Cutting planes from two rows of a simplex tableau

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May 2, 2006

Abstract

1 Introduction

In this paper a basic geometric object is investigated that allows us to derive cutting planes for general mixed integer linear programs by considering two rows of a simplex tableau simultaneously. Throughout this paper the mixed integer linear program (MIP) is given in equation form, *i.e.*, the optimization problem is defined for a set I of integer variables, a set C of continuous variables, a rational matrix A, a rational vector b of right hand sides and a rational objective function vector c (of suitable dimensions)

$$\text{(MIP)} \qquad \max\ c^T x \text{ subject to } Ax = b,\ x \geq 0,\ x_i \in \mathbb{Z} \text{ for } i \in I.$$

Let LP denote the linear programming relaxation of MIP, *i.e.*, the linear program obtained by replacing the condition $x_i \in \mathbb{Z}$ with the weaker condition $x_i \in \mathbb{R}$. In cutting plane theory, the goal is to determine linear inequalities $\sum_{i \in I \cup C} a_i x_i \geq a_0$ that satisfy

- every feasible solution x to MIP satisfies $\sum_{i \in I \cup C} a_i x_i \geq a_0$, and
- a vertex x^* of LP is cut off by the linear inequality, i.e., $\sum_{i \in I \cup C} a_i x_i^* < a_0$.

From linear programming theory, it follows that a vertex x^* of LP corresponds to a basic feasible solution of a simplex tableau associated with subsets B and N of so-called basic and nonbasic variables

$$x_i + \sum_{j \in N} \bar{a}_{i,j} x_j = \bar{b}_i \text{ for } i \in B.$$

Any row associated with an index $i \in B \cap I$ such that $\bar{b}_i \notin \mathbb{Z}$ gives rise to a set

$$X(i) := \left\{ x \in \mathbb{R}^{|N|} \;\middle|\; \bar{b}_i - \sum_{j \in N} \bar{a}_{i,j} x_j \in \mathbb{Z},\ x_j \geq 0 \text{ for all } j \in N \right\}$$

whose analysis provides inequalities that are violated by x^*. Indeed, Gomory's mixed integer cuts [5] and mixed integer rounding cuts [7] are derived from such a basic set $X(i)$, that is – associated with exactly *one* index $i \in B \cap I$ such that $\bar{b}_i \notin \mathbb{Z}$. Interestingly, unlike in the pure integer case, no finite convergence proof of a cutting plane algorithm is known when Gomory's mixed integer cuts or mixed integer rounding cuts are applied only. More drastically, in [4], an interesting mixed integer program in three variables is presented, and it is shown that none of the known classes of general cutting planes contain the cut needed to solve this problem.

Example 1: Consider the mixed integer set

$$\begin{aligned} t &\leq x_1, \\ t &\leq x_2, \\ x_1 + x_2 + t &\leq 2, \\ x \in \mathbb{Z}^2 &\text{ and } t \in \mathbb{R}^1. \end{aligned}$$

The projection of this set onto the space of x_1 and x_2 variables is given by $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 \leq 2\}$ and is illustrated in Fig. 1. A simple analysis shows that the inequality $x_1 + x_2 \leq 2$, or equivalently $t \leq 0$, is valid. In [4] it is, however, shown that with the objective function $z = \max t$, traditional mixed integer cutting plane algorithms do not converge finitely. ■

In this paper we provide some insight into why cutting plane algorithms based on Gomory's mixed integer cuts, split cuts, lift-and-project cuts [3] and MIR cuts do not always converge in finite time. Roughly speaking, the reason is that all the traditional cutting planes arise from sets $X(i)$ associated with one integer variable x_i.

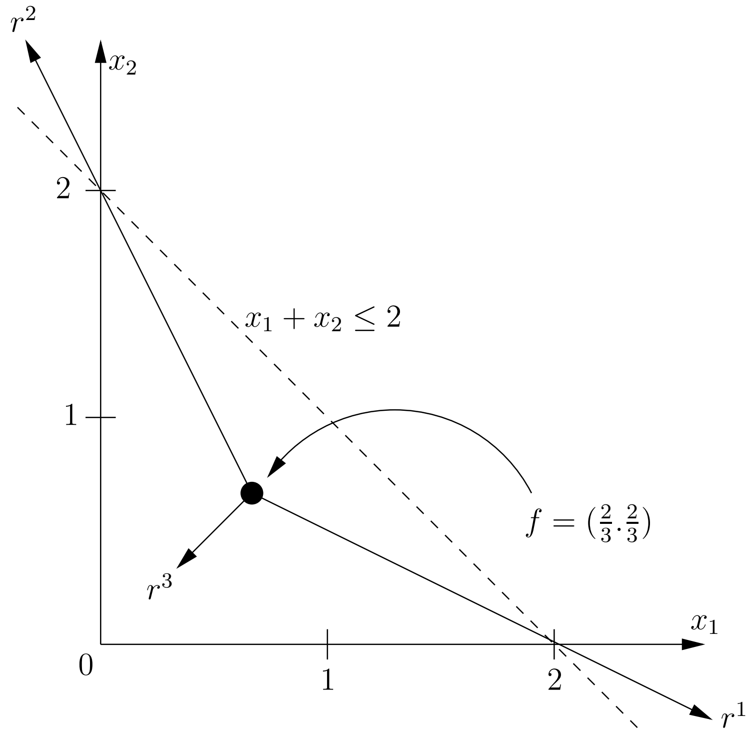


Figure 1: The Instance in [4]

We go one step further in this paper by considering two indices $i_1, i_2 \in B \cap I$ simultaneously. It turns out that the underlying basic geometric object is significantly more complex than its one-variable counterpart. The set that we denote by $X(i_1, i_2)$ is described as

$$X(i_1, i_2) := \big\{x \in \mathbb{R}^{|N|} \mid \ \bar{b}_i - \sum_{j \in N} \bar{a}_{i,j} x_j \in \mathbb{Z} \text{ for } i = i_1, i_2, \ x_j \geq 0 \text{ for all } j \in N\big\}.$$

Setting

$$\begin{array}{lcl} f & := & \big(\bar{b}_{i_1}, \bar{b}_{i_2}\big) \in \mathbb{R}^2, \text{ and} \\ r^j & := & \big(\bar{a}_{i_1}, \bar{a}_{i_2}\big) \in \mathbb{R}^2, \end{array}$$

the set obtained from two rows of a simplex tableau can be represented as

$$P_I := \{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} s_j r^j\},$$

where f is fractional and $r^j \in \mathbb{R}^2$ for all $j \in N$.

Example 1 (revisited): For the instance of Example 1, introduce slack variables, s_1, s_2 and y_1 in the three constraints. Then, solving as a linear program, the constraints of the optimal simplex tableau are

$$\begin{array}{ccccccccc} t & & & +\frac{1}{3}s_1 & +\frac{1}{3}s_2 & +\frac{1}{3}y_1 & = & \frac{2}{3} \\ & x_1 & & -\frac{2}{3}s_1 & +\frac{1}{3}s_2 & +\frac{1}{3}y_1 & = & \frac{2}{3} \\ & & x_2 & +\frac{1}{3}s_1 & -\frac{2}{3}s_2 & +\frac{1}{3}y_1 & = & \frac{2}{3} \end{array}$$

Taking the last two rows, and rescaling $s'_i = s_i/3$ for $i = 1, 2$, we obtain the set P_I

$$\begin{array}{cccccccc} x_1 & & -2s'_1 & +1s'_2 & +\frac{1}{3}y & = & +\frac{2}{3} \\ & x_2 & +1s'_1 & -2s'_2 & +\frac{1}{3}y & = & +\frac{2}{3} \\ \multicolumn{3}{c}{x \in \mathbb{Z}^2, s \in \mathbb{R}^2_+, y_1 \in \mathbb{R}^1_+.} \end{array}$$

Letting $f = (\frac{2}{3}, \frac{2}{3})^T$, $r^1 = (2, -1)^T$, $r^2 = (-1, 2)^T$ and $r_3 = (-\frac{1}{3}, -\frac{1}{3})$ (see Fig. 1), one easily obtains the desired cut for $\text{conv}(P_I)$

$$x_1 + x_2 + y_1 \geq 2 \text{ or equivalently } t \leq 0,$$

which, when used in a cutting plane algorithm, yields immediate termination. ■

Our main contribution is to characterize geometrically all the facets of $\text{conv}(P_I)$. All facets are *intersection cuts* [2], *i.e.*, they can be obtained from a (two-dimensional) convex body that does not contain any integer points in its interior. Our geometric approach is based on two important facts that we prove in this paper

- every facet is derivable from at most four nonbasic variables.
- with every facet F one can associate three or four particular vertices of $\text{conv}(P_I)$. The classification of F depends on how these $k = 3, 4$ vertices can be partitioned into k sets of cardinality at most two.

More precisely, the facets of $\text{conv}(P_I)$ can be distinguished with respect to the number of sets that contain two integer points. Since $k = 3$ or $k = 4$, the following interesting situations occur

- no sets with cardinality two: all the $k \in \{3, 4\}$ sets contain exactly one tight integer point. We call cuts of this type *disection cuts*.
- exactly one set has cardinality two: in this case we show that the inequality can be derived from lifting a cut associated with a two-variable subproblem to k variables. We call these cuts *lifted two-variable cuts*.
- two sets have cardinality two. In this case we show that the corresponding cuts are *split cuts*.

Furthermore, we show that inequalities of the first two families are *never* split cuts. Our geometric approach allows us to generalize the cut introduced in Example 1. More specifically, the cut of Example 1 is a degenerate case in the sense that it is "almost" a disection cut and "almost" a lifted two-variable cut: by perturbing the vectors r^1, r^2 and r^3 slightly, the cut in Example 1 can become both a disection cut and a lifted two-variable cut.

We review some basic facts about the structure of $\text{conv}(P_I)$ in Section 2. In Section 3 we explore the geometry of all the feasible points that are tight for a given facet of $\text{conv}(P_I)$. Section 4 explains our main result and presents the classification of all the facets of $\text{conv}(P_I)$. Finally, Section 5 presents a combinatorial polynomial-time algorithm to compute all the vertices of $\text{conv}(P_I)$. This algorithm can be used to construct a polynomial time separation algorithm.

2 Basic structure of $\text{conv}(P_I)$

The basic mixed-integer set considered in this paper is

$$P_I := \{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} s_j r^j\}, \tag{1}$$

where $N := \{1, 2, \ldots, n\}$, $f \in \mathbb{Q}^2 \backslash \mathbb{Z}^2$ and $r^j \in \mathbb{Q}^2$ for all $j \in N$. The set $P_{LP} := \{(x, s) \in \mathbb{R}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} s_j r^j\}$ denotes the LP relaxation of P_I. The j^{th} unit vector in $\mathbb{R}^n$ is denoted e_j. In this section, we describe some basic properties of $\text{conv}(P_I)$. The vectors $\{r^j\}_{j \in N}$ are called *rays*, and we assume $r^j \neq 0$ for all $j \in N$. We first characterize the case when P_I is empty.

Lemma 1 *The set P_I is empty if and only if*

(i) All rays $\{r^j\}_{j\in N}$ are parallel.

(ii) The lines $\{f + s_j r^j : s_j \in \mathbb{R}\}$ for $j \in N$ do not contain any integer points.

Proof: Clearly (i) and (ii) are sufficient. Suppose $P_I = \emptyset$. If there are $j, k \in N$ such that r^j and r^k are not parallel, the set $f + \mathrm{cone}(\{r^j, r^k\})$ contains integer points. Hence all rays $\{r^j\}_{j\in N}$ are parallel, and the sets $\{f + s_j r^j : s_j \in \mathbb{R}\}$ for $j \in N$ do not contain any integer points. ∎

In the remainder of the paper we assume $P_I \neq \emptyset$. The next lemma gives a characterization of $\mathrm{conv}(P_I)$ in terms of vertices and extreme rays.

Lemma 2

(i) The dimension of $\mathrm{conv}(P_I)$ is n.

(ii) The extreme rays of $\mathrm{conv}(P_I)$ are (r^j, e_j) for $j \in N$.

(iii) The vertices (x^I, s^I) of $\mathrm{conv}(P_I)$ take the following two forms:

*(a) $(x^I, s^I) = (x^I, s^I_j e_j)$, where $x^I = f + s^I_j r^j \in \mathbb{Z}^2$ and $j \in N$
(an integer point on the ray $\{f + s_j r^j : s_j \geq 0\}$).*

*(b) $(x^I, s^I) = (x^I, s^I_j e_j + s^I_k e_k)$, where $x^I = f + s^I_j r^j + s^I_k r^k \in \mathbb{Z}^2$ and $j, k \in N$
(an integer point in the set $f + \mathrm{cone}(\{r^j, r^k\})$).*

Proof: Let $(\bar{x}, \bar{s}) \in P_I$ be arbitrary. (ii): For any $j \in N$, since $r^j \in \mathbb{Q}^2$, there exists a positive integer q_j such that $q_j r^j$ is integer. Hence we have $f + \sum_{j\in N} \bar{s}_j r_j + q_j r_j \in \mathbb{Z}^2$. This proves (r^j, e_j) is a ray of $\mathrm{conv}(P_I)$ for all $j \in N$. In addition, every other ray of $\mathrm{conv}(P_I)$ can be expressed as a conic combination of these rays. (i): The $(n+1)$ points $(\bar{x}, \bar{s})$ and $\{(\bar{x} + q_j r^j, \bar{s} + q_j e_j)\}_{j\in N}$ are in P_I and affinely independent. (iii): If $(\bar{x}, \bar{s})$ is a vertex of $\mathrm{conv}(P_I)$, then $\bar{x}$ is integer, and $\bar{s}$ is a basic solution to the system $\bar{x} = f + \sum_{j\in N} s_j r^j$ and $s_j \geq 0$ for all $j \in N$. ∎

Using Lemma 2, we now give a simple form for the valid inequalities for $\mathrm{conv}(P_I)$ considered in the remainder of the paper.

