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A NEW VARIATIONAL PRINCIPLE FOR FINITE ELASTIC DISPLACEMENTS†

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Abstract—Variational principles for finite elastic displacements have been formulated in terms of Green strain and Kirchhoff–Trefftz stress tensors. The first is a functional of the displacement field only and implies stationarity of the total potential. The second is a canonical principle, in the sense of Friedrichs [1], involving both stresses and displacements and generalizing Reissner's principle [5]. In contrast with the geometrically linearized elasticity theory, it cannot be reduced to a complementary energy principle involving equilibrium stresses only [9, 10]. The paper discusses the Levinson [9] and Zubov [11] formulation in terms of displacement gradients and the Piola stress tensor, which, however interesting from a theoretical viewpoint, does not appear suitable for practical applications. A new set of variational principles, of displacement, canonical or complementary energy types, is found to derive from the use of the polar decomposition theorem of the jacobian. It involves the engineering strain tensor and its conjugate stress tensor is to be regarded as a function of the Piola tensor and the material rotation. The complementary energy formulation is discussed in terms of first order stress functions. The presence of the rotational degrees of freedom opens the possibility of discretizing the rotational equilibrium equations in approximate solutions.

1. EULERIAN AND LAGRANGIAN STRESS TENSORS

ASSUME, for simplicity, the initial or reference configuration of an elastic body and its strained or final configuration to be referred to the same cartesian frame. Let x_i ($i = 1, 2, 3$) denote the material coordinates, or coordinates of a material point in the initial configuration, ξ_i its coordinates in the final configuration. The components of the displacement vector

$$\partial \xi_j / \partial x_i = D_i \xi_j. \tag{2}$$

The determinant of this matrix is denoted by J.

An oriented volume element $\mathrm{d}V$ in the initial configuration is constructed on three infinitesimal vectors $d_{(1)}\vec{x}$, $d_{(2)}\vec{x}$ and $d_{(3)}\vec{x}$ as

$$\xi_i - x_i = u_i(x_j) \tag{1}$$

are considered to be functions of the material coordinates. In this Lagrangian point of view, the local transformation from initial to final configurations is governed by the Jacobian matrix

$$\mathrm{d}V = e_{mnp}\, d_{(1)}x_m\, d_{(2)}x_n\, d_{(3)}x_p$$

where e_{mnp} is the alternating tensor. In the final configuration they become respectively $d_{(1)}\vec{\xi}$, $d_{(2)}\vec{\xi}$ and $d_{(3)}\vec{\xi}$ and generate the volume element

$$\begin{aligned} d\Omega &= e_{ijk}\, d_{(1)}\xi_i\, d_{(2)}\xi_j\, d_{(3)}\xi_k \\ &= e_{ijk} D_m \xi_i D_n \xi_j D_p \xi_k\, d_{(1)}x_m\, d_{(2)}x_n\, d_{(3)}x_p \\ &= J\, e_{mnp}\, d_{(1)}x_m\, d_{(2)}x_n\, d_{(3)}x_p = J\, dV. \end{aligned} \tag{3}$$

†The Russian version of this paper was dedicated to Professor B. G. Galerkin's centenary.

Hence J is a measure of the change in volume. Physically the change in configuration takes place continuously and the volume element cannot collapse to zero nor change its orientation, so that

$$J > 0 \quad \text{everywhere.} \tag{4}$$

Thus the Jacobian matrix is invertible; the elements of its inverse are

$$\partial x_i / \partial \xi_m = \partial_m x_i \tag{5}$$

and yield the following useful formulas

$$\partial_m x_i D_i \xi_J = \delta_{mj} \qquad D_i \xi_j \partial_j x_m = \delta_{im}. \tag{6}$$

The oriented surface elements in the initial and final configuration are respectively

$$e_{mnp} d_{(1)} x_n d_{(2)} x_p = n_m \, dS \tag{7}$$

$$e_{ijk} d_{(1)} \xi_j d_{(2)} \xi_k = \nu_i \, d\Sigma \tag{8}$$

where n_m and ν_i denote the direction cosines of the normal, taken positive outwards when the surface belongs to the boundary of the elastic body under consideration. Since

$$(D_m \xi_i) \nu_i \, d\Sigma = e_{ijk} D_m \xi_i D_n \xi_j D_p \xi_k d_{(1)} x_n d_{(2)} x_p$$
$$= J e_{mnp} d_{(1)} x_n d_{(2)} x_p = J n_m \, dS$$

the surface elements are related by

$$n_m \, dS = \frac{1}{J} (D_m \xi_i) \nu_i \, d\Sigma \qquad \nu_i \, d\Sigma = J (\partial_i x_m) n_m \, dS. \tag{9}$$

In the strained configuration, the 'true' or 'Eulerian' stress tensor τ_{ij} is defined by the force element

$$dF_j = \tau_{ij} \nu_i \, d\Sigma \tag{10}$$

acting on the oriented surface element. This equation contains the statement of equilibrium of an infinitesimal tetrahedron, three faces of which are parallel to the coordinate planes. It can also be written in the equivalent 'surface equilibrium' form

$$\tau_j = \tau_{ij} \nu_i \tag{11}$$

where the left-hand side represents the 'surface traction' components.

