta_{jj''} \delta_{JJ'} \delta_{II} \delta \frac{N_c(3N_c + 4)}{16} \sqrt{2S'' + 1} \frac{5(\ell + 1)(2\ell - 1)(2\ell + 1)(2\ell + 3)}{6} \sum_{S' S''} \left( \begin{array}{c|c|c} \ell & \ell & J \\ \hline S & S & J' \end{array} \right) \left( \begin{array}{c} 1 \ 1 \ 2 \\ \hline S' \ S'' \end{array} \right) \left( \begin{array}{c} N_c - 1, 1 \\ \hline SI \end{array} \right)_1 \left( \begin{array}{c} 21^2 \end{array} \right) \left( \begin{array}{c} N_c - 1, 1 \\ \hline S'' I'' \end{array} \right)_1 \left( \begin{array}{c} \ell' \ \ell'' \end{array} \right)_1, \tag{D1}
\]
TABLE VII: Young tableaux and the corresponding basis vectors of the irrep [41] of $S_5$ for the configuration $s^4p$.

<table>
<thead>
<tr>
<th>Young tableau</th>
<th>Young-Yamanouchi basis vectors of [41]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 4 5</td>
<td>$\frac{1}{\sqrt{20}} (4sssp - sssps - sspss - pssss)$</td>
</tr>
<tr>
<td>1 2 3 5 4</td>
<td>$\frac{1}{\sqrt{12}} (3ssps - sspss - spsss - pssss)$</td>
</tr>
<tr>
<td>1 2 4 5 3</td>
<td>$\frac{1}{\sqrt{6}} (2sspss - spsss - pssss)$</td>
</tr>
<tr>
<td>1 3 4 5 2</td>
<td>$\frac{1}{\sqrt{2}} (spsss - pssss)$</td>
</tr>
</tbody>
</table>

$$
\langle \ell' J' J_3'; I' I_3' | \ell T^a G^{ia} | \ell S J J_3; I I_3 \rangle = \delta_{\ell' J} \delta_{J_3' J_3} \delta_{I' I_3} \delta_{I J_3 I_3} (-1)^{J+\ell+S'} \frac{N_c(3N_c+4)}{8} \sqrt{2S'+1} 
\times \sqrt{\ell(\ell+1)(2\ell+1)} \left\{ \begin{array}{c} \ell \\ \ell' \\ S' \\ J' \end{array} \right\} \left( \begin{array}{c} [N_c - 1, 1] \\ SI \\ 11 \\ S' I \end{array} \right) \left( \begin{array}{c} [N_c - 1, 1] \\ SI \\ 21^2 \\ S' I \end{array} \right) \left( \begin{array}{c} [N_c - 1, 1] \\ S' I \\ 01 \\ S' I \end{array} \right) \right) \right) \right), \quad (D2)
$$

and

$$
\langle \ell' S' J' J_3'; I' I_3' | S^i T^a G^{ia} | \ell S J J_3; I I_3 \rangle = \delta_{\ell' J} \delta_{J_3' J_3} \delta_{I' I_3} \delta_{I J_3 I_3} 
\times \frac{1}{4} \sqrt{N_c(3N_c+4)} \sqrt{I(I+1)} \sqrt{S(S+1)} 
\left( \begin{array}{c} [N_c - 1, 1] \\ SI \\ 11 \\ S' I' \end{array} \right) \right) \right) \right). \quad (D3)
$$

Acknowledgments The work of one of us (N. M.) was supported by the Institut Interuniversitaire des Sciences Nucléaires (Belgium).

[34] see Ref. 29, Sec 4.5.