Corollary 1 *Every non-trivial valid inequality for P_I that is tight at a point $(\bar{x}, \bar{s}) \in P_I$ can be written in the form*

$$\sum_{j\in N} \alpha_j s_j \geq 1, \tag{2}$$

where $\alpha_j \geq 0$ for all $j \in N$.

Proof: Let $\sum_{j\in N} \alpha_j s_j \geq \beta$ be a non-trivial valid inequality for $\mathrm{conv}(P_I)$, and let $(\bar{x}, \bar{s}) \in P_I$ be a tight feasible point, *i.e.*, $\sum_{j\in N} \alpha_j \bar{s}_j = \beta$. Suppose there exists $j \in N$ such that $\alpha_i < 0$. From Lemma 2.(ii), we have $(\bar{x} + r^j, \bar{s} + e_j) \in conv(P_I)$, which contradicts the validity of $\sum_{j\in N} \alpha_j s_j \geq \beta$ for $conv(P_I)$. Hence $\alpha_j \geq 0$ for all $j \in N$. Since $\sum_{j\in N} \alpha_j \bar{s}_j \geq 0$, we can not have $\beta < 0$. If $\beta = 0$, $\sum_{j\in N} \alpha_j s_j \geq \beta$ is a non-negative combination of the inequalities $s_j \geq 0$ for $j \in N$, which contradicts the assumption that $\sum_{j\in N} \alpha_j s_j \geq \beta$ is non-trivial. ∎

For an inequality $\sum_{j\in N} \alpha_j s_j \geq 1$ of the form (2), let $N^0_\alpha := \{j \in N : \alpha_j = 0\}$ denote the variables with coefficient zero, and let $N^{\neq 0}_\alpha := N \setminus N^0_\alpha$ denote the remainder of the variables. We now introduce an object that is associated with the inequality $\sum_{j\in N} \alpha_j s_j \geq 1$. We will use this object to obtain a two dimensional representation of the facets of $\mathrm{conv}(P_I)$.

Lemma 3 *Let $\sum_{j\in N}\alpha_j s_j \geq 1$ be a valid inequality for* $\mathrm{conv}(P_I)$ *of the form (2). Define $v^j := f + \frac{1}{\alpha_j} r^j$ for $j \in N_\alpha^{\neq 0}$. Consider the convex polyhedron in $\mathbb{R}^2$*

$$L_\alpha = \{x \in \mathbb{R}^2 : \text{ there exists } s \in \mathbb{R}^n_+ \text{ s.t. } (x,s) \in P_{LP} \text{ and } \sum_{j\in N} \alpha_j s_j \leq 1\}.$$

(i) $L_\alpha = \mathrm{conv}(\{f\} \cup \{v^j\}_{j\in N_\alpha^{\neq 0}}) + \mathrm{cone}(\{r^j\}_{j\in N_\alpha^0})$.

(ii) $\mathrm{interior}(L_\alpha)$ *does not contain any integer points.*

(iii) If $\mathrm{interior}(L_\alpha) \neq \emptyset$, *then* $f \in \mathrm{interior}(L_\alpha)$.

Proof: (i) First suppose $\bar{x} \in L_\alpha$. This means there exists $\bar{s} \in \mathbb{R}^n_+$ such that $\sum_{j\in N}\alpha_j \bar{s}_j \leq 1$ and $\bar{x} = f + \sum_{j\in N} \bar{s}_j r^j$. Therefore, we may write $\bar{x} = \lambda_0 f + \sum_{j\in N_\alpha^0} \lambda_j v^j + \sum_{j\in N_\alpha^{\neq 0}} \bar{s}_j r^j$, where $\lambda_j = \bar{s}_j\alpha_j$ for $j \in N_\alpha^{\neq 0}$ and $\lambda_0 = 1 - \sum_{j\in N^{\neq 0}} \lambda_j$, which shows $\bar{x} \in \mathrm{conv}(\{f\} \cup \{v^j\}_{j\in N_\alpha^{\neq 0}}) + \mathrm{cone}(\{r^j\}_{j\in N_\alpha^0})$. Conversely, if $\bar{x} \in \mathrm{conv}(\{f\} \cup \{v^j\}_{j\in N_\alpha^{\neq 0}}) + \mathrm{cone}(\{r^j\}_{j\in N_\alpha^0})$, then $\bar{x} = \lambda_0 f + \sum_{j\in N_\alpha^{\neq 0}} \lambda_j v^j + \sum_{j\in N_\alpha^0} \mu_j r^j$, where $\{\mu_j\}_{j\in N_\alpha^0} \subset \mathbb{R}_+$, $\lambda_0 \geq 0$, $\{\lambda_j\}_{N_\alpha^{\neq 0}} \subset \mathbb{R}_+$ and $\lambda_0 + \sum_{j\in N}\lambda_j = 1$. Hence, we may write $\bar{x} = f + \sum_{j\in N} \bar{s}_j r^j$, where $\bar{s}_j = \frac{\lambda_j}{\alpha_j}$ for $j \in N_\alpha^{\neq 0}$ and $\bar{s}_j = \mu_j$ for $j \in N_\alpha^0$, and therefore $\bar{x} \in L_\alpha$. (ii) If $\bar{x} \in \mathrm{interior}(L_\alpha)$, then there exists $\bar{s} \in \mathbb{R}^n_+$ such that $(\bar{x}, \bar{s}) \in P_{LP}$ and $\sum_{j\in N}\alpha_j\bar{s}_j < 1$. Since $\sum_{j\in N}\alpha_j s_j \geq 1$ is valid for P_I, we can not have that $\bar{x}$ is integer. (iii) This is clear from the facts that f can be represented using $s = 0$ and $\mathrm{interior}(L_\alpha)$ can be written in the form $\{x \in \mathbb{R}^2 : \exists s \in \mathbb{R}^n_+ \text{ s.t. } (x,s) \in P_{LP} \text{ and } \sum_{j\in N}\alpha_j s_j < 1\}$. ■

Example 2: Consider the set

$$P_I = \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^5_+ : \quad x = f + \begin{pmatrix} 2 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} -3 \\ 2 \end{pmatrix} s_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_4 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} s_5\},$$

where $f = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$, and consider the inequality

$$2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7}s_5 \geq 1. \tag{3}$$

The corresponding set L_α is shown in Fig. 2. As can be seen from the figure, L_α does not contain any integer points in its interior. It follows that (3) is valid for $\mathrm{conv}(P_I)$. Note that, conversely, the coefficients α_j for $j = 1, 2, \ldots, 5$ can be obtained from the polygon L_α as follows: α_j is the ratio between the length of r^j and the distance between f and v^j. In particular, if the length of r^j is 1, then α_j is the inverse of the distance from f to v^j. ■

The interior of L_α gives a two-dimensional representation of the points $x \in \mathbb{R}^2$ that are affected by the addition of the inequality $\sum_{j\in N}\alpha_j s_j \geq 1$ to the LP relaxation P_{LP} of P_I. In other words, for any $(x,s) \in P_{LP}$ that satisfies $\sum_{j\in N}\alpha_j s_j < 1$, we have $x \in \mathrm{interior}(L_\alpha)$. Furthermore, for a facet defining inequality $\sum_{j\in N}\alpha_j s_j \geq 1$ of $\mathrm{conv}(P_I)$, there exists n affinely independent points $(x^i, s^i) \in P_I$, $i = 1, 2, \ldots, n$, such that $\sum_{j\in N}\alpha_j s^i_j = 1$. The integer points $\{x^i\}_{i\in N}$ are on the boundary of L_α, *i.e.*, they belong to the following integer set:

$$X_\alpha := \{x \in \mathbb{Z}^2 : \exists s \in \mathbb{R}^n_+ \text{ s.t. } (x,s) \in P_{LP} \text{ and } \sum_{j\in N}\alpha_j s_j = 1\}.$$

We have $X_\alpha = L_\alpha \cap \mathbb{Z}^2$, and $X_\alpha \neq \emptyset$ whenever $\sum_{j\in N}\alpha_j s_j \geq 1$ defines a facet of $\mathrm{conv}(P_I)$. In Sect. 3 we characterize $\mathrm{conv}(X_\alpha)$. This characterization is then used in Sect. 4 to characterize L_α. We first characterize the facets of $\mathrm{conv}(P_I)$ that have zero coefficients.

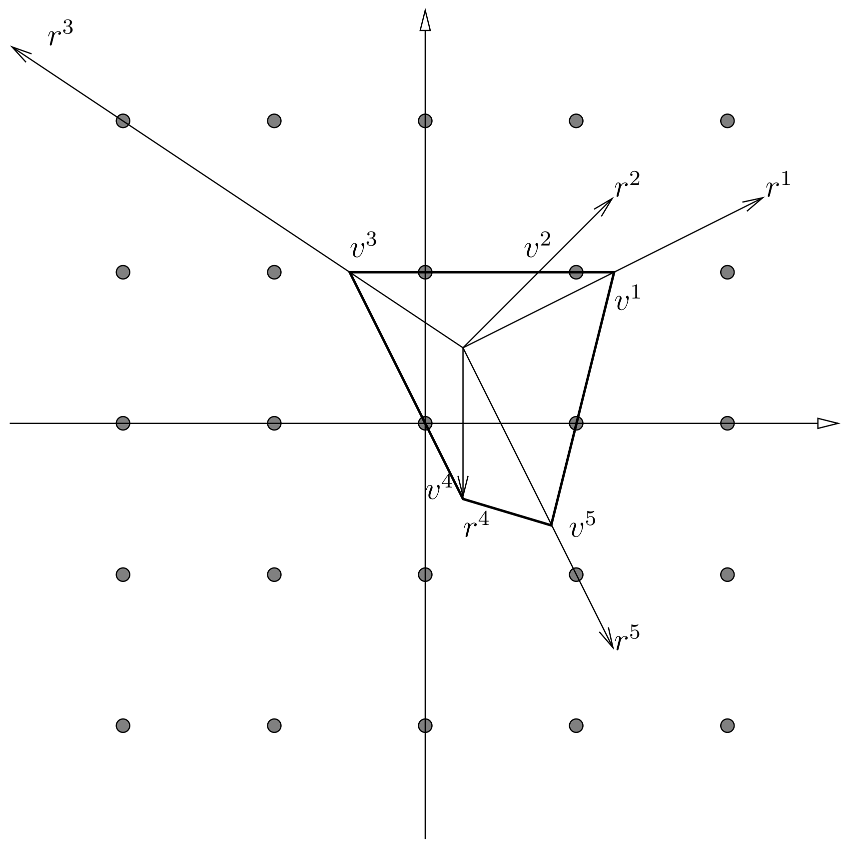


Figure 2: The set L_α for a valid inequality for $\mathrm{conv}(P_I)$

Lemma 4 *Any facet defining inequality $\sum_{j\in N}\alpha_j s_j \geq 1$ for $\mathrm{conv}(P_I)$ of the form (2) that has some zero coefficients is a split cut. In other words, if $N^0_\alpha \neq \emptyset$, there exists $(\pi,\pi_0)\in\mathbb{Z}^2\times\mathbb{Z}$ such that $L_\alpha \subseteq \{(x_1,x_2) : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$.*

Proof: Let $k \in N^0_\alpha$ be arbitrary. Then the line $\{f+\mu r^k : \mu\in\mathbb{R}\}$ does not contain any integer points. Furthermore, if $j\in N^0_\alpha$, $j\neq k$, is such that r^k and r^j are *not* parallel, then $f+\mathrm{cone}(\{r^k,r^j\})$ contains integer points. It follows that all rays $\{r^j\}_{j\in N^0_\alpha}$ are parallel. By letting $\pi' := (r^k)^\perp = \binom{-r^k_2}{r^k_1}$ and $\pi'_0 := (\pi')^T f$, we have $\{f+\mu r^k : \mu\in\mathbb{R}\} = \{\binom{x_1}{x_2} : \pi'_1 x_1 + \pi'_2 x_2 = \pi'_0\}$. Now define:

$$
\begin{aligned}
\pi^1_0 &:= \max\{\pi'_1 x_1 + \pi'_2 x_2 : \pi'_1 x_1 + \pi'_2 x_2 \leq \pi'_0 \text{ and } x\in\mathbb{Z}^2\}, \text{ and}\\
\pi^2_0 &:= \min\{\pi'_1 x_1 + \pi'_2 x_2 : \pi'_1 x_1 + \pi'_2 x_2 \geq \pi'_0 \text{ and } x\in\mathbb{Z}^2\}.
\end{aligned}
$$

We have $\pi^1_0 < \pi'_0 < \pi^2_0$, and the set $S_\pi := \{x\in\mathbb{R}^2 : \pi^1_0 \leq \pi'_1 x_1 + \pi'_2 x_2 \leq \pi^2_0\}$ does not contain any integer points in its interior. We now show $L_\alpha \subseteq S_\pi$ by showing that every vertex $v^m = f + \frac{1}{\alpha_m} r^m$ of L_α, where $m\in N^{\neq 0}_\alpha$, satisfies $v^m \in S_\pi$. Suppose v^m satisfies $\pi'_1 v^m_1 + \pi'_2 v^m_2 < \pi^1_0$ (the case $\pi'_1 v^m_1 + \pi'_2 v^m_2 > \pi^2_0$ is symmetric). By definition of π^1_0, there exists $x^I\in\mathbb{Z}^2$ such that $\pi'_1 x^I_1 + \pi'_2 x^I_2 = \pi^1_0$, and $x^I = \lambda v^m + (1-\lambda)(f+\delta r^k)$, where $\lambda \in]0,1[$, for some $\delta > 0$. We then have $x^I = f + \frac{\lambda}{\alpha_m} r^m + \delta(1-\lambda) r^k$. Inserting this representation of x^I into the inequality $\sum_{j\in N}\alpha_j s_j \geq 1$ then gives $\alpha_m \frac{\lambda}{\alpha_m} + \alpha_k \delta(1-\lambda) = \lambda < 1$, which contradicts the validity of $\sum_{j\in N}\alpha_j s_j \geq 1$ for P_I. Hence $L_\alpha \subseteq S_\pi$.