In some infinitesimal change of configuration δu_j, the virtual work performed by the surface tractions, acting on a closed surface Σ bounding the volume Ω, is measured by

$$\int_{\Sigma} \mathrm{d}F_j \delta u_j = \int_{\Sigma} \nu_i \tau_{ij} \delta u_j \,\mathrm{d}\Sigma = \int_{\Omega} \partial_i(\tau_{ij}\delta u_j)\,\mathrm{d}\Omega, \tag{12}$$

the last form deriving from an application of the divergence theorem. The statement of conservation of energy for the hyperelastic body can thus be written in the form

$$\delta \int_{\Omega} \rho U \,\mathrm{d}\Omega = \int_{\Omega} \{\rho g_j \delta u_j + \partial_i(\tau_{ij}\delta u_j)\}\,\mathrm{d}\Omega \tag{13}$$

where U is the specific strain energy (per unit mass), ρ the mass per unit volume in the strained configuration and g_j the acceleration components of body forces. Noting that $\rho\,\mathrm{d}\,\Omega$, which is the mass of a particle, is an invariant,

$$\delta \int_{\Omega} \rho U \,\mathrm{d}\Omega = \int_{\Omega} \rho \delta U \,\mathrm{d}\Omega$$

and (13) implies

$$\rho\,\delta U = \rho g_j \delta u_j + \partial_i(\tau_{ij}\delta u_j). \tag{14}$$

In particular, when δu_j is a constant vector, the relative translation of the volume element produces no increase in strain energy and we get the translational equilibrium equations

$$\rho g_j + \partial_i \tau_{ij} = 0, \tag{15}$$

that reduce (14) to the simpler form

$$\rho \delta U = \tau_{ij}\partial_i \delta u_j. \tag{16}$$

There is likewise no strain energy increase if

$$\delta u_j = e_{jmn}(\mathrm{d}\omega_m)\xi_n, \tag{17}$$

representing an infinitesimal rigid body rotation about the origin, superimposed on the strained configuration and characterized by the constant but arbitrary rotation vector $\mathrm{d}\omega_m$. In this case (16) reduces to the rotational equilibrium conditions:

$$\tau_{ij} e_{jmn} \partial_i \xi_n = \tau_{ij} e_{jmi} = 0 \quad (m = 1, 2, 3), \tag{18}$$

which are equivalent to the symmetry statements

$$\tau_{ij} = \tau_{ji} \quad i \neq j. \tag{19}$$

Because the equilibrium conditions for the Eulerian stress tensor reduce to simple symmetry (19) or to differential equations (15) that are linear in space (final) coordinates, the Eulerian formulation, so commonly used in Fluid Mechanics, is also tempting at first sight to deal with Elasticity problems involving finite displacements. It is however, like Fluid Mechanics, restricted to isotropic constitutive equations. For fibrous materials, for instance, the constitutive equations can only be set up in a configuration where the local orientation of fibers is known; this is usually the initial configuration. In the strained configuration the fibers are reoriented by the local material rotations, whose amplitudes are not restricted in the non linear theory. As a consequence the constitutive equations depend not only on the Eulerian or Almansi strain measure but also on the local rotations, which complicates matters considerably. Furthermore the boundary conditions are also modified by the motion of the bounding surfaces.

These considerations motivate a return to material coordinates. This is achieved by expressing the force element in (10) in terms of the initial oriented surface element. The combination of (9) and (10)

$$\mathrm{d}F_j = \tau_{ij} J\,(\partial_i x_m) n_m\,\mathrm{d}S$$

introduces the Lagrangian or Piola stress tensor t_{mj}

$$t_{mj} = \tau_{ij} J\,\partial_i x_m \tag{20}$$

so that

$$\mathrm{d}F_j = t_{mj} n_m\,\mathrm{d}S. \tag{21}$$

In this operation the force element has been simply translated from the final to the initial surface element and the virtual work has been kept invariant. Application of the divergence theorem on the initial configuration yields now

$$\int_S \mathrm{d}F_j \delta u_j = \int_S n_m t_{mj} \delta u_j\,\mathrm{d}S = \int_V D_m(t_{mj}\delta u_j)\,\mathrm{d}V. \tag{22}$$

With ρ_0 denoting the mass per unit volume in the initial configuration, conservation of energy can be stated in the form

$$\delta \int_V \rho_0 U\,\mathrm{d}V = \int_V \{\rho_0 g_j \delta u_j + D_m(t_{mj}\delta u_j)\}\,\mathrm{d}V$$

and implies

$$\delta W = \rho_0 g_j \delta u_j + D_m(t_{mj}\delta u_j) \tag{23}$$

where, for concision of notation, the strain energy per unit initial volume $W = \rho_0 U$ was introduced.

