11ème cours de Mécanique Analytique (25/11/2010)

Comète Mc Naught 2006


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• 2.3 Le principe variationnel d'Hamilton modifié

$$L(q, \dot{q}(q,p,t), t) = p_i \dot{q}_i(q,p,t) - H(q,p,t) \qquad (2.11)$$

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q,p,t))\, dt = 0 \qquad (2.12)$$

$$\frac{d}{dt}\left(\frac{\partial(p_i\dot{q}_i - H)}{\partial \dot{q}_k}\right) = \frac{\partial(p_i\dot{q}_i - H)}{\partial q_k} \Rightarrow \boxed{\dot{p}_k = -\frac{\partial H}{\partial q_k}} \qquad (2.13)$$

$$\frac{d}{dt}\left(\frac{\partial(p_i\dot{q}_i - H)}{\partial \dot{p}_k}\right) = \frac{\partial(p_i\dot{q}_i - H)}{\partial p_k} \Rightarrow \boxed{0 = \dot{q}_k - \frac{\partial H}{\partial p_k}} \qquad (2.14)$$

- 2.4 Transformations canoniques

$$\begin{cases} Q_i = Q_i(q, p, t) \\ P_i = P_i(q, p, t) \end{cases} \quad (2.19)$$

$$\boxed{\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \qquad (i = 1, \ldots, f)} \quad (2.20)$$

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q, p, t))\, dt = 0$$

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q, P, t))\, dt = 0 \quad (2.21)$$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt} \quad (2.22)$$

• 2.4 Transformations canoniques

a) Le premier choix possible est :

$$F = F_1(q, Q, t) \quad (2.23)$$

$$\begin{aligned} p_i \dot{q}_i - H &= P_i \dot{Q}_i - K + \frac{dF_1}{dt} \\ &= P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i \end{aligned} \quad (2.24)$$

$$p_i = \frac{\partial F_1}{\partial q_i} \quad (2.25a)$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} \quad (2.25b)$$

$$K = H + \frac{\partial F_1}{\partial t} \quad (2.25c)$$

$$\boxed{\begin{cases} Q_i = Q_i(q, p, t) \\ P_i = P_i(q, p, t) \end{cases} \quad (2.19)}$$

• 2.4 Transformations canoniques

b) Un autre choix intéressant possible pour la fonction F est :

$$F = F_2(q, P, t) - Q_i P_i \qquad (2.26)$$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt} \qquad (2.27)$$

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad (2.28a)$$

$$Q_i = \frac{\partial F_2}{\partial P_i} \qquad (2.28b)$$

$$K = H + \frac{\partial F_2}{\partial t} \qquad (2.28c)$$

$$\begin{cases} Q_i = Q_i(q, p, t) \\ P_i = P_i(q, p, t) \end{cases} \qquad (2.19)$$

• 2.4 Transformations canoniques

c) $F = q_i p_i + F_3(p, Q, t)$ (2.29)

$$q_i = -\frac{\partial F_3}{\partial p_i} \quad (2.30a)$$

$$P_i = -\frac{\partial F_3}{\partial Q_i} \quad (2.30b)$$

$$K = H + \frac{\partial F_3}{\partial t} \quad (2.30c)$$

- 2.4 Transformations canoniques

d) $$F = q_i p_i - Q_i P_i + F_4(p, P, t)$$ (2.31)

$$q_i = -\frac{\partial F_4}{\partial p_i}$$ (2.32a)

$$Q_i = \frac{\partial F_4}{\partial P_i}$$ (2.32b)

$$K = H + \frac{\partial F_4}{\partial t}$$ (2.32c)

• 2.4 Transformations canoniques

$$\begin{cases} Q_i = Q_i(q,p,t) \\ P_i = P_i(q,p,t) \end{cases} \quad (2.19)$$

$$\boxed{\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \qquad (i = 1, \ldots, f)} \quad (2.20)$$