To finish the proof, we show that we may write $S_\pi = \{x\in\mathbb{R}^2 : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$ for some $(\pi,\pi_0)\in\mathbb{Z}^2\times\mathbb{Z}$. First observe that we can assume (by scaling) that π', π^1_0 and π^2_0 are integers. Next observe that any common divisor of π'_1 and π'_2 also divides both π^1_0 and π^2_0 (this follows from the fact that there exists $x^1, x^2 \in \mathbb{Z}^2$ such that $\pi'_1 x^1_1 + \pi'_2 x^1_2 = \pi^1_0$ and $\pi'_1 x^2_1 + \pi'_2 x^2_2 = \pi^2_0$). Hence we can assume that π'_1 and π'_2 are relative prime. Now the Integral Farkas Lemma (see [9]) implies that the set $\{x\in\mathbb{Z}^2 : \pi'_1 x_1 + \pi'_2 x_2 = 1\}$ is non-empty. It follows that we must have $\pi^2_0 = \pi^1_0 + 1$, since otherwise the point $\bar{y} := x' + x^1 \in \mathbb{Z}^2$, where $x' \in \{x\in\mathbb{Z}^2 : \pi'_1 x_1 + \pi'_2 x_2 = 1\}$ and $x^1 \in \{x\in\mathbb{Z}^2 : \pi'_1 x_1 + \pi'_2 x_2 = \pi^1_0\}$, satisfies $\pi^1_0 < \pi'_1 \bar{y}_1 + \pi'_2 \bar{y}_2 < \pi^2_0$, which contradicts that S_π does not contain any integer points in its interior. ■

3 A characterization of $\mathrm{conv}(X_\alpha)$

As a preliminary step of our analysis, we first characterize the set $\mathrm{conv}(X_\alpha)$. We assume $\alpha_j > 0$ for all $j\in N$. Clearly $\mathrm{conv}(X_\alpha)$ is a convex polygon with only integer vertices, and since $X_\alpha \subseteq L_\alpha$, $\mathrm{conv}(X_\alpha)$ does not have any integer points in its interior. We first limit the number of vertices of $\mathrm{conv}(X_\alpha)$ to four.

Lemma 5 *Let $P\subset\mathbb{R}^2$ be a convex polygon with integer vertices that has no integer points in its interior.*

(i) P has at most four vertices

(ii) If P has four vertices, then at least two of its four facets are parallel.

(iii) If P is not a triangle with integer points in the interior of all three facets (see Fig. 4.(c)), then there exists parallel lines $\pi x = \pi_0$ and $\pi x = \pi_0 + 1$, $(\pi,\pi_0)\in\mathbb{Z}^3$, such that P is contained in the corresponding split set, i.e., $P \subseteq \{x\in\mathbb{R}^2 : \pi_0 \leq \pi x \leq \pi_0 + 1\}$.

Proof: (i)-(ii): Let $v_1, v_2, \ldots, v_p$ denote the vertices of P. Wlog assume $v_1 = 0$, and that v_2 and v_3 are such that $P \subset \mathrm{cone}(v_2, v_3)$. Observe that $\mathrm{cone}(v_2, v_3)$ can be written as the disjoint union of the following parallelograms

$$\Pi_{i,j} := \{z \in \mathbb{R}^2 : z = (i+\lambda_2)v_2 + (j+\lambda_3)v_3, \text{ where } \lambda_2, \lambda_3 \in [0,1[\},$$

where $i, j \geq 0$. The parallelograms $\Pi_{i,j}$ are illustrated in Fig. 3. Note that the following translation property holds: for any integer point $w_{i,j}$ in parallelogram $\Pi_{i,j}$, there is a corresponding integer point $w_{i',j'}$ in parallelogram $\Pi_{i',j'}$ at the same position. Specifically, if the point $w_{i,j} = (i+\lambda_2)v_2 + (j+\lambda_3)v_3 = (iv_2 + jv_3) + (\lambda_2 v_2 + \lambda_3 v_3) \in \Pi_{i,j}$ is integer, then $\lambda_2 v_2 + \lambda_3 v_3$ is integer, and therefore $w_{i',j'} := (i'v_2 + j'v_3) + (\lambda_2 v_2 + \lambda_3 v_3) = (i'+\lambda_2)v_2 + (j'+\lambda_3)v_3 \in \Pi_{i',j'}$ is integer (see Fig. 3).

We first claim that every vertex v_k, $k \geq 4$, must be either on the line $L_2 = \{v_2 + \lambda v_3 | \lambda > 0\}$, or on the line $L_3 = \{\lambda v_2 + v_3 | \lambda > 0\}$. First suppose $v_k \in \Pi_{i,j}$, where $i, j \geq 1$. Then either $v_k \in L_2$ or $v_k \in L_3$, since otherwise the integer point $v_2 + v_3$ is an interior point of P.

Now suppose $v_k \in \Pi_{0,j}$, where $j \geq 1$ (the case when $v_k \in \Pi_{i,0}$, $i \geq 1$, is symmetric). This means there exists $\lambda_2, \lambda_3 \in [0,1[$ such that $v_k = \lambda_2 v_2 + (j+\lambda_3)v_3$. If $\lambda_2 = 0$, v_3 is on the line between v_1 and v_k, which is a contradiction, so we have $\lambda_2 > 0$. From the translation property discussed above, we have $w := \lambda_2 v_2 + (j-1+\lambda_3)v_3 \in \mathbb{Z}^2$. We claim w is in the interior of P. Indeed, we may write

$$w = \frac{j-1+\lambda_3}{j+\lambda_3} v_k + \frac{\lambda_2}{j+\lambda_3} v_2,$$

where $\mu_2 := \frac{\lambda_2}{j+\lambda_3} \in]0,1[$ and $\mu_k := \frac{j-1+\lambda_3}{j+\lambda_3} \in]0,1[$. Hence we have $w = \mu_2 v_2 + \mu_k v_k + (1-\mu_2-\mu_k)v_1$, where $\mu_2 + \mu_k = \frac{(j+\lambda_3)-(1-\lambda_2)}{j+\lambda_3} \in]0,1[$, and this shows w is in the interior of P.

Finally suppose $v_k \in \Pi_{0,0}$. We may therefore write $v_k = \lambda_2 v_2 + \lambda_3 v_3$, where $\lambda_2, \lambda_3 \in [0,1[$. If $\lambda_2 = 0$, v_k is on the line between v_1 and v_3, and if $\lambda_3 = 0$, v_k is on the line between v_1 and v_2, so we can assume $\lambda_2, \lambda_3 > 0$. If $\lambda_2 + \lambda_3 < 1$, then v_k is in the interior of $\text{conv}(\{v_1, v_2, v_3\})$, and if $\lambda_2 + \lambda_3 = 1$, v_k can not be a vertex. Finally, if $\lambda_2 + \lambda_3 > 1$, the point $w := (v_2 + v_3) - v_k = (1-\lambda_2)v_2 + (1-\lambda_3)v_3$ is integer and in the interior of $\text{conv}(\{v_1, v_2, v_3\})$. It follows that v_k must be either in L_2 or L_3.

To finish the proof of (i) and (ii), we only need to argue that $L_2 \cup L_3$ can only contain one vertex of P. Clearly L_2 and L_3 can only contain one vertex each, so we may assume $p = 5$, $v_4 = v_2 + \lambda_3 v_3 \in L_2$ and $v_5 = v_3 + \lambda_2 v_2 \in L_3$. Furthermore, we have either $\lambda_2, \lambda_3 > 1$ or $\lambda_2, \lambda_3 < 1$, since otherwise either v_4 is in the interior of $\text{conv}(\{v_1, v_2, v_3, v_5\})$, or v_5 is in the interior of $\text{conv}(\{v_1, v_2, v_3, v_4\})$. If $\lambda_2, \lambda_3 > 1$, the point $v_2 + v_3$ is in the interior of P, and if $\lambda_2, \lambda_3 < 1$, the point $w := \lambda_2 v_2 + \lambda_3 v_3$ is integer and in the interior of P.

(iii) Consider again the parallelogram $\Pi_{0,0}$. If $|\det(v^2, v^3)| = 1$, it is well known that the vertex 0 is the only integer point contained in $\Pi_{0,0}$. Hence all parallelograms $\Pi_{i,j}$ do not have any integer point in their interior. In particular $\cup_{j\in\mathbb{Z}}\Pi_{0,j}$ and $\cup_{i\in\mathbb{Z}}\Pi_{i,0}$ do not have any integer point in their interior. Furthermore P is always contained in at least one of these two sets since a potential vertex v_4 would be either in L_2 or in L_3.

Now suppose $|\det(v^2, v^3)| > 1$. Therefore $\Pi_{0,0}$ contains more than one integer point. If P is a quadrangle, observe that these points must be on the boundary of $\Pi_{0,0}$, since otherwise P would contain integer points in its interior. If P is a triangle, we assume that the edge (v_2, v_3) does not contain an integer point in its interior. Otherwise we can re-index the vertices so that this holds. In this case, it is also true that the integer points of $\Pi_{0,0}$ are on its boundary. We claim that all integer points of $\Pi_{0,0}$ must be on the same side of $\Pi_{0,0}$, *i.e.*, either in $S_2 := \{\lambda v_2 : 0 \leq \lambda < 1\}$, or in $S_3 := \{\lambda v_3 : 0 \leq \lambda < 1\}$. Suppose we have $w_2 = \lambda_2 v_2 \in \mathbb{Z}^2$ with $0 < \lambda_2 < 1$ and $w_3 = \lambda_3 v_3 \in \mathbb{Z}^2$ with $0 < \lambda_3 < 1$. Note that λ_2 and λ_3 can be chosen in $]0, 1/2]$ by possibly considering $v_2 - w_2$ or $v_3 - w_3$, which are also integer. Therefore $w_2 + w_3 = \lambda_2 v_2 + \lambda_3 v_3$ is integer and in the interior of P if P is a quadrangle. This contradicts the hypothesis that the edge (v_2, v_3) does not contain an integer point if it is a triangle. Hence either S_2 or S_3 does not contain any integer point. This implies that either $\cup_{j\in\mathbb{Z}}\Pi_{0,j}$ or $\cup_{i\in\mathbb{Z}}\Pi_{i,0}$ does not contain an integer point in its interior. ■

From the proof of Lemma 5, it follows that the polygons in Fig. 4 include all possible polygons that can be included in the set L_α in the case when L_α is bounded and of dimension 2. The dashed lines in Fig. 4 indicate the possible split sets that include P. We excluded from Fig. 4 the cases when X_α is of dimension 1. We note that Lemma 5.(iii) (existence of split sets) proves that there cannot be any triangles where two facets have interior integer points, and also that no quadrangle can have more than two facets that have integer points in the interior.

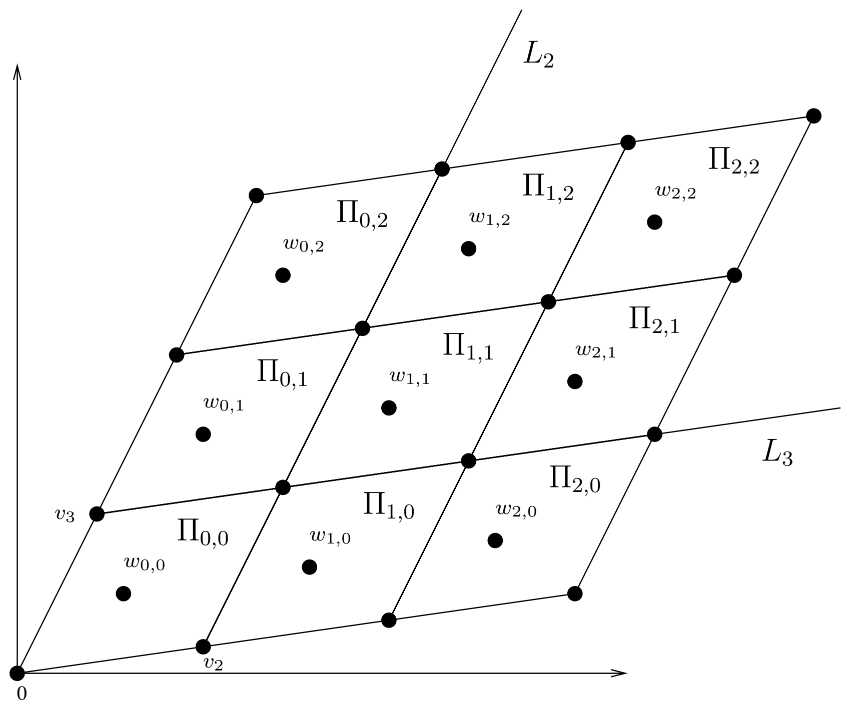


Figure 3: Partitioning of cone(v_2, v_3) into parallelograms

4 A characterization of the facets of $\text{conv}(P_I)$

In this section we focus on the set L_α. As in the previous section, we assume $\alpha_j > 0$ for all $j \in N$. Due to the direct correspondence between the set L_α and a facet defining inequality $\sum_{j\in N} \alpha_j s_j \geq 1$ for $\text{conv}(P_I)$, this gives a characterization of the facets of $\text{conv}(P_I)$. The main result in this section is that L_α can have at most four vertices. In other words, we prove