The translational equilibrium equations are again obtained as the special case $\delta W = 0$, when the δu_j are arbitrary constants, yielding

$$\rho_0 g_j + D_m t_{mj} = 0 \quad (j = 1, 2, 3). \tag{24}$$

Hence the Piola tensor satisfies linear differential equations in material coordinates; they reduce (23) to

$$\delta W = t_{qj} D_q \delta u_j. \tag{25}$$

The rotational equilibrium equations are however non-linear; they are obtained by substitution of (17) with again $\delta W = 0$, whence

$$e_{jmn} t_{qj} D_q \xi_n = 0 \quad (m = 1, 2, 3). \tag{26}$$

The last factor is no more a simple Kronecker delta but, in view of (1),

$$D_q \xi_m = D_q x_m + D_q u_m = \delta_{qm} + D_q u_m. \tag{27}$$

Geometrical linearization of elasticity theory implies that we consider the displacement gradients as negligible before unity

$$|D_q u_m| \ll 1, \tag{28}$$

in which case the second term in the right-hand side of (27) can be neglected when substituted into (26) and we would again obtain a statement of symmetry for the Piola stress tensor. As a matter of fact the linearizing assumption is easily seen to make the Piola tensor and the Eulerian stress tensor undistinguishable. Even in the case of infinitesimal strains, finite rotations causes assumptions (28) not to be satisfied and the general form (26) of rotational equilibrium, showing the Piola tensor not to be symmetrical, must be retained. It can be stated in the equivalent form

$$t_{qj} D_q \xi_m = t_{qm} D_q \xi_j. \tag{29}$$

2. KIRCHHOFF–TREFFTZ TENSOR

Material coordinates allow to perform all integrations and to state boundary conditions on the known and fixed initial configuration; in addition they allow to commute partial derivatives and variations

$$D_q \delta u_j = D_q\{\hat{u}_j(x_m) - u_j(x_m)\} = D_q \hat{u}_j - D_q u_j = \delta D_q u_j$$

so that equation (25) of conservation of energy becomes that of a perfect differential

$$\delta W = t_{qj} \delta e_{qj} \tag{30}$$

$$e_{qj} = D_q u_j. \tag{31}$$

The energy per unit volume is thus a function of the nine displacement gradients e_{qj} and

$$t_{qj} = \partial W / \partial e_{qj}. \tag{32}$$

Additional information as to the structure of the function W is provided by the rotational equilibrium equations. One way to obtain this information is to rewrite (29), using (27), (31) and (32), as

$$\frac{\partial W}{\partial e_{mj}} + e_{qm} \frac{\partial W}{\partial e_{qj}} = \frac{\partial W}{\partial e_{jm}} + e_{qj} \frac{\partial W}{\partial e_{qm}}. \tag{33}$$

For $m = j$ those equations are trivially satisfied, for $m \neq j$ they constitute a set of three linear partial differential equations for W, that are easily solved by the method of characteristics.

For instance, for $m = 1$ and $= 2$, the differential equations of the characteristics are

$$\frac{\mathrm{d}e_{12}}{1+e_{11}} = \frac{-\mathrm{d}e_{21}}{1+e_{22}} = \frac{\mathrm{d}e_{22}}{e_{21}} = \frac{\mathrm{d}e_{32}}{e_{31}} = \frac{-\mathrm{d}e_{11}}{e_{12}} = \frac{-\mathrm{d}e_{31}}{e_{32}}$$

$$= \frac{\mathrm{d}e_{13}}{0} = \frac{\mathrm{d}e_{23}}{0} = \frac{\mathrm{d}e_{33}}{0} = \frac{\mathrm{d}W}{0}. \tag{34}$$