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q,p,t))\, dt = 0$$

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q,P,t))\, dt = 0 \quad (2.21)$$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt} \quad (2.22)$$

$$F = F_1(q,Q,t)$$

$$F = F_2(q,P,t) - Q_i P_i$$

$$F = q_i p_i + F_3(p,Q,t)$$

$$F = q_i p_i - Q_i P_i + F_4(p,P,t)$$

- 2.5 Exemples de transformations canoniques

a) $F_2(q,P) = q_i P_i$ (2.33)

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i \quad (2.34a)$$

$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i \quad (2.34b)$$

$$K = H \quad (2.34c)$$

b) $F_1 = q_k Q_k$ (2.35)

$$p_i = \frac{\partial F_1}{\partial q_i} = Q_i \quad (2.36a)$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i \quad (2.36b)$$

$$K(Q,P) = H(-P,Q) \quad (2.36c)$$

• 2.5 Exemples de transformations canoniques

c) Considérons le problème classique de l'*oscillateur harmonique*

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \qquad (f = 1) \qquad (2.37)$$

$$F_1 = \frac{m\omega q^2}{2} \operatorname{cotg} Q \qquad (2.38)$$

$$p = \frac{\partial F_1}{\partial q} = m\omega q \operatorname{cotg} Q \qquad (2.39a)$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2 \sin^2 Q} \qquad (2.39b)$$

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \qquad (2.40a)$$

$$p = \sqrt{2m\omega P} \cos Q \qquad (2.40b)$$

$$K = \omega P \qquad (2.41)$$

• 2.5 Exemples de transformations canoniques

$$K = \omega P \qquad (2.41)$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \qquad (2.42a)$$

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \qquad (2.42b)$$

$P = \alpha =$ constante et : $Q = \omega t + \beta$ (2.43)

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \; + \beta) \qquad (2.44)$$

$$p = \sqrt{2m\omega P} \cos Q$$

• 2.6 L'approche symplectique des transformations canoniques

vecteur colonne η à $2f$ éléments $\quad \eta_i = q_i \quad , \quad n_{i+f} = p_i \quad i \leq f$

(2.45)

le vecteur colonne $\dfrac{\partial H}{\partial \eta}$ $\quad \left(\dfrac{\partial H}{\partial \eta}\right)_i = \dfrac{\partial H}{\partial q_i} \quad , \quad \left(\dfrac{\partial H}{\partial \eta}\right)_{i+f} = \dfrac{\partial H}{\partial p_i} \quad i \leq f$

(2.46)

J la matrice carrée $2f \times 2f$

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \quad (2.47)$$

$$\dot{\eta} = J \frac{\partial H}{\partial \eta} \quad (2.48)$$

• 2.6 L'approche symplectique des transf. canoniques

J la matrice carrée $2f \times 2f$

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \quad (2.47)$$

$$\begin{aligned} J^2 &= -\mathbf{1} && (2.49a) \\ J\tilde{J} &= \mathbf{1} && (2.49b) \\ \tilde{J} &= -J = J^{-1} && (2.49c) \\ dtm\,(J) &= +1 && (2.49d) \end{aligned}$$

- 2.6 L’approche symplectique des transf. canoniques

$$\mathbf{J}_{ij} = 1\,\delta_{i,j-f}\,\Pi_{[1\le i\le f,\,f+1\le j\le 2f]} - 1\,\delta_{i-f,j}\,\Pi_{[f+1\le i\le 2f,\,1\le j\le f]}$$

$$\tilde{\mathbf{J}}_{ij} = \mathbf{J}_{ji}$$

$$= 1\,\delta_{j,i-f}\,\Pi_{[1\le j\le f,\,f+1\le i\le 2f]} - 1\,\delta_{j-f,i}\,\Pi_{[f+1\le j\le 2f,\,1\le i\le f]}$$