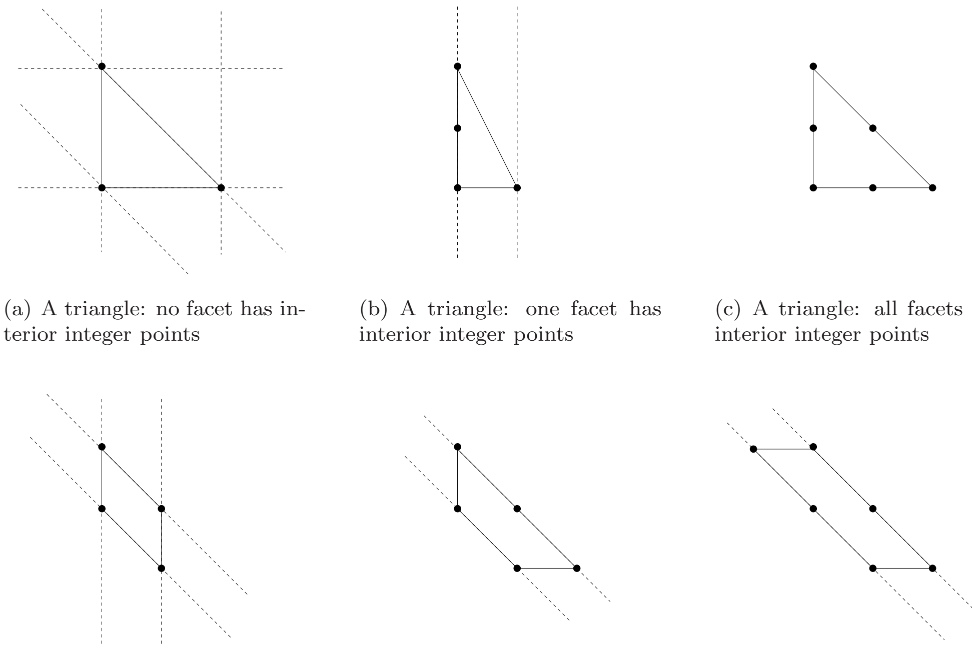
Theorem 2 *Let $\sum_{j\in N} \alpha_j s_j \geq 1$ be a facet defining inequality for $\text{conv}(P_I)$ that satisfies $\alpha_j > 0$ for all $j \in N$. Then L_α is a polygon with at most four vertices.*

Theorem 2 shows that there exists a set $S \subseteq N$ such that $|S| \leq 4$ and $\sum_{j\in S} \alpha_j s_j \geq 1$ is facet defining for $\text{conv}(P_I(S))$, where

$$P_I(S) := \{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}_+^{|S|} : x = f + \sum_{j\in S} s_j r^j\}.$$

Throughout this section we assume that no two rays point in the same direction. If two variables $j_1, j_2 \in N$ are such that $j_1 \neq j_2$ and $r^{j_1} = \delta r^{j_2}$ for some $\delta > 0$, then the halflines $\{x \in \mathbb{R}^2 : x = f + s_{j_1} r^{j_1}, s_{j_1} \geq 0\}$ and $\{x \in \mathbb{R}^2 : x = f + s_{j_2} r^{j_2}, s_{j_2} \geq 0\}$ intersect the boundary of L_α at the same point, and therefore $L_\alpha = \text{conv}(\{f\} \cup \{v^j\}_{j\in N}) = \text{conv}(\{f\} \cup \{v^j\}_{j\in N\setminus\{j_2\}})$, where $v^j := f + \frac{1}{\alpha_j} r^j$ for $j \in N$. This assumption does therefore not effect the validity of Theorem 2.

The proof of Theorem 2 is based on characterizing the vertices $\text{conv}(P_I)$ that are tight for $\sum_{j\in N} \alpha_j s_j \geq 1$. Since $\text{conv}(P_I)$ has dimension n, there exists n affinely independent vertices



(a) A triangle: no facet has interior integer points

(b) A triangle: one facet has interior integer points

(c) A triangle: all facets have interior integer points

(d) A quadrangle: no facet has interior integer points

(e) A quadrangle: one facet has interior integer points

(f) A quadrangle: two facets have interior integer points

Figure 4: All integer polygons that do not have interior integer points

of $\mathrm{conv}(P_I)$ that satisfy $\sum_{j\in N} \alpha_j s_j \geq 1$ with equality. In the remainder of this section, the set $\{(y^i, t^i)\}_{i\in M} \subseteq P_I$ denotes n such points, where $M := \{1, 2, \ldots, n\}$. In other words we have

$$\sum_{j\in N} \alpha_j t^i_j = 1, \quad \text{for } i \in M. \tag{4}$$

Since $\sum_{j\in N} \alpha_j s_j \geq 1$ is facet defining for $\mathrm{conv}(P_I)$, the values $\{\alpha_j\}_{j\in N}$ are the unique solution to (4). It follows that every variable $j \in N$ is involved in system (4). We will show that there exists a subset $S \subseteq N$ of variables and a set of $|S|$ affinely independent vertices of $\mathrm{conv}(P_I)$ such that $|S| \leq 4$ and $\{\alpha_j\}_{j\in S}$ is the unique solution to the equality system defined by these vertices.

Observe that the set $\{y^i\}_{i\in M}$ is the set of vertices of $\mathrm{conv}(X_\alpha)$. We can therefore partition the equalities (4) according to which vertex of $\mathrm{conv}(X_\alpha)$ is involved in each equality. The following notation will be used intensively in the remainder of this section.

Notation 1

(i) *The number* $k \leq 4$ *denotes the number of vertices of* $\mathrm{conv}(X_\alpha)$*.*

(ii) *The set* $\{x^v\}_{v\in K}$ *denotes the vertices of* $\mathrm{conv}(X_\alpha)$*, where* $K := \{1, 2, \ldots, k\}$*.*

(iii) *For every* $v \in K$*, the set* $M^v := \{i \in M : y^i = x^v\}$*, denotes the equalities of (4) for which* x^v *is the vertex of* $\mathrm{conv}(X_\alpha)$ *involved in the corresponding equality of (4).*

The system (4) can now be rewritten as

$$\sum_{j\in N} \alpha_j t^i_j = 1, \quad \text{for every vertex } x^v \text{ of } \mathrm{conv}(X_\alpha) \text{ and } i \in M^v. \tag{5}$$

Lemma 2.(iii) demonstrates that for a vertex $(\bar{x}, \bar{s})$ of $\text{conv}(P_I)$, $\bar{s}$ is positive on at most two coordinates $j_1, j_2 \in N$, and $\bar{x} \in f + \text{cone}(\{r^{j_1}, r^{j_2}\})$. If for some $i \in M$, t^i is positive on only one coordinate, then $y^i = f + t^i_j r^j$ for some $j \in N$, and the system (5) includes the equality $\alpha_j t^i_j = 1$, which simply states $\alpha_j = \frac{1}{t^i_j}$. A point $\bar{x} \in \mathbb{Z}^2$ that satisfies $\bar{x} \in \{x \in \mathbb{R}^2 : x = f + s_j r^j, s_j \geq 0\}$ for some $j \in N$ will be called a *ray point* in the remainder of the paper. In order to characterize the equalities of (5) that correspond to vertices x^v of $\text{conv}(X_\alpha)$ that are *not* ray points, we introduce the following concepts.

Definition 1 *Let $\sum_{j \in N} \alpha_j s_j \geq 1$ be valid for* $\text{conv}(P_I)$. *Suppose $\bar{x} \in \mathbb{Z}^2$ is* not *a ray point, and that $\bar{x} \in f + \text{cone}(\{r^{j_1}, r^{j_2}\})$, where $j_1, j_2 \in N$. This implies $\bar{x} = f + s_{j_1} r^{j_1} + s_{j_2} r^{j_2}$, where $s_{j_1}, s_{j_1} > 0$ are unique.*

(a) The pair (j_1, j_2) is said to give a representation *of $\bar{x}$.*

(b) If $\alpha_{j_1} s_{j_1} + \alpha_{j_2} s_{j_2} = 1$, (j_1, j_2) is said to give a tight *representation of $\bar{x}$ wrt. $\sum_{j \in N} \alpha_j s_j \geq 1$.*

(c) If $(i_1, i_2) \in N \times N$ satisfies $\text{cone}(\{r^{i_1}, r^{i_2}\}) \subseteq \text{cone}(\{r^{j_1}, r^{j_2}\})$, the pair (i_1, i_2) is said to define a subcone *of (j_1, j_2).*

Example 2 (continued): Consider again the set

$$P_I = \{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^5_+ : \quad x = f + \begin{pmatrix} 2 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} -3 \\ 2 \end{pmatrix} s_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_4 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} s_5\},$$

where $f = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$, and the valid inequality $2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7} s_5 \geq 1$ for $\text{conv}(P_I)$. The point $\bar{x} = (1, 1)$ is on the boundary of L_α (see Fig. 2). We have that $\bar{x}$ can be written in any of the following forms

$$\begin{aligned} \bar{x} &= f + \frac{1}{4} r^1 + \frac{1}{4} r^2, \\ \bar{x} &= f + \frac{3}{7} r^1 + \frac{1}{28} r^3, \\ \bar{x} &= f + \frac{3}{4} r^2 + \frac{1}{4} r^4. \end{aligned}$$

It follows that $(1, 2), (1, 3)$ and $(2, 4)$ all give representations of $\bar{x}$. Note that $(1, 2)$ and $(1, 3)$ give tight representations of $\bar{x}$ wrt. the inequality $2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7} s_5 \geq 1$, whereas $(2, 4)$ does not. Finally note that $(1, 5)$ defines a subcone of $(2, 4)$. ■

Observe that, for a vertex x^v of $\text{conv}(X_\alpha)$ and an equality $i \in M^v$, if t^i has support two, the equality $\sum_{j \in N} \alpha_j t^i_j = 1$ of system (5) can be written as $\alpha_{j_1} t^i_{j_1} + \alpha_{j_2} t^i_{j_2} = 1$, where $j_1, j_2 \in N$ satisfy $t^i_{j_1}, t^i_{j_2} > 0$. Hence, the equality $\sum_{j \in N} \alpha_j t^i_j = 1$ corresponds to the tight representation (j_1, j_2) of x^v wrt. $\sum_{j \in N} \alpha_j s_j \geq 1$. Furthermore, if t^i is positive on only one coordinate, then x^v is a ray point. We now characterize the set of tight representations of an arbitrary integer point $\bar{x} \in \mathbb{Z}^2$, which is not a ray point

$$T_\alpha(\bar{x}) := \{(j_1, j_2) \in N \times N : (j_1, j_2) \text{ gives a tight representation of } \bar{x} \text{ wrt. } \sum_{j \in N} \alpha_j s_j \geq 1\}.$$

We will show (Lemma 6 and Lemma 7) that $T_\alpha(\bar{x})$ contains a unique maximal representation $(j_1^{\bar{x}}, j_2^{\bar{x}}) \in T_\alpha(\bar{x})$ that satisfies:

(i) Every subcone (j_1, j_2) of $(j_1^{\bar{x}}, j_2^{\bar{x}})$ that gives a representation of $\bar{x}$ satisfies $(j_1, j_2) \in T_\alpha(\bar{x})$.

(ii) Conversely, every $(j_1, j_2) \in T_\alpha(\bar{x})$ defines a subcone of $(j_1^{\bar{x}}, j_2^{\bar{x}})$.

We then use the maximal representations of the vertices of $\operatorname{conv}(X_\alpha)$ to rewrite (5) (Lemma 8). From this system, we construct a subset S of variables such that $|S| = k$ and $\sum_{j\in S} \alpha_j s_j \geq 1$ is facet defining for $\operatorname{conv}(P_I(S))$.

To prove (i) and (ii), there are two cases to consider. For two representations (i_1, i_2) and (j_1, j_2) of $\bar{x}$, either one of the two cones (i_1, i_2) and (j_1, j_2) is contained in the other, or their intersection defines a subcone of both (i_1, i_2) and (j_1, j_2) (note that we cannot have that their intersection is empty, since they both give a representation of $\bar{x}$). We first consider the case when one cone defines a subcone of another cone.

Lemma 6 *Let $\sum_{j\in N} \alpha_j s_j \geq 1$ be a facet defining inequality for $\operatorname{conv}(P_I)$ that satisfies $\alpha_j > 0$ for all $j \in N$, and let $\bar{x} \in \mathbb{Z}^2$. Then $(j_1, j_2) \in T_\alpha(\bar{x})$ implies $(i_1, i_2) \in T_\alpha(\bar{x})$ for every subcone (i_1, i_2) of (j_1, j_2) that gives a representation of $\bar{x}$.*

Proof: Suppose $(j_1, j_2) \in T_\alpha(\bar{x})$. Observe that it suffices to prove the following: for any $j_3 \in N$ such that $r^{j_3} \in \operatorname{cone}(\{r^{j_1}, r^{j_2}\})$ and (j_1, j_3) gives a representation of $\bar{x}$, the representation (j_1, j_3) is tight wrt. $\sum_{j\in N} \alpha_j s_j \geq 1$. The result for all remaining subcones of (j_1, j_2) follows from repeated application of this result. For simplicity we assume $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$.