Simple first integrals of those equations, that is functions of the e_{mn} that are arbitrary constants under those differential relations, are readily found to be

$$2e_{11} + e_{11}^2 + e_{12}^2,\ 2e_{22} + e_{21}^2 + e_{22}^2,\ e_{12} + e_{21} + e_{11}e_{21} + e_{12}e_{22}$$

$$e_{31} + e_{11}e_{31} + e_{12}e_{32},\ e_{32} + e_{21}e_{31} + e_{22}e_{32},\ e_{31}^2 + e_{32}^2$$

and

$$e_{13}, e_{23}, e_{33}.$$

The symmetrical Green or Lagrangian strain tensor g_{mn}, issued from the quadratic form

$$\mathrm{d}\xi_j\,\mathrm{d}\xi_j - \mathrm{d}x_n\,\mathrm{d}x_n = 2g_{mn}\,\mathrm{d}x_m\,\mathrm{d}x_n, \tag{35}$$

has six elements

$$g_{nm} = g_{mn} = \tfrac{1}{2}(D_m\xi_i D_n\xi_i - \delta_{mn}) = \tfrac{1}{2}(e_{mn} + e_{nm} + e_{mi}e_{ni}) \tag{36}$$

that are algebraic combinations of the preceding first integrals and are, consequently, first integrals themselves. They are the only combinations that remain first integrals of the two other sets ($m = 2, j = 3$ and $m = 3, j = 1$).

Moreover W is seen to be constant itself. The conclusion is that the strain energy per unit initial volume is a function of the nine displacements gradients e_{qj} through the six independent coordinates of the Green strain tensor g_{mn}.

A different proof of this result introduces in a natural way the symmetrical Kirchhoff–Trefftz stress tensor s_{qp}. The rotational equilibrium equations (29) are satisfied by setting

$$t_{qj} = s_{qp}D_p\xi_j \tag{37}$$

provided

$$s_{pq} = s_{qp}. \tag{38}$$

Equation (30) becomes then

$$\delta W = s_{qp}D_p\xi_j\delta e_{qj} \tag{39}$$

but also, exchanging the dummy subscripts q and p and using the symmetry of s_{qp},

$$\delta W = s_{qp} D_q \xi_j \delta e_{pj}$$

or, adding both results,

$$2\delta W = s_{qp}(D_p \xi_j \delta e_{qj} + D_q \xi_j \delta e_{pj}).$$

The bracket, however, is, in view of (27),

$$\delta e_{qp} + e_{pj} \delta e_{qj} + \delta e_{pq} + e_{qj} \delta e_{pj}$$

$$= \delta(e_{qp} + e_{pq} + e_{pj} e_{qj}) = 2\delta g_{qp}$$

and we obtain finally

$$\delta W = s_{qp} \delta g_{qp}. \tag{40}$$

Provided we convene to distinguish the order of the subscripts in g_{qp} and to express W symmetrically in terms of the six quantities $\frac{1}{2}(g_{qp} + g_{pq})$, the constitutive equations or stress–strain relations can be summarized by

$$s_{qp} = \partial W / \partial g_{qp}. \tag{41}$$

They contain the symmetry statements $s_{pq} = s_{qp}$, equivalent to the rotational equilibrium conditions. By the chain rule of differentiation and use of this symmetry

$$t_{qj} = \frac{\partial W}{\partial g_{mn}} \frac{\partial g_{mn}}{\partial e_{qj}} = s_{mn} \tfrac{1}{2}(\delta_{mq}\delta_{nj} + \delta_{nq}\delta_{mj} + \delta_{mq}e_{nj} + \delta_{nq}e_{mj})$$

$$= s_{mn}\delta_{mq}(\delta_{nj} + e_{nj}) = s_{qn} D_n \xi_j$$

we retrieve definition (37).

Trefftz [3] gave the geometrical interpretation of the stress tensor s_{qp}. In the strained configuration the convected material coordinates constitute a set of curvilinear coordinates with local base vectors

$$\vec{G}_i = D_i \vec{\xi} = D_i \xi_j \vec{e}_j. \tag{42}$$

The corresponding fundamental metric tensor is

$$G_{im} = \vec{G}_i \,.\, \vec{G}_m = D_i \xi_j D_m \xi_k \vec{e}_j \,.\, \vec{e}_k$$

and, since the base vectors $\vec{e}_j$ of the cartesian frame are orthonormal,

$$G_{im} = D_i \xi_j D_m \xi_j = 2g_{im} + \delta_{im}. \tag{43}$$

This fundamental metric tensor is in fact Cauchy's definition of the strain measure and is thus not fundamentally different from Green's strain tensor. Using in succession equations (21), (37) and (42), the force element expression that is obtained

$$\mathrm{d}\vec{F} = \mathrm{d}F_j\vec{e}_j = t_{qj}n_q\mathrm{d}S\,\vec{e}_j = s_{qp}n_q\mathrm{d}S\,D_p\xi_j\vec{e}_j = s_{qp}n_q\mathrm{d}S\,\vec{G}_p \qquad (44)$$

still associates the Kirchhoff–Trefftz tensor with a definition per unit initial area, but this time in the metric induced by the deformation.