$$= -\left(1\,\delta_{j-f,i}\,\Pi_{[f+1\le j\le 2f,\,1\le i\le f]} - 1\,\delta_{j,i-f}\,\Pi_{[1\le j\le f,\,f+1\le i\le 2f]}\right)$$

$$= -\left(1\,\delta_{i,j-f}\,\Pi_{[1\le i\le f,\,f+1\le j\le 2f]} - 1\,\delta_{i-f,j}\,\Pi_{[f+1\le i\le 2f,\,1\le j\le f]}\right)$$

$$= -\mathbf{J}_{ij}$$

$$\Rightarrow \quad \tilde{\mathbf{J}} = -\mathbf{J}$$

- 2.6 L'approche symplectique des transf. canoniques

$$\mathbf{J}_{ij} = 1\ \delta_{i,j-f}\ \Pi_{[1\leq i\leq f, f+1\leq j\leq 2f]} - 1\ \delta_{i-f,j}\ \Pi_{[f+1\leq i\leq 2f, 1\leq j\leq f]}$$

$$(\mathbf{J}\ \tilde{\mathbf{J}})_{ij} = \mathbf{J}_{ik}\ \mathbf{J}_{jk} =$$
$$(\delta_{i,k-f}\ \Pi_{[1\leq i\leq f, f+1\leq k\leq 2f]} - \delta_{i-f,k}\ \Pi_{[f+1\leq i\leq 2f, 1\leq k\leq f]}) \cdot$$
$$(\delta_{j,k-f}\ \Pi_{[1\leq j\leq f, f+1\leq k\leq 2f]} - \delta_{j-f,k}\ \Pi_{[f+1\leq j\leq 2f, 1\leq k\leq f]})$$

$$= (\delta_{i,k-f}\ \delta_{j,k-f}\ \Pi_{[1\leq i\leq f, f+1\leq k\leq 2f]}\ \Pi_{[1\leq j\leq f, f+1\leq k\leq 2f]}$$
$$+\ \delta_{i-f,k}\ \delta_{j-f,k}\ \Pi_{[f+1\leq i\leq 2f, 1\leq k\leq f]}\ \Pi_{[f+1\leq j\leq 2f, 1\leq k\leq f]}$$

$$= \delta_{ij} = \mathbf{1}_{ij} \qquad \Rightarrow \qquad \mathbf{J}\ \tilde{\mathbf{J}} = \mathbf{1}$$

• 2.6 L'approche symplectique des transf. canoniques

$$\mathbf{J}\,\tilde{\mathbf{J}} = \mathbf{1}, \quad \tilde{\mathbf{J}}\,\mathbf{J} = \mathbf{1} \;\Rightarrow\; \tilde{\mathbf{J}} = -\mathbf{J} = \mathbf{J}^{-1}$$

$$\Downarrow$$

$$\mathrm{dtm}(\mathbf{J}) = +1 \;\Leftarrow\; \mathbf{J}^2 = \mathbf{J}\,\mathbf{J} = -\tilde{\mathbf{J}}\,\mathbf{J} = -\mathbf{1}$$

$$\Downarrow$$

$\Rightarrow$ c.q.f.d.

$$J^2 = -\mathbf{1} \qquad (2.49a)$$

$$J\tilde{J} = \mathbf{1} \qquad (2.49b)$$

$$\tilde{J} = -J = J^{-1} \qquad (2.49c)$$

$$dtm\,(J) = +1 \qquad (2.49d)$$

• 2.6 L'approche symplectique des transf. canoniques

vecteur colonne ζ à $2f$ éléments

nouvelles variables canoniques $Q_i,\ P_i$ $\qquad \zeta = \zeta(\eta)$ (2.50)

$$\dot{\zeta}_i = \frac{\partial \zeta_i}{\partial \eta_j}\dot{\eta}_j \qquad (i, j = 1, \ldots, 2f)$$

$$\dot{\zeta} = M\dot{\eta} \quad (2.51) \qquad M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j} \quad (2.52)$$