Since $\bar{x} \in f + \operatorname{cone}(\{r^1, r^2\})$, $\bar{x} \in f + \operatorname{cone}(\{r^1, r^3\})$ and $r^3 \in \operatorname{cone}(\{r^1, r^2\})$, we may write $\bar{x} = f + u_1 r^1 + u_2 r^2$, $\bar{x} = f + v_1 r^1 + v_3 r^3$ and $r^3 = w_1 r^1 + w_2 r^2$, where $u_1, u_2, v_1, v_3, w_1, w_2 \geq 0$. Furthermore, since $(1, 2)$ gives a tight representation of $\bar{x}$ wrt. $\sum_{j\in N} \alpha_j s_j \geq 1$, we have $\alpha_1 u_1 + \alpha_2 u_2 = 1$. Finally we have $\alpha_1 v_1 + \alpha_3 v_3 \geq 1$, since $\sum_{j\in N} \alpha_j s_j \geq 1$ is valid for P_I. If also $\alpha_1 v_1 + \alpha_3 v_3 = 1$, we are done, so suppose $\alpha_1 v_1 + \alpha_3 v_3 > 1$.

We first argue that this implies $\alpha_3 > \alpha_1 w_1 + \alpha_2 w_2$. Since $\bar{x} = f + u_1 r^1 + u_2 r^2 = f + v_1 r^1 + v_3 r^3$, it follows that $(u_1 - v_1) r^1 = v_3 r^3 - u_2 r^2$. Now, using the representation $r^3 = w_1 r^1 + w_2 r^2$, we get $(u_1 - v_1 - v_3 w_1) r^1 + (u_2 - v_3 w_2) r^2 = 0$. Since r^1 and r^2 are linearly independent, we obtain:

$$(u_1 - v_1) = v_3 w_1 \text{ and } u_2 = v_3 w_2.$$

Now we have $\alpha_1 v_1 + \alpha_3 v_3 > 1 = \alpha_1 u_1 + \alpha_2 u_2$, which implies $(v_1 - u_1)\alpha_1 - \alpha_2 u_2 + \alpha_3 v_3 > 0$. Using the identities derived above, we get $-v_3 w_1 \alpha_1 - \alpha_2 v_3 w_2 + \alpha_3 v_3 > 0$, or equivalently $v_3(-w_1 \alpha_1 - \alpha_2 w_2 + \alpha_3) > 0$. It follows that $\alpha_3 > \alpha_1 w_1 + \alpha_2 w_2$.

We now derive a contradiction to the identity $\alpha_3 > \alpha_1 w_1 + \alpha_2 w_2$. Since $\sum_{j\in N} \alpha_j s_j \geq 1$ defines a facet of $\operatorname{conv}(P_I)$, there must exist $x' \in \mathbb{Z}^2$ and $k \in N$ such that $(3, k)$ gives a tight representation of x' wrt. $\sum_{j\in N} \alpha_j s_j \geq 1$. In other words, there exists $x' \in \mathbb{Z}^2$, $k \in N$ and $\delta_3, \delta_k \geq 0$ such that $x' = f + \delta_3 r^3 + \delta_k r^k$ and $\alpha_3 \delta_3 + \alpha_k \delta_k = 1$. Furthermore, we can choose x', δ_3 and δ_k such that r^3 is used in the representation of x', *i.e.*, we can assume $\delta_3 > 0$.

Now, using the representation $r^3 = w_1 r^1 + w_2 r^2$ then gives $x' = f + \delta_3 r^3 + \delta_k r^k = f + \delta_3 w_1 r^1 + \delta_3 w_2 r^2 + \delta_k r^k$. Since $\sum_{j\in N} \alpha_j s_j \geq 1$ is valid for P_I, we have $\alpha_1 \delta_3 w_1 + \alpha_2 \delta_3 w_2 + \alpha_k \delta_k \geq 1 = \alpha_3 \delta_3 + \alpha_k \delta_k$. This implies $\delta_3(\alpha_3 - \alpha_1 w_1 - \alpha_2 w_2) \leq 0$, and therefore $\alpha_3 \leq \alpha_1 w_1 - \alpha_2 w_2$, which is a contradiction. ∎

To finish the proof of (i) and (ii), we need to consider the case of two cones $(j_1, j_2), (j_3, j_4) \in T_\alpha(\bar{x})$, where neither of the two cones is a subcone of the other cone. This case is captured in the following lemma.

Lemma 7 *Let $\sum_{j\in N} \alpha_j s_j \geq 1$ be a facet defining inequality for $\operatorname{conv}(P_I)$ satisfying $\alpha_j > 0$ for $j \in N$, and suppose $\bar{x} \in \mathbb{Z}^2$ is* not *a ray point. Also suppose the intersection between the cones $(j_1, j_2), (j_3, j_4) \in T_\alpha(\bar{x})$ is given by the subcone (j_2, j_3) of both (j_1, j_2) and (j_3, j_4). Then $(j_1, j_4) \in T_\alpha(\bar{x})$, i.e., (j_1, j_4) also gives a tight representation of $\bar{x}$.*

Proof: For simplicity assume $j_1 = 1$, $j_2 = 2$, $j_3 = 3$ and $j_4 = 4$. Since the cones $(1, 2)$ and $(3, 4)$ intersect in the subcone $(2, 3)$, we have $r^3 \in \text{cone}(\{r^1, r^2\})$, $r^2 \in \text{cone}(\{r^3, r^4\})$, $r^4 \notin \text{cone}(\{r^1, r^2\})$ and $r^1 \notin \text{cone}(\{r^3, r^4\})$. We first represent $\bar{x}$ in the translated cones in which we have a tight representation of $\bar{x}$. In other words, we can write

$$\bar{x} = f + u_1 r^1 + u_2 r^2, \tag{6}$$

$$\bar{x} = f + v_3 r^3 + v_4 r^4 \text{ and} \tag{7}$$

$$\bar{x} = f + z_2 r^2 + z_3 r^3, \tag{8}$$

where $u_1, u_2, v_3, v_4, z_2, z_3 > 0$. Note that Lemma 6 proves that (8) gives a tight represention of $\bar{x}$. Using (6)-(8), we obtain the relation

$$\begin{pmatrix} T_{1,1} I_2 & T_{1,2} I_2 \\ T_{2,1} I_2 & T_{2,2} I_2 \end{pmatrix} \begin{pmatrix} r^2 \\ r^3 \end{pmatrix} = \begin{pmatrix} u_1 r^1 \\ v_4 r^4 \end{pmatrix}, \tag{9}$$

where T is the 2×2 matrix $T := \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} = \begin{pmatrix} (z_2 - u_2) & z_3 \\ z_2 & (z_3 - v_3) \end{pmatrix}$ and I_2 is the 2×2 identity matrix. On the other hand, the tightness of the representations (6)-(8) leads to the following identities

$$\alpha_1 u_1 + \alpha_2 u_2 = 1, \tag{10}$$

$$\alpha_3 v_3 + \alpha_4 v_4 = 1 \text{ and} \tag{11}$$

$$\alpha_2 z_2 + \alpha_4 z_3 = 1, \tag{12}$$

where, again, the last identity follows from Lemma 6. Using (10)-(12), we obtain the relation

$$\begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} u_1 \alpha_1 \\ v_4 \alpha_4 \end{pmatrix}. \tag{13}$$

We now argue that T is non-singular. Suppose, for a contradiction, that $T_{1,1} T_{2,2} = T_{1,2} T_{2,1}$. From (7) and (8) we obtain $v_4 r^4 = (z_3 - v_3) r^3 + z_2 r^2$, which implies $z_3 < v_3$, since $r^4 \notin \text{cone}(\{r^1, r^2\}) \supseteq \text{cone}(\{r^2, r^3\})$. Multiplying the first equation of (13) with $T_{2,2}$ gives $T_{2,2} T_{1,1} \alpha_2 + T_{2,2} T_{1,2} \alpha_3 = u_1 T_{2,2} \alpha_1$, which implies $T_{1,2}(T_{2,1} \alpha_2 + T_{2,2} \alpha_3) = u_1 T_{2,2} \alpha_1$. By using the definition of T, this can be rewritten as $z_3(\alpha_2 z_2 + (z_3 - v_3)\alpha_3) = u_1 \alpha_1 (z_3 - v_3)$. Since $z_2 \alpha_2 + z_3 \alpha_3 = 1$, this implies $z_3(1 - v_3 \alpha_3) = u_1 \alpha_1 (z_3 - v_3)$. However, from (11) we have $v_3 \alpha_3 \in]0, 1[$, so $z_3(1 - v_3 \alpha_3) > 0$ and $u_1 \alpha_1 (z_3 - v_3) < 0$, which is a contradiction. Hence T is non-singular.

We now solve (9) for an expression of r^2 and r^3 in terms of r^1 and r^4. The inverse of the coefficient matrix on the left hand side of (9) is given by $\begin{pmatrix} T^{-1}_{1,1} I_2 & T^{-1}_{1,2} I_2 \\ T^{-1}_{2,1} I_2 & T^{-1}_{2,2} I_2 \end{pmatrix}$, where $T^{-1} := \begin{pmatrix} T^{-1}_{1,1} & T^{-1}_{1,2} \\ T^{-1}_{2,1} & T^{-1}_{2,2} \end{pmatrix}$ denotes the inverse of T. We therefore obtain

$$r^2 = \lambda_1 r^1 + \lambda_4 r^4 \text{ and} \tag{14}$$

$$r^3 = \mu_1 r^1 + \mu_4 r^4, \tag{15}$$

where $\lambda_1 := u_1 T^{-1}_{1,1}, \lambda_4 := v_4 T^{-1}_{1,2}$, $\mu_1 := u_1 T^{-1}_{2,1}$ and $\mu_4 := v_4 T^{-1}_{2,2}$. Similarly, solving (13) to express α_2 and α_3 in terms of α_1 and α_4 gives

$$\alpha_2 = \lambda_1 \alpha_1 + \lambda_4 \alpha_4 \text{ and} \tag{16}$$

$$\alpha_3 = \mu_1 \alpha_1 + \mu_4 \alpha_4. \tag{17}$$

Now, using for instance (6) and (14), we obtain

$$\bar{x} = f + (u_1 + u_2\lambda_1)r^1 + (u_2\lambda_4)r^4, \quad \text{and:}$$

$$\begin{array}{ll} (u_1 + u_2\lambda_1)\alpha_1 + (u_2\lambda_4)\alpha_4 & = \quad \text{(using (10))} \\ (1 - u_2\alpha_2) + u_2\lambda_1\alpha_1 + (u_2\lambda_4)\alpha_4 & = \\ 1 + u_2(\lambda_1\alpha_1 + \lambda_4\alpha_4 - \alpha_2) & = 1. \ \text{(using (16))} \end{array}$$

To finish the proof, we only need to argue that we indeed have $\bar{x} \in f + \text{cone}(\{r^1, r^4\})$, *i.e.*, that $\bar{x} = f + \delta_1 r^1 + \delta_4 r^4$ with $\delta_1 = u_1 + u_2\lambda_1$ and $\delta_4 = u_2\lambda_4$ satisfying $\delta_1, \delta_4 \geq 0$. If $\delta_1 \leq 0$ and $\delta_4 > 0$, we have $\bar{x} = f + \delta_1 r^1 + \delta_4 r^4 = f + u_1 r^1 + u_2 r^2$, which means $\delta_4 r^4 = (u_1 - \delta_1)r^1 + u_2 r^2 \in \text{cone}(\{r^1, r^2\})$, which is a contradiction. Similarly, if $\delta_1 > 0$ and $\delta_4 \leq 0$, we have $\bar{x} = f + \delta_1 r^1 + \delta_4 r^4 = f + v_3 r^3 + v^4 r^4$, which implies $\delta_1 r^1 = v_3 r^3 + (v_4 - \delta_4) r^4 \in \text{cone}(\{r^3, r^4\})$, which is also a contradiction. Hence we can assume $\delta_1, \delta_4 \leq 0$. However, since $\delta_1 = u_1 + u_2\lambda_1$ and $\delta_4 = u_2\lambda_4$, this implies $\lambda_1, \lambda_4 \leq 0$, and this contradicts what was shown above, namely that the representation $\bar{x} = f + \delta_1 r^1 + \delta_4 r^4$ satisfies $\alpha_1\delta_1 + \alpha_4\delta_4 = 1$. ■

By using Lemma 6 and Lemma 7, *i.e.*, the characterization of the tight representations of an integer point, we now rewrite the subsystems of (5) that correspond to vertices x^v of $\text{conv}(X_\alpha)$ that satisfy $|M^v| > 1$.

Lemma 8 *Let $\sum_{j \in N} \alpha_j s_j \geq 1$ be a facet defining inequality for $\text{conv}(P_I)$ s.t. $\alpha_j > 0$ for $j \in N$. Consider a vertex x^v of $\text{conv}(X_\alpha)$ that satisfies $|M^v| > 1$, and the subsystem of (5) corresponding to x^v*

$$\sum_{j \in N} \alpha_j t_j^i = 1, \quad \textit{for every } i \in M^v. \tag{18}$$

Let $J^v := \{j \in N : t_j^i > 0$ for some $i \in M^v\}$ denote the variables that appear in (18). There exist two variables $j_1^v, j_2^v \in J^v$ such that

(i) (j_1^v, j_2^v) gives a tight representation of x^v wrt. $\sum_{j \in N} \alpha_j s_j \geq 1$, i.e., there exists $w_{j_1^v}, w_{j_2^v} > 0$ such that $x^v = f + w_{j_1^v} r^{j_1^v} + w_{j_2^v} r^{j_2^v}$ and $w_{j_1^v}\alpha_{j_1^v} + w_{j_2^v}\alpha_{j_2^v} = 1$.

(ii) For every $j \in J^v \setminus \{j_1^v, j_2^v\}$, we have $r^j \in cone(r^{j_1^v}, r^{j_2^v})$, i.e., there exists $w_1^j, w_2^j > 0$ such that $r^j = w_1^j r^{j_1^v} + w_2^j r^{j_2^v}$.