3. THE PRINCIPLE OF VARIATION OF DISPLACEMENTS

With kinematical boundary conditions imposed on displacements

$$u_j = \bar{u}_j \rightarrow \delta u_j = 0 \quad \text{on} \quad S_1 \qquad (45)$$

and Lagrangian surface tractions t_j imposed on the complementary part S_2 of the surface S, bounding the simply connected domain V, the sum of strain energy and potential energy of loads

$$P = \int_V W\,\mathrm{d}V - \int_V \rho_0 g_j u_j\,\mathrm{d}V - \int_{S_2} t_j u_j\,\mathrm{d}S \qquad (46)$$

is a functional of the displacement field that is stationary under arbitrary variations δu_j respecting (45). As proof we can use (25) and obtain, after integration by parts, the Euler equations

$$\mathrm{D}_q t_{qj} + \rho_0 g_j = 0 \quad (j = 1, 2, 3)$$

and natural boundary conditions

$$n_q t_{qj} = t_j \quad (j = 1, 2, 3) \quad \text{on} \quad S_2, \qquad (47)$$

expressing all equilibrium conditions, except rotational equilibrium. As observed earlier, rotational equilibrium becomes integrated in the principle provided W be considered to depend on the displacement gradients through the Green strain tensor. This is equivalent to replace (25) by (39) and (31), or

$$\delta W = s_{qp} D_p \xi_j D_q \delta u_j \qquad (48)$$

and produces the other equilibrium equations in the form given by Signorini:

$$D_q(s_{qp} D_p \xi_j) + \rho_0 g_j = 0 \qquad (49)$$

$$n_q s_{qp} D_p \xi_j = t_j \quad \text{on} \quad S_2. \qquad (50)$$

4. THE CANONICAL PRINCIPLE BASED ON KIRCHHOFF–TREFFTZ STRESSES AND GREEN STRAINS

Starting from this last formulation, a 'canonical' variational principle in the sense of Friedrichs [1] (see also Courant and Hilbert [2]) is obtained by addition of a 'dislocation potential'

$$D = \int_V \tfrac{1}{2} s_{qp}(D_q u_p + D_p u_q + D_q u_j D_p u_j - 2g_{qp})\,\mathrm{d}V$$

in which the s_{qp} are momentarily considered to be a set of Lagrangian multipliers. Their object is to transfer the differential constraints between Green strains and displacements, obtained by substitution of (31) into (36), from the status of *a priori* equations necessary to obtain (48) to a status of Euler equations. They are indeed the variational equations associated to variation of the multipliers in the extended principle $\delta(P+D)=0$.

This principle is a three-field principle, since, in addition to the field of Lagrangian multipliers, the field of the Green tensor g_{mn} has now become independant from the displacement field. Consequently W is now considered to depend on the g_{mn}. The variational equations associated to variations on the g_{mn} turn out to be identical to equations (41), thus identifying the Lagrangian multipliers with the Kirchhoff–Trefftz stresses related to the strains in the constitutive equations.

This allows the three-field principle to be simplified by accepting the constitutive equations to be satisfied *a priori*. Indeed, grouping all terms containing the Green strain tensor, the following contact transformation is suggested

$$s_{qp}g_{qp}-W=S(s_{mn}) \tag{51}$$

which would remove the presence of the g_{mn} and lead to the 'canonical form'

$$\int_V \{\tfrac{1}{2}s_{qp}(D_q u_p+D_p u_q+D_q u_j D_p u_j)-S\}\,\mathrm{d}V$$
$$-\int_V \rho_0 g_j u_j\,\mathrm{d}V-\int_{S_2} t_j u_j\,\mathrm{d}S \quad \text{stationary.} \tag{52}$$

The function S of the Kirchhoff–Trefftz tensor is a 'complementary energy density'. Its existence and uniqueness imply that the constitutive equations (41) establish a one to one correspondance between stresses and strains, they must have a unique inverse. This inverse can then be written as

$$g_{qp}=\partial S/\partial s_{qp} \tag{53}$$

provided S be expressed in a symmetrical form.

The variational equations of the canonical two-field principle (52) are the Signorini equilibrium equations (49) and (50) and, associated to the variations of the Kirchhoff–Trefftz stresses, the compatibility equations

$$\tfrac{1}{2}(D_q u_p+D_p u_q+D_q u_j D_p u_j)=\partial S/\partial s_{pq}. \tag{54}$$

This canonical principle obviously constitutes a generalization to finite displacements of the Reissner principle of linear elasticity theory. It was discovered independantly by Reissner [5] and the author.