$$\dot{\zeta} = MJ\frac{\partial H}{\partial \eta} \quad (2.53)$$

$$\frac{\partial H}{\partial \eta_i} = \frac{\partial H}{\partial \zeta_j}\frac{\partial \zeta_j}{\partial \eta_i} \qquad \frac{\partial H}{\partial \eta} = \tilde{M}\frac{\partial H}{\partial \zeta} \quad (2.54)$$

$$\dot{\zeta} = MJ\tilde{M}\frac{\partial H}{\partial \zeta} \quad (2.55)$$

- 2.6 L'approche symplectique des transf. canoniques

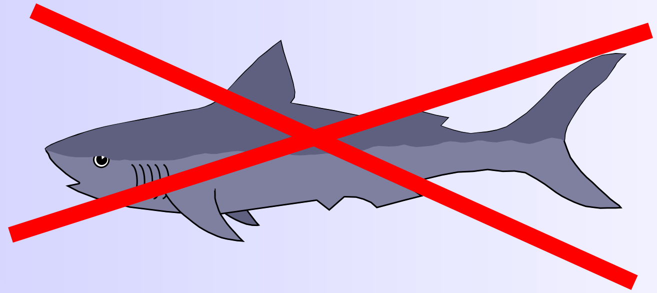
$$\dot{\zeta} = M J \tilde{M} \frac{\partial H}{\partial \zeta} \qquad (2.55)$$

$$\dot{\zeta} = J \frac{\partial H}{\partial \zeta} \qquad (2.56)$$

$$M J \tilde{M} = J \qquad (2.57a)$$

$$\tilde{M} J M = J \qquad (2.57b)$$

- 2.7 Les crochets de Poisson

[]

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} \qquad (2.58)$$

$$[u, v]_{\eta} = \frac{\widetilde{\partial u}}{\partial \eta} J \frac{\partial v}{\partial \eta} \qquad (2.59)$$

$$Si \quad u_i = \frac{\tilde{\partial} u}{\partial \eta_i} \quad et \quad v_k = \frac{\partial v}{\partial \eta_k}$$

$$= u_i \, J_{ik} \, v_k$$

$$= u_i \, [\delta_{i,k-f} \, \Pi_{[1 \leq i \leq f, f+1 \leq k \leq 2f]} - \delta_{i-f,k} \, \Pi_{[f+1 \leq i \leq 2f, 1 \leq k \leq f]}] v_k$$

- 2.7 Les crochets de Poisson

$$= u_i \, [\delta_{i,k-f} \, \Pi_{[1 \leq i \leq f, f+1 \leq k \leq 2f]} - \delta_{i-f,k} \, \Pi_{[f+1 \leq i \leq 2f, 1 \leq k \leq f]}] v_k$$

$$= \frac{\partial u}{\partial q_i} \cdot \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i}$$

$$\boxed{[u, v]_{q,p} = \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i}} \qquad (2.58)$$

$$\begin{cases} [q_j, q_k]_{q,p} = 0 \\ [p_j, p_k]_{q,p} = 0 \\ [q_j, p_k]_{q,p} = -[p_j, q_k]_{q,p} = \delta_{jk} \end{cases} \qquad (2.60)$$

• 2.7 Les crochets de Poisson

matrice crochet de Poisson

désignée par $[\eta, \eta]$ dont l'élément (ℓm) est $[\eta_\ell, \eta_m]$

$$[\eta, \eta]_\eta = J \qquad (2.61)$$

$$\zeta = \zeta(\eta, t)$$

$$[\zeta, \zeta]_\eta = \frac{\partial \zeta}{\partial \eta} J \widetilde{\frac{\partial \zeta}{\partial \eta}} \qquad (2.62)$$

$$[\zeta, \zeta]_\eta = M J \tilde{M} \qquad (2.63)$$

$$[\zeta, \zeta]_\eta = J \qquad (2.64)$$

$$[\zeta, \zeta]_\zeta = J \qquad (2.65)$$