(iii) $\{\alpha_j\}_{j \in N}$ is the unique solution to the system (5') obtained from (5) by replacing (18) with

$$w_{j_1^v}\alpha_{j_1^v} + w_{j_2^v}\alpha_{j_2^v} = 1, \tag{19}$$

$$\alpha_j = w_1^j \alpha_{j_1^v} + w_2^j \alpha_{j_2^v}, \qquad \textit{for every } j \in J^v \setminus \{j_1^v, j_2^v\}. \tag{20}$$

Proof: Observe that, since $|M^v| > 1$, there exists $i \in M^v$ such that t^i is non-zero on two coordinates. Wlog suppose this is true for *all* $i \in M^v$. For every $i \in M^v$, let $j_1^i, j_2^i \in J^v$ be such that $t_{j_1^i}^i, t_{j_2^i}^i > 0$. Then (j_1^i, j_2^i) gives a tight representation of x^v wrt. $\sum_{j \in N} \alpha_j s_j \geq 1$ for every $i \in M^v$. It follows from Lemma 6 and Lemma 7 that there exists $j_1^v, j_2^v \in J^v$ such that $j_1^v \neq j_2^v$, (j_1^v, j_2^v) gives a tight representation of x^v wrt. $\sum_{j \in N} \alpha_j s_j \geq 1$, and (j_1^i, j_2^i) defines a subcone of (j_1^v, j_2^v) for all $i \in M^v$. This shows (i) and (ii).

We now show (iii). Clearly equality (19) is implied by (5), since (j_1^v, j_2^v) gives a tight representation of x^v wrt. $\sum_{j \in N} \alpha_j s_j \geq 1$ (note that (19) might not be part of the system (5)). Now consider a variable $j \in J^v \setminus \{j_1^v, j_2^v\}$, and define the point $x' := f + \frac{1}{w_1^j \alpha_{j_1^v} + w_2^j \alpha_{j_2^v}} r^j$ on the halfline $\{f + s_j r^j : s_j \geq 0\}$. By using (ii), we obtain $x' = f + \frac{w_1^j}{w_1^j \alpha_{j_1^v} + w_2^j \alpha_{j_2^v}} r^{j_1^v} + \frac{w_2^j}{w_1^j \alpha_{j_1^v} + w_2^j \alpha_{j_2^v}} r^{j_2^v}$.

Inserting this representation of x' into $\sum_{j\in N}\alpha_j s_j \geq 1$ then gives $\frac{w_1^j\alpha_{j_1^v}}{w_1^j\alpha_{j_1^v}+w_2^j\alpha_{j_2^v}} + \frac{w_2^j\alpha_{j_2^v}}{w_1^j\alpha_{j_1^v}+w_2^j\alpha_{j_2^v}} = 1$. This shows x' is on the boundary of L_α, and since the intersection point between the halfline $\{f + s_j r^j : s_j \geq 0\}$ and the boundary of L_α is unique, we have $x' = v^j = f + \frac{1}{\alpha_j} r^j$, which implies $\alpha_j = w_1^j\alpha_{j_1^v} + w_2^j\alpha_{j_2^v}$. Hence the system (19)-(20) is implied by (5).

We now show the converse, *i.e.*, that $\{\alpha_j\}_{j\in N}$ is the unique solution to the system (5') obtained from (5) by replacing (18) with (19)-(20). Therefore let $i \in M^v$ be arbitrary, and let $(j_1, j_2) := (j_1^i, j_2^i)$ for simplicity. We need to show $t_{j_1}^i\alpha_{j_1} + t_{j_2}^i\alpha_{j_2} = 1$, where $x^v = f + t_{j_1}^i r^{j_1} + t_{j_2}^i r^{j_2}$. From (19)-(20) we know $\alpha_{j_1} = w_1^{j_1}\alpha_{j_1^v} + w_2^{j_1}\alpha_{j_2^v}$, $\alpha_{j_2} = w_1^{j_2}\alpha_{j_1^v} + w_2^{j_2}\alpha_{j_2^v}$ and $\alpha_{j_1^v}w_{j_1^v} + \alpha_{j_2^v}w_{j_2^v} = 1$, where $r^{j_1} = w_1^{j_1}r^{j_1^v} + w_2^{j_1}r^{j_2^v}$, $r^{j_2} = w_1^{j_2}r^{j_1^v} + w_2^{j_2}r^{j_2^v}$ and $x^v = f + w_{j_1^v}r^{j_1^v} + w_{j_2^v}r^{j_2^v}$. Inserting the expressions for r^{j_1} and r^{j_2} into the identity $x^v = f + t_{j_1}^i r^{j_1} + t_{j_2}^i r^{j_2}$ then gives $x^v = f + (t_{j_1}^i w_1^{j_1} + t_{j_2}^i w_1^{j_2})r^{j_1^v} + (t_{j_1}^i w_2^{j_1} + t_{j_2}^i w_2^{j_2})r^{j_2^v}$. Since the representation of x^v in the set $f + \text{cone}(r^{j_1^v}, r^{j_2^v})$ is unique, it follows that $w_{j_1^v} = t_{j_1}^i w_1^{j_1} + t_{j_2}^i w_1^{j_2}$ and $w_{j_2^v} = t_{j_1}^i w_2^{j_1} + t_{j_2}^i w_2^{j_2}$. Now, from the equality $\alpha_{j_1^v}w_{j_1^v} + \alpha_{j_2^v}w_{j_2^v} = 1$, we get $1 = \alpha_{j_1^v}(t_{j_1}^i w_1^{j_1} + t_{j_2}^i w_1^{j_2}) + \alpha_{j_2^v}(t_{j_1}^i w_2^{j_1} + t_{j_2}^i w_2^{j_2}) = t_{j_1}^i(w_1^{j_1}\alpha_{j_1^v} + w_2^{j_1}\alpha_{j_2^v}) + t_{j_2}^i(w_1^{j_2}\alpha_{j_1^v} + w_2^{j_2}\alpha_{j_2^v})$. Since $\alpha_{j_1} = w_1^{j_1}\alpha_{j_1^v} + w_2^{j_1}\alpha_{j_2^v}$ and $\alpha_{j_2} = w_1^{j_2}\alpha_{j_1^v} + w_2^{j_2}\alpha_{j_2^v}$, we obtain $t_{j_1}^i\alpha_{j_1} + t_{j_2}^i\alpha_{j_2} = 1$. ∎

By using Lemma 8, we are now able to finish the proof of Theorem 2.

Lemma 9 *Suppose $\sum_{j\in N}\alpha_j s_j \geq 1$ is facet defining for $\text{conv}(P_I)$ and $\alpha_j > 0$ for $j \in N$. Choose a set $S \subseteq N$ of variables as follows. Initialize $S = \emptyset$, and for every vertex x^v of $\text{conv}(X_\alpha)$:*

(i) If $M^v = \{\bar{i}\}$, update $S := S \cup J^v$, where $J^v := \{j \in N : t_j^{\bar{i}} > 0\}$ denotes the (at most two) variables appearing in equality $\bar{i}$ of system (5).

(ii) If $|M^v| > 1$, update $S := S \cup \{j_1^v, j_2^v\}$, where (j_1^v, j_2^v) denotes the tight representation of x^v given in Lemma 8.

Then the inequality $\sum_{j\in S}\alpha_j s_j \geq 1$ is facet defining for $P_I(S)$, $|S| = k$ and we have $L_\alpha = \text{conv}(\{f\} \cup \{v^j\}_{j\in S})$, where $v^j = f + \frac{1}{\alpha_j}$ for $j \in S$.

Proof: From Lemma 8 and the fact that $\sum_{j\in N}\alpha_j s_j \geq 1$ is facet defining for $\text{conv}(P_I)$, we know that $\{\alpha_j\}_{j\in N}$ is the unique solution to the system

$$\sum_{j\in N}\alpha_j t_j^{\bar{i}} = 1, \quad \text{for all } v \in K \text{ s.t. } M^v = \{\bar{i}\}, \tag{21}$$

$$w_{j_1^v}\alpha_{j_1^v} + w_{j_2^v}\alpha_{j_2^v} = 1, \quad \text{for all } v \in K \text{ s.t. } |M^v| > 1, \tag{22}$$

$$w_1^j\alpha_{j_1^v} + w_2^j\alpha_{j_2^v} = \alpha_j, \text{ for all } v \in K \text{ s.t. } |M^v| > 1 \text{ and } j \in J^v \setminus \{j_1^v, j_2^v\}, \tag{23}$$

Observe that, for a vertex x^v of $\text{conv}(X_\alpha)$ such that $|M^v| > 1$ and a variable $j \in J^v \setminus \{j_1^v, j_2^v\}$, α_j only appears in system (23). It follows that $\{\alpha_j\}_{j\in S}$ is the unique solution to the subsystem (21)-(22) of (21)-(23). Now, since the system (21)-(22) consists of exactly $|K| = k$ equalities, it follows that $|S| = k$. ∎

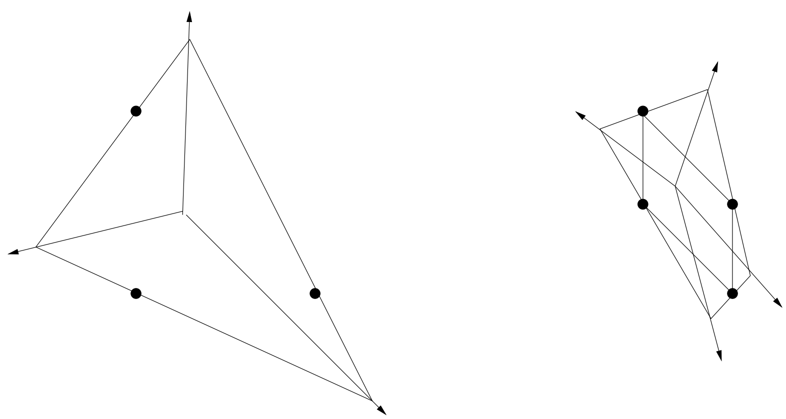
We note that, from a facet defining inequality $\sum_{j\in S}\alpha_j s_j \geq 1$ for $\text{conv}(P_I(S))$, where $|S| = k$, the coefficients on the variables $j \in N \setminus S$ can be simultaneously lifted from equality (20) of Lemma 8, *i.e.*, by computing the intersection point between the halfline $\{f + s_j r^j : s_j \geq 0\}$ and the boundary of L_α.

We now use Theorem 2 to categorize the facet defining inequalities $\sum_{j\in N}\alpha_j s_j \geq 1$ for $\text{conv}(P_I)$. For simplicity, we only consider the most general case, namely when none of the vertices of $\text{conv}(X_\alpha)$ are ray points. Furthermore, we only consider $k = 3$ and $k = 4$. When $k = 2$, $\sum_{j\in N}\alpha_j s_j \geq 1$ is a facet defining inequality for a cone defined by two rays. We divide the facets of $\text{conv}(P_I)$ into the following three main categories.

(i) *Disection cuts (Fig. 5):*
Every vertex of $\text{conv}(X_\alpha)$ belongs to a different facet of L_α.

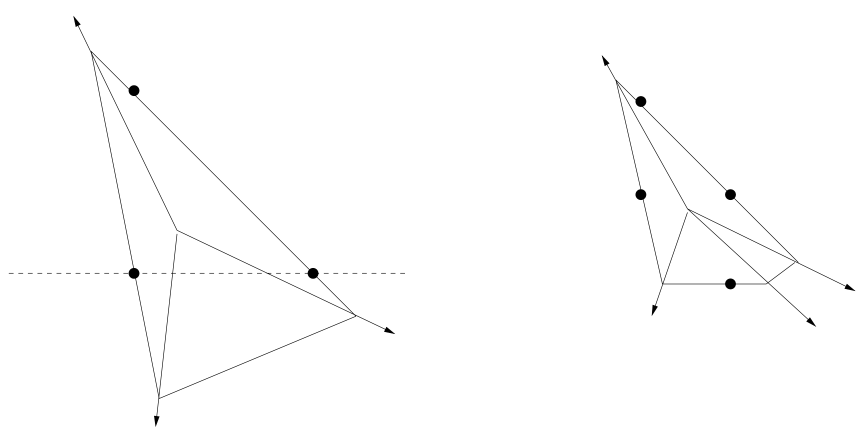
(ii) *Lifted two-variable cuts (Fig. 6):*
Exactly one facet of L_α contains two vertices of $\text{conv}(X_\alpha)$. Observe that this implies that there is a set $S' \subset S$, $|S'| = 2$, such that $\sum_{j \in S'} \alpha_j s_j \geq 1$ is facet defining for $\text{conv}(P_I(S'))$.

(iii) *Split cuts:*
Two facets of L_α each contain two vertices of $\text{conv}(X_\alpha)$.



(a) Disection cut from a triangle (b) Disection cut from a quadrangle

Figure 5: Disection cuts



(a) Lifted two-variable cut from triangle (b) Lifted two-variable cut from quadrangle

Figure 6: Lifted two-variable cuts

An example of a cut that is not a split cut was given in [4] (see Fig. 1). This cut is the only cut when $\text{conv}(X_\alpha)$ is the triangle of Fig. 4.(c), and, necessarily, $L_\alpha = \text{conv}(X_\alpha)$ in this case. Hence, *all* three rays that define this triangle are ray points. As mentioned in the introduction, the cut in [4] can be viewed as being on the boundary between disection cuts and lifted two-variable cuts. Indeed, by perturbing the three rays that define the triangle slightly, both a disection cut and a lifted two-variable cut can be obtained.

Since the cut presented in [4] is not a split cut, and this cut can be viewed as being a "boundary cut" between disection cuts and lifted two-variable cuts, a natural question is whether or not disection cuts and lifted two-variable cuts are split cuts. We finish this section by answering this question.