In the linearized case, assumptions (28) allow to discard the term $D_q u_j D_p u_j$ in (52) and to replace $D_p\xi_j$ by the Kronecker δ_{pj} in the Signorini equations. Then, if the functional (52) is integrated by parts and if the stresses are assumed to satisfy the Signorini equations *a priori*, one obtains the complementary energy principle

$$-\int_V S\,\mathrm{d}V+\int_{S_1} n_q s_{qp}\bar{u}_p\,\mathrm{d}S \quad \text{stationary.}$$

It is a single-field principle involving only stresses in equilibrium. Its generalization, when assumptions (28) are discarded, is known to fail when attempted on the present formulation [9, 10].

5. VARIATIONAL FORMULATIONS BASED ON THE PIOLA AND DISPLACEMENT GRADIENT TENSORS

Levinson [9] pointed out that a complementary energy principle for finite displacements could be obtained on the condition that the contact transformation

$$t_{qj} e_{qj} - W = T(t_{mn}) \tag{55}$$

would exist, in which case

$$e_{qj} = \partial T / \partial t_{qj}. \tag{56}$$

This would furnish a canonical principle, akin to (52)

$$\int_V (t_{qj} D_q u_j - T)\, \mathrm{d}V - \int_V \rho_0 g_j u_j\, \mathrm{d}V - \int_{S_2} t_j u_j\, \mathrm{d}S \quad \text{stationary.} \tag{57}$$

Then, integration by parts and *a priori* satisfaction of the equilibrium equations (24) and (47) would give the complementary energy principle

$$-\int_V T\, \mathrm{d}V + \int_{S_1} n_q t_{qj} \bar{u}_j\, \mathrm{d}S \quad \text{stationary.} \tag{58}$$

Zubov [11] proposed a method for computing T in the case of an isotropic medium and performed the computation for a semilinear material. However, in the general case, it is not known whether a one to one correspondance between g_{mn} and s_{mn} will be sufficient to guarantee the existence of contact transformation (55). In this connexion the following observations can be made. Equations (30) or (32) hold true, whether or not the rotational equilibrium equations are satisfied. If they are, we know that W has a special structure, which must be reflected in a corresponding structure for T. Indeed, if T exists, we can use (56) to place the rotational equilibrium equations (29), or, equivalently,

$$t_{qj}(\delta_{qm} + e_{qm}) = t_{qm}(\delta_{qj} + e_{qj})$$

in the form

$$t_{mj} + t_{qj} \frac{\partial T}{\partial t_{qm}} = t_{jm} + t_{qm} \frac{\partial T}{\partial t_{qj}} \quad j \neq m \tag{59}$$

of a set of 3 partial differential equations to be satisfied by T. Integration by the method of characteristics yields the result

$$T(t_{mn}) = -(t_{11} + t_{22} + t_{33}) + F(t_{mi} t_{ni}). \tag{60}$$

Thus the structure of T comprises a function to be determined in the elements of the symmetrical tensor

$$T_{mn} = t_{mi} t_{ni}. \tag{61}$$

In view of (37) and (53) it can be expressed in terms of the Kirchhoff–Trefftz tensor

$$T_{mn} = s_{mq} s_{np} D_q \xi_i D_p \xi_i = s_{mq} s_{np} (2 g_{qp} + \delta_{qp}), \text{ or,}$$

$$T_{mn} = 2 s_{mq} s_{np} \frac{\partial S}{\partial s_{qp}} + s_{mq} s_{nq}. \tag{62}$$

Furthermore, Levinson's contact transformation can be rewritten in the form

$$t_{qj} D_q \xi_j - W = t_{qj}(e_{qj} + \delta_{qj}) - W = T + t_{qq} = F$$

and, using (37) once again,

$$s_{qp} D_p \xi_j D_q \xi_j - W = s_{qp}(2 g_{qp} + \delta_{qp}) - W = F$$

or

$$F = 2 s_{qp} g_{qp} + s_{qq} - W. \tag{63}$$

If we accept the existence of the contact transformation (51), the function F, is expressible in terms of the Kirchhoff–Trefftz stresses. We have thus obtained the following result: the existence of contact transformation (51), which is in the nature of a physical assumption, guarantees the existence of contact transformation (55), provided equations (62) are invertible. The existence and unicity problem for the inversion of non-linear equations (62) can be handled theoretically by the implicit functions theorem. The result would however be useless for practical applications and this approach will not be pursued.

6. A CANONICAL VARIATIONAL PRINCIPLE BASED ON THE POLAR DECOMPOSITION

The Jacobian matrix (2) has a unique decomposition

$$D_i \xi_j = \alpha_{jm} (\delta_{mi} + h_{mi}) \tag{64}$$

in terms of a Lagrangian measure of strain

$$h_{mi} = h_{im} \tag{65}$$

and a rigid body rotation operator α_{jm}:

$$\alpha_{jm} \alpha_{jn} = \delta_{mn} \quad \alpha_{jm} \alpha_{pm} = \delta_{jp} \quad |\alpha_{jm}| = 1. \tag{66}$$

Through this decomposition a neighborhood of the medium is first strained and afterwards rotated. The strain tensor h_{mj} is related to the Green tensor through

$$\begin{aligned} D_i\xi_j D_p\xi_j &= \alpha_{jm}\alpha_{jn}(\delta_{mi}+h_{mi})(\delta_{np}+h_{np}) \\ &= \delta_{mn}(\delta_{mi}+h_{mi})(\delta_{np}+h_{np}) \\ &= (\delta_{ni}+h_{ni})(\delta_{np}+h_{np}) \end{aligned}$$

or, finally, using the symmetry properties (65),

$$2g_{ip} = 2h_{ip}+h_{in}h_{np}. \tag{67}$$

Because of (4) the relation is known to be one–one. The tensors are co-axial and, in a set of principal axes, their diagonal elements are related as follows:

$$2g_i = 2h_i+h_i^2 \qquad (i=1,2,3). \tag{68}$$

Then, because (4) implies

$$1+2g_i > 0 \qquad \text{and} \qquad 1+h_i > 0, \tag{69}$$

the unique inverse to (68) is

$$h_i = -1+\sqrt{1+2g_i}. \tag{70}$$

For commodity the tensor h_{mi} will be called the 'engineering strain tensor'; this is justified by the fact that the principal strains can be defined, as in the linearized case, by the usual engineering formula

$$h_i = \frac{\mathrm{d}\sigma_i-\mathrm{d}s_i}{\mathrm{d}s_i},$$

$\mathrm{d}s_i$ and $\mathrm{d}\sigma_i$ being the elementary distances measured before and after straining along the principal directions. In fact, for infinitesimal strains, (67) shows that the Green and engineering strain measures become identical.

The one to one correspondance that exists, even for finite strains, establishes that W can also be considered to be a function of the engineering strains and that a new symmetrical stress tensor can be introduced by

$$r_{mi} = \partial W/\partial h_{mi}. \tag{71}$$

From (67)

$$\delta g_{ip} = \delta h_{ip}+\tfrac{1}{2}h_{in}\delta h_{np}+\tfrac{1}{2}h_{np}\delta h_{in}$$

and (40) can be manipulated as follows

$$\delta W = s_{ip}\delta g_{ip} = s_{ip}\delta h_{ip}+\tfrac{1}{2}s_{ip}h_{in}\delta h_{np}+\tfrac{1}{2}s_{ip}h_{np}\delta h_{in}.$$

In the second right hand term exchange the dummy subscripts n and i, and p and n in the third, then

$$\delta W = (s_{ip} + \tfrac{1}{2}s_{np}h_{ni} + \tfrac{1}{2}s_{in}h_{pn})\,\delta h_{ip}.$$

The bracket is clearly symmetrical in the subscripts i and p, so that by identification with

$$\delta W = r_{ip}\,\delta h_{ip} \tag{72}$$

the following relationship is established between the new stress tensor and the Kirchhoff–Trefftz one:

$$r_{ip} = s_{ip} + \tfrac{1}{2}h_{in}s_{np} + \tfrac{1}{2}s_{in}h_{np}. \tag{73}$$

In the case of infinitesimal strains, but even for finite rotations, both tensors are clearly coalescent. Then, invertible stress–strain relations of type (41) are equivalent to invertible relations of type (71) and the following diagram of one to one relationship

$$s_{ip} \leftrightarrow g_{ip} \leftrightarrow h_{ip} \leftrightarrow r_{ip} \tag{74}$$

shows (73) to be invertible.

Introducing the polar decomposition (64) into definition (37)

$$t_{qj} = s_{qp}\alpha_{jm}(\delta_{mp} + h_{mp}) \tag{75}$$

showing that, in view of (74), the nine components of the Piola tensor are determined by the six components of the Kirchhoff–Trefftz tensor and the three effective parameters of the material rotation.