Lemma 10 *Let* $\sum_{j\in N} \alpha_j s_j \geq 1$ *be a facet defining inequality for* $\mathrm{conv}(P_I)$ *satisfying* $\alpha_j > 0$ *for* $j \in N$*, and suppose none of the vertices of* $\mathrm{conv}(X_\alpha)$ *are ray points. If* $\sum_{j\in N} \alpha_j s_j \geq 1$ *is a disection cut or a lifted two-variable cut, then* $\sum_{j\in N} \alpha_j s_j \geq 1$ *is* not *a split cut.*

Proof: Observe that, if $\sum_{j\in N} \alpha_j s_j \geq 1$ is a split cut, then there exists $(\pi, \pi_0) \in \mathbb{Z}^2 \times \mathbb{Z}$ such that L_α is contained in the split set $S_\pi := \{x \in \mathbb{R}^2 : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$. Furthermore, all points $x \in X_\alpha$ and all vertices of L_α must be either on the line $\pi^T x = \pi_0$, or on the line $\pi^T x = \pi_0 + 1$. However, this implies that there must be two facets of L_α that do not contain any integer points. ∎

Example 3: Consider the set

$$P_I = \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^5_+ : \quad x = f + \begin{pmatrix} 5 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 5 \\ 13 \end{pmatrix} s_2 + \begin{pmatrix} -4 \\ 3 \end{pmatrix} s_3 + \begin{pmatrix} -2 \\ -5 \end{pmatrix} s_4 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_5\},$$

where $f = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \end{pmatrix}$. We computed all the facets of $\mathrm{conv}(P_I)$ for this example. In Table 1, we report how many of the 25 facets belong to each of the categories defined above.

	Disection (triangle)	Disection (quadrangle)	Lifted 2-var (triangle)	Lifted 2-var (quadrangle)	split cut
# facets	3	3	1	14	4
percentage	12	12	4	56	16

Table 1: A categorization of the facets for an example

5 An algorithm to compute the vertices of $\mathrm{conv}(P_I)$

In sections 3 and 4 we described the structure of the facets of $\mathrm{conv}(P_I)$. In particular, we demonstrated that every facet of $\mathrm{conv}(P_I)$ can be obtained from a choice of at most four vertices of $\mathrm{conv}(P_I)$. Therefore, to compute the facets of $\mathrm{conv}(P_I)$, a method for generating the vertices of $\mathrm{conv}(P_I)$ is needed, and this is the topic of this section.

It is well known that the facets of any mixed integer set can be computed from its polar. The polar of $\mathrm{conv}(P_I)$ can be represented as a polyhedron by including one inequality for each extreme point and each extreme ray of $\mathrm{conv}(P_I)$. In our case, every extreme point (x^v, s^v) of $\mathrm{conv}(P_I)$ can be represented by two rays r^{j_v} and r^{k_v}, *i.e.*, $x^v = s_{j_v} r^{j_v} + s_{k_v} r^{k_v}$, where $s_{j_v}, s_{k_v} \geq 0$ (Lemma 2.(iii)). Therefore, the polar of $\mathrm{conv}(P_I)$ is the set of $u \in \mathbb{R}^n$ that satisfy

$$\begin{aligned} & s^v_{j_v} u_{j_v} + s^v_{k_v} u_{k_v} \geq 1, && \text{for all vertices } (x^v, s^v) \text{ of } \mathrm{conv}(P_I), \\ & u_j \geq 0, && \text{for all } j \in N, \end{aligned}$$

where non-negativity follows from the extreme rays of $\mathrm{conv}(P_I)$. To set up and optimize over such a linear system, the vertices of $\mathrm{conv}(P_I)$ must be computed. In this section, we present a combinatorial algorithm for achieving this. Since the number of vertices of $\mathrm{conv}(P_I)$ is polynomial, an overall algorithm that is based on optimizing over the polar of $\text{conv}(P_I)$ runs in polynomial time.

Consider two arbitrary rays r^{i_1} and r^{i_2}, where $i_1, i_2 \in N$. We now show how to compute all vertices of $\text{conv}(P_I)$ that can be obtained from r^{i_1} and r^{i_2}. For simplicity assume $i_1 = 1, i_2 = 2$, and let

$$P_I(f,R) = \{x \in \mathbb{Z}^2 : x = f + r^1 s_1 + r^2 s_2,\ s_1, s_2 \geq 0\}, \tag{24}$$

be the integer set generated from r^1 and r^2, where $R := \text{cone}(\{r^1, r^2\})$.

Observe that (24) can be reformulated by using a unimodular transformation. A unimodular transformation maps integer vertices to integer vertices and therefore preserves integrality.

Observation 1 *Let $T \in \mathbb{Q}^{2\times 2}$ be a unimodular matrix, and let $\{x^v\}_{v\in V}$ denote the vertices of* $\text{conv}(P_I(f,R))$*. Define $\bar{f} := Tf$, $\bar{R} := \text{cone}(\{Tr^1, Tr^2\})$ and $\bar{x}^v := Tx^v$ for $v \in V$. Then*

$$\text{conv}(P_I(f,R)) = \text{conv}(\{x^v\}_{v\in V}) + R,$$

and

$$\text{conv}(P_I(\bar{f},\bar{R})) = \text{conv}(\{\bar{x}^v\}_{v\in V}) + \bar{R}.$$

To simplify the exposition, we use a unimodular transformation to represent the problem in a standard form. This is achieved with a variant of the Hermite normal form (see [9]).

Observation 2 *Let $C \in \mathbb{Z}^{2\times 2}$ be nonsingular. There exists a unimodular matrix $T \in \mathbb{Q}^{2\times 2}$ such that*

(i) $TC = \begin{pmatrix} 1 & p_0 \\ 0 & q_0 \end{pmatrix}$*, where $q_0 \geq p_0 \geq 0$.*

(ii) The size of all entries in TC is polynomially bounded by the encoding length of C.

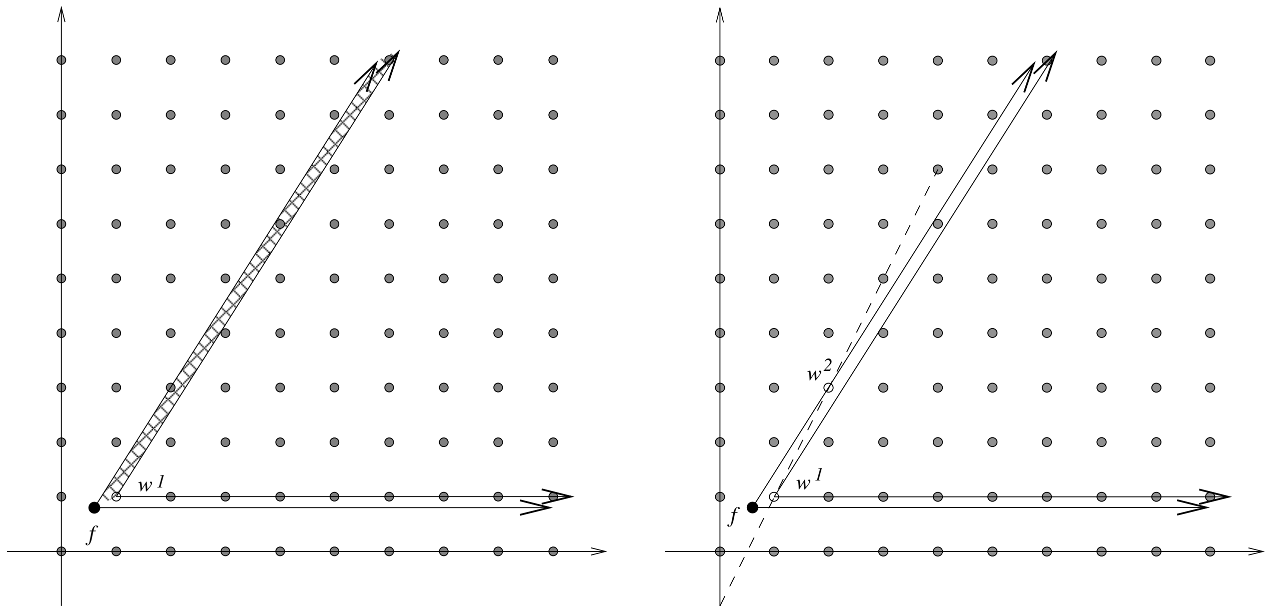
From Observations 1 and 2 it follows that we can assume $r^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $r^2 = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$. We now describe how to successively construct the vertices of $\text{conv}(P_I(f,R))$. The algorithm constructs a new vertex in every iteration, where the first vertex is trivially obtained as follows.

Lemma 11 *The point $w^1 := (\lceil f_1 \rceil, \lceil f_2 \rceil)^T$ is a vertex of* $\text{conv}(P_I(f,R))$*. Furthermore, the inequality $x_2 \geq \lceil f_2 \rceil$ is facet defining for* $\text{conv}(P_I(f,R))$*.*

Proof: This follows directly from the fact that $R = \text{cone}(\{(1,0)^T, (p_0,q_0)^T)\}$, where $p, q \geq 0$. ∎

The remaining vertices of $\text{conv}(P_I(f,R))$ are constructed in a particular order, which we now discuss. If there exists a vertex w^2 of $\text{conv}(P_I(f,R))$ with the same first coordinate as w^1, then we consider w^2 as the "next" vertex of $\text{conv}(P_I(f,R))$. Observe that there can only be one such vertex, and that this vertex, if it exists, can be constructed by simple rounding. In the following, we therefore assume for simplicity that no such vertex exists.

Now consider the set $R_2 := w^1 + R$ with vertex w^1 and extreme rays r^1 and r^2 (see Fig. 7.(a)). Observe that w^1 is the only vertex of $\text{conv}(P_I(f,R))$ contained in R_2. Therefore all the remaining vertices of $\text{conv}(P_I(f,R))$ are contained in the set $(f+R)\setminus R_2$ (the hatched area of Fig. 7.(a)). Furthermore, observe that the slope of the line between w^1 and any remaining vertex w of $\text{conv}(P_I(f,R))$ is larger than the slope $\frac{q_0}{p_0}$ of r^2 (see Fig. 7.(b)). The "next" vertex w^2 is then defined to be the vertex w of $\text{conv}(P_I(f,R))$ for which this slope is as large as possible. Similarily, by induction, for any $t \in \{2,3,\ldots,r\}$, $w^{(t-1)}$ is the only vertex of $\text{conv}(P_I(f,R))$ in the set $R_t := w^{(t-1)} + R$, and the "next" vertex w^t is defined to be the vertex w of $\text{conv}(P_I(f,R))$, among the vertices that have not been chosen so far, for which the slope of the line between $w^{(t-1)}$ and w is as large as possible.



(a) Initial vertex w^1 and the set R_2 (b) The vertex w^2 and the corresponding facet

Figure 7: Construction the vertex w^2

We now introduce the following notation. For any $k \in \{2, 3, \dots, r\}$, let $(p_k, q_k)^T := w^k - w^{(k-1)}$. Then the slope of the line between $w^{(k-1)}$ and w^k is given by $\frac{q_k}{p_k}$, and by the choice of the ordering $w^1, w^2, \dots, w^r$ of the vertices of $\operatorname{conv}(P_I(f, R))$, we have that the slopes are ordered in decreasing order

$$\frac{q_2}{p_2} > \frac{q_3}{p_3} > \dots > \frac{q_r}{p_r}.$$

Also note that all slopes are upper approximations to the slope $\frac{q_0}{p_0}$ of r^2, *i.e.*, $\frac{q_k}{p_k} > \frac{q_0}{p_0}$ for all $k = 2, 3, \dots, r$, and that the approximation becomes better as k increases. The next lemma demonstrates that vertex w^k can be obtained from the previous vertices $w^1, w^2, \dots, w^{(k-1)}$ by solving an optimization problem.

Lemma 12 *Let $2 \leq k \leq r$. The pair (p_k, q_k) is an optimal solution to the optimization problem:*

$$\begin{aligned} \max\ & \frac{q}{p} \\ s.t.\ & \frac{q}{p} \geq \frac{q_0}{p_0} \\ & w^{(k-1)} + \begin{pmatrix} p \\ q \end{pmatrix} \in f + R \\ & p, q \in \mathbb{Z}_+. \end{aligned} \tag{25}$$

Furthermore, there exists an optimal solution (p^h, q^h) to (25) and an integer $\delta^k \in \mathbb{Z}_+$ such that $(p_k, q_k) = \delta^k(p^h, q^h)$, and the set $\{(p, q) \in \mathbb{Z}^2 : (p, q) = \delta(p^h, q^h),\ \delta \in \mathbb{Z}_+ \text{ and } 1 \leq \delta \leq \delta^k\}$ describes all *optimal solutions to (25).*

We now relate problem (25) to the *directed simultaneous diophantine approximation problem*. The directed simultaneous diophantine approximation problem $D(\frac{\hat{q}}{\hat{p}}, N)$ is the problem of finding the best upper approximation $\frac{q}{p}$ of a rational number $\frac{\hat{q}}{\hat{p}}$ under the constraint that $p \leq N$

$$\begin{aligned} & \min \frac{q}{p} \\ & s.t.\ \frac{q}{p} \geq \frac{\hat{q}}{\hat{p}}, \\ & p \leq N, \\ & p, q \in \mathbb{Z}_+. \end{aligned} \tag{26}$$

The following theorem gives the relationship between the directed simultaneous diophantine approximation problem and problem (25).

Theorem 3 *Assuming* $\operatorname{conv}(P_I(f,R))$ *does not have a vertex with the same first coordinate as* w^1*, the problem* (25) *has the same set of optimal solutions as the directed simultaneous diophantine approximation problem* $D(\frac{q_0}{p_0}, N)$ *with* $N = p_k$.

Proof: Consider the optimal solution (p_k, q_k) to (25). Also consider an optimal solution $(\bar{p}, \bar{q})$ to $D(\frac{q_0}{p_0}, p_k)$. We will prove that $\frac{q_k}{p_k} = \frac{\bar{q}}{\bar{p}}$. Note that this implies that every optimal solution (p^*, q^*) to (25) must be optimal for $D(\frac{q_0}{p_0}, p_k)$ (since (p^*, q^*) is feasible for $D(\frac{q_0}{p_0}, p_k)$ and attains the same objective value as $(\bar{p}, \bar{q})$). Conversely every optimal solution (p', q') to $D(\frac{q_0}{p_0}, p_k)$ satisfies $\frac{q'}{p'} = \frac{q_k}{p_k}$, and since $p' \leq p_k$, (p', q') is feasible for (25).

We now prove $\frac{q_k}{p_k} = \frac{\bar{q}}{\bar{p}}$. Suppose, for a contradiction, that $\frac{\bar{q}}{\bar{p}} < \frac{q_k}{p_k}$. Clearly we have $p_k - \bar{p} \geq 0$, since $(\bar{p}, \bar{q})$ is feasible for (26), and we also have $q_k - \bar{q} \geq 0$, since $\frac{\bar{q}}{\bar{p}} < \frac{q_k}{p_k}$. We first argue that

$$w^{(k-1)} + \begin{pmatrix} p_k - \bar{p} \\ q_k - \bar{q} \end{pmatrix} \in f + R. \tag{27}$$

Suppose (27) is *not* satisfied, *i.e.*, we have the situation in Fig. 8. Observe that the slope of the line between the points $w^{(k-1)}$ and $w^{(k-1)} + (\bar{p}, \bar{q})$ is identical to the slope of the line between $w^{(k-1)} + (p_k - \bar{p}, q_k - \bar{q})$ and $w^{(k-1)} + (p_k, q_k)$, and that this slope is given by $\frac{\bar{q}}{\bar{p}}$. Hence, if (27) is not satisfied, then we have $\frac{\bar{q}}{\bar{p}} < \frac{q_0}{p_0}$. However, this contradicts the fact that $(\bar{p}, \bar{q})$ is feasible for $D(\frac{q_0}{p_0}, p_k)$. Hence (27) is satisfied

Next suppose $\bar{p} = p_k$. If also $\bar{q} = q_k$, we are done, and if $q_k > \bar{q}$, (27) implies that the vertical line through $w^{(k-1)}$ contains integer points different from $w^{(k-1)}$ that are contained $f + R$. However, this contradicts the assumption that $\operatorname{conv}(P_I(f,R))$ does not have a vertex with the same first coordinate as w^1.

Finally suppose $p_k > \bar{p}$ and $q_k > \bar{q}$. Since $\frac{\bar{q}}{\bar{p}} < \frac{q_k}{p_k}$, we have

$$\frac{q_k - \bar{q}}{p_k - \bar{p}} > \frac{q_k}{p_k}. \tag{28}$$

Indeed (28) can be written as

$$p_k q_k - p_k \bar{q} > p_k q_k - q_k \bar{p}. \tag{29}$$

It now follows from (28) and (27) that $(p_k - \bar{p}, q_k - \bar{q})$ is a feasible solution to (25) with a better objective value than (p_k, q_k), which is a contradiction. ■

The optimal solutions of the directed simultaneous diophantine approximation problem $D(\frac{q_0}{p_0}, N)$ can be characterized from a minimal *Hilbert basis* of the cone $R' = \operatorname{cone}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}\right)$, where a Hilbert basis of a rational polyhedral cone is defined as follows.

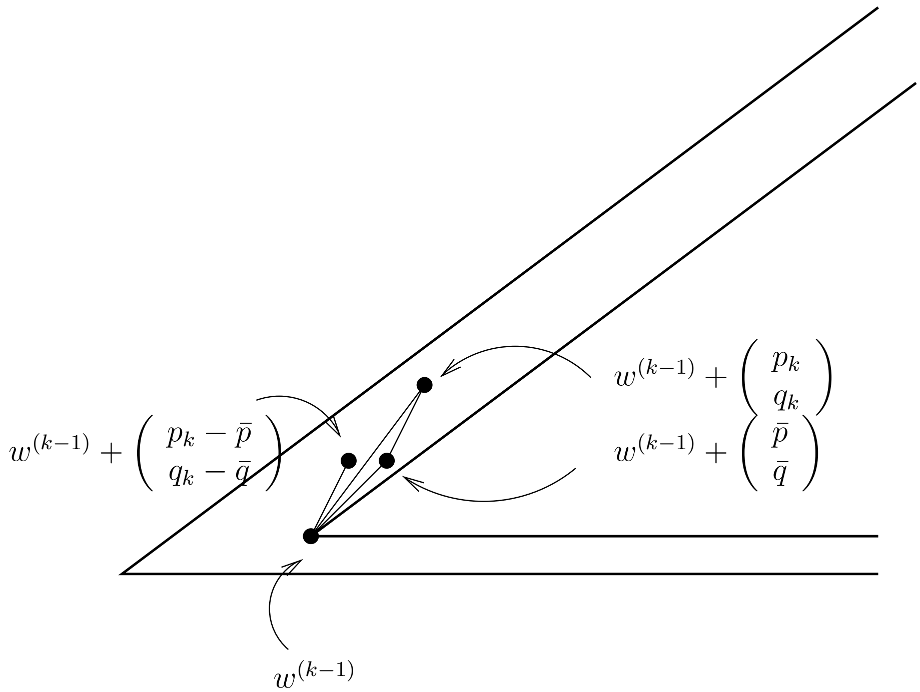


Figure 8: Existence of $(\bar{p}, \bar{q})$

Definition 2 *Let $C \subseteq \mathbb{R}^n$ be a rational polyhedral cone. A finite set $\mathcal{H}(C) = \{h^1, \ldots, h^t\} \subset C \cap \mathbb{Z}^n$ is called a* Hilbert basis *of C if every $z \in C \cap \mathbb{Z}^n$ has a representation of the form*

$$z = \sum_{i=1}^{t} \lambda_i h^i,$$

where $\lambda_i \in \mathbb{Z}_+$. Furthermore, if C is pointed, there exists a unique Hilbert basis $\mathcal{H}^*(C)$ of C of minimum cardinality.

The following result is due to Henk and Weismantel [6].

Theorem 4 *Let $\hat{p}, \hat{q} \in \mathbb{Z}_+$. For every $N \in \mathbb{Z}_+$, there exists an optimal solution to the problem $D(\frac{\hat{q}}{\hat{p}}, N)$ defined by (26) which is a member of the Hilbert basis $\mathcal{H}^*(R')$ of the cone $R' =$ cone$\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} \right)$. Conversely, for every member h of the Hilbert basis $\mathcal{H}^*(C)$ of R', there exists $N \in \mathbb{Z}_+$ such that h is an optimal solution of $D(\frac{\hat{q}}{\hat{p}}, N)$.*

An algorithm to construct the vertices of $\text{conv}(P_I(f, R))$ now follows directly from Theorem 3 and Theorem 4. We start from the vertex $w^1 = (\lceil f_1 \rceil, \lceil f_2 \rceil)$. To construct the next vertex, we consider the Hilbert basis elements of $\mathcal{H}^*(R')$, where $R' = \text{cone}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \right)$. We sort the elements of $\mathcal{H}^*(R')$ in terms of decreasing slopes. The first element $h \in \mathcal{H}^*(R')$ for which $w^1 + h \in f + R$ leads to the next vertex. Notice that, if there is a vertex of $\text{conv}(P_I(f, R))$ with the same first coordinate as w^1, then this vertex can be found using the element $(0, 1)$ of $\mathcal{H}^*(R')$, which is always contained in $\mathcal{H}^*(R')$. Also note that a similar approach has been used recently by Agra and Constantino [1] to determine a complete description for multidimensional knapsack problems in two variables. The cones analyzed in [1], however, slightly differ from the family of cones studied here.

The algorithm above, however, does not run in polynomial time. It is well known that a Hilbert basis of a two-dimensional cone can be of exponential size. An algorithm, which is based on checking every element of $\mathcal{H}^*(R')$, can therefore have exponential running time. Fortunately, we can overcome this difficulty by exploiting the special structure of Hilbert bases of two-dimensional cones. A minimal Hilbert basis of a two-dimensional cone can be described by a polynomial number of *vertices* and *edges* (see [8]). The following theorem gives the structure of a minimal Hilbert basis of a two-dimensional cone in terms of vertices and edges.

Theorem 5 *Let $C = \mathrm{cone}(r^1, r^2) \subseteq \mathbb{R}^2$ be a rational polyhedral cone of dimension two. There exists a polynomial number of vectors $e_1, \ldots, e_s \in \mathbb{Z}^2$, called the* edges, *and numbers $k_1, \ldots, k_s \in \mathbb{Z}_+$ such that, if we denote the partial sums $\sigma_{1i} = \sum_{j=1}^{i} k_j e_j$,*

$$\begin{aligned}\mathcal{H}^*(R) = \{r^1, &r^1 + e_1, \ldots, r^1 + k_1 e_1 = r^1 + \sigma_{11},\\ &r^1 + \sigma_{11} + e_2, \ldots, r^1 + \sigma_{11} + k_2 e_2 = r^1 + \sigma_{12},\\ &\vdots\\ &r^1 + \sigma_{1(s-1)} + e_s, \ldots, r^1 + \sigma_{1s} = r^2 \ \}.\end{aligned}$$

We can now modify the initial algorithm as follows. Instead of checking every element of $\mathcal{H}^*(R')$ in turn, we sequentially check every edge of $\mathcal{H}^*(R')$. This can be done in polynomial time using binary search. To do this, consider the inverse of the matrix $\begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}$ given by $\begin{pmatrix} 1 & -\frac{p}{q} \\ 0 & \frac{1}{q} \end{pmatrix}$. To check whether the elements $e_0 + je$ of the Hilbert basis $\mathcal{H}^*(R')$ of R' leads to a new vertex of $\mathrm{conv}(P_I(f, R))$ from a previous vertex $w^{(k-1)}$ of $\mathrm{conv}(P_I(f, R))$, where $j \in \mathbb{Z}_+$ and $e, e_0 \in \mathbb{Z}^2_+$ are edges of $\mathcal{H}^*(R')$, one needs to check the following condition

$$\begin{pmatrix} 1 & -\frac{p}{q} \\ 0 & \frac{1}{q} \end{pmatrix} (w^{(k-1)} + e_0 + je - f) \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{30}$$

The second inequality of (30) is trivially satisfied since $w_2^{(k-1)} > f_2$. Hence the first inequality of (30) determines whether an element of the Hilbert basis $\mathcal{H}^*(R')$ along the edge $e_0 + je$ provides a new vertex of $\mathrm{conv}(P_I(f, R))$.

Example 4 Consider the cone

$$R = \mathrm{cone}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 237 \\ 1033 \end{pmatrix} \right),$$

and $f = (2/3, 5/7)$. The edges of the Hilbert basis $\mathcal{H}^*(R') = \mathcal{H}^*(\mathrm{cone}((0, 1), (1033, 237)))$ are given by

$$(0, 1), (1, 4), (3, 13), (14, 61), (53, 231), (145, 632),$$

with $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 2, k_5 = k_6 = 1$. The algorithm takes the following steps.

Step 0 Construct the initial vertex $w^1 = (\lceil f_1 \rceil, \lceil f_2 \rceil) = (1, 1)$.

Step 1 Check the edge $e_1 = (0, 1)$.
Find j such that $(1, 1 + j) \in f + R$.
We obtain $j \leq 1.1$, *i.e.*, $w^2 = (1, 2)$.

Step 2 Check the edge $e_2 = (1, 4)$, *i.e.*, Hilbert basis elements $(j, 1 + 4j)$.
Find minimal j such that $(1, 2) + (j, 1 + 4j) \in f + R$.
We obtain $j \geq 2.4$, *i.e.*, we do not find a new vertex because j must be smaller than $k_2 = 2$.

Step 3 Check the edge $e_3 = (3, 13)$, *i.e.*, Hilbert basis elements $(2 + 3j, 9 + 13j)$.
Find minimal j such that $(1, 2) + (2 + 3j, 9 + 13j) \in f + R$.
We obtain $j \geq 1.52$. Hence $j = 2$, and $w^3 = (9, 37)$ is the next vertex.

Step 4 Check the edge $e_4 = (14, 61)$, *i.e.*, Hilbert basis elements $(11 + 14j, 48 + 61j)$.
Find minimal j such that $(9, 37) + (11 + 14j, 48 + 61j) \in f + R$.
We obtain $j \geq 0.88$. Hence $j = 1$, and $w^4 = (34, 146)$.

Step 5 Check the edge $e_5 = (53, 231)$, *i.e.*, Hilbert basis elements $(39 + 53j, 170 + 231j)$.
Find minimal j such that $(34, 146) + (39 + 53j, 170 + 231j) \in f + R$.
We obtain $j \geq 1.19$. Hence no new vertex is found.

Step 6 Edge $e_6 = (145, 632)$ provides $(237, 1033)$ as only Hilbert basis element. This is the extreme ray of the translated cone.

The separation problem The method presented in this section can be used to compute all vertices of $\text{conv}(P_I)$ in polynomial time. For this, every pair of rays ($\mathcal{O}(n^2)$) must be considered, and the vertices that can be obtained from this pair must be computed. Each subproblem provides a polynomial number of such vertices. To solve the separation problem, one can set up the polar of $\text{conv}(P_I)$ from the vertices of $\text{conv}(P_I)$. The polar includes an inequality for every vertex that is not a ray point, and a nonnegativity constraints for every extreme ray.

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