From (75) follows also

$$t_{qj}\,\alpha_{jn} = s_{qp}(\delta_{np} + h_{np}) = s_{qn} + s_{qp}h_{pn}$$

and, in view of (73),

$$r_{qn} = \tfrac{1}{2}(t_{qj}\alpha_{jn} + t_{nj}\alpha_{jq}). \tag{76}$$

To the functional P, given by (46), we add a dislocation potential that removes the differential constraints (64) of the polar decomposition and consider W to be a function of the engineering strain tensor. Since the use of (71) implies that W be expressed symmetrically with respect to h_{mi} and h_{im}, we take care to do the same for the dislocation potential:

$$\Delta = \int_V t_{ij}\left\{D_i\xi_j - \alpha_{jm}\left(\delta_{mi} + \frac{h_{mi}+h_{im}}{2}\right)\right\}\mathrm{d}V. \tag{77}$$

The variational equation corresponding to δh_{qn} in $\delta(P+\Delta) = 0$ is

$$\partial W/\partial h_{qn} = \tfrac{1}{2}(t_{qj}\,\alpha_{jn} + t_{nj}\,\alpha_{jq})$$

which corresponds to (76), r_{qn} being regarded as given by the constitutive equations in the form of (71). This can serve to identify the set of multipliers of the dislocation potential as the Piola stress tensor. The variational equations corresponding to the rotational degrees of freedom are obtained as follows:

$$\alpha_{jm}\alpha_{jn} = \delta_{mn} \rightarrow \alpha_{jn}\delta\alpha_{jm} = -\alpha_{jm}\delta\alpha_{jn} = \delta\omega_{mn} \quad \text{skew-symmetric}$$

and consequently,

$$\delta\alpha_{jm} = \alpha_{jn}\delta\omega_{mn}. \tag{78}$$

Substituting this into $\delta(P+\Delta) = 0$, there comes

$$t_{ij}\,\alpha_{jn}(\delta_{mi}+h_{mi})\,\delta\omega_{mn} = 0$$

or, on account of the skew-symmetry of $\delta\omega_{mn}$,

$$t_{ij}\,\alpha_{jn}(\delta_{mi}+h_{mi}) = t_{ij}\,\alpha_{jm}(\delta_{ni}+h_{ni}).$$

Multiplying by $\alpha_{pn}\alpha_{qm}$, it becomes

$$t_{ip}\alpha_{qm}(\delta_{mi}+h_{mi}) = t_{iq}\alpha_{pn}(\delta_{ni}+h_{ni})$$

and, in view of (64), reduces to the known rotational equilibrium form

$$t_{ip}D_i\xi_q = t_{iq}D_i\xi_p.$$

Thus, as could be suspected, the rotational degrees of freedom in the principle have the rotational equilibrium conditions as variational equations.

We now simplify the principle by grouping the terms containing the h_{mi}

$$t_{ij}\,\alpha_{jm}\frac{h_{mi}+h_{im}}{2} - W.$$

Exchange the dummy subscripts i and m in the second term, to obtain in view of (76)

$$h_{mi}\tfrac{1}{2}(t_{ij}\,\alpha_{jm}+t_{mj}\,\alpha_{ji}) - W = h_{mi}r_{mi} - W$$

which suggests again a contact transformation:

$$h_{mi}r_{mi} - W = R(r_{pq}) \quad h_{mi} = \partial R/\partial r_{mi}. \tag{79}$$

For infinitesimal strains this transformation is identical to (51) and will be assumed to exist; in other words (71) is assumed to have a unique inverse. Note also that for $q = n = m$, (76) gives

$$r_{11}+r_{22}+r_{33} = r_{mm} = t_{mj}\,\alpha_{jm},$$

and that

$$t_{ij}D_i\xi_j = t_{ij}(\delta_{ij}+D_iu_j) = t_{jj}+t_{ij}D_iu_j.$$

The canonical form of the variational principle can now be given as

$$\int_V (t_{ij}D_iu_j + t_{jj} - r_{mm} - R)\,\mathrm{d}V - \int_V \rho_0 g_j u_j\,\mathrm{d}V - \int_{S_2} t_j u_j\,\mathrm{d}S \quad \text{stationary.} \tag{80}$$

It contains as independant variables, the displacements u_j, the Piola stress tensor t_{ij} and the 3 rotational degrees of freedom that appear in the 'auxiliary' variables (76). Its variational equations are (24) and (47), together with

$$D_p u_q + \delta_{pq} - \alpha_{qp} - h_{ij}\frac{\partial r_{ij}}{\partial t_{pq}} = 0 \quad \text{for} \quad \delta t_{pq},$$

that is easily recognized to be a disguised form of the polar decomposition

$$D_p \xi_q = \alpha_{qn}(\delta_{np} + h_{nq}).$$

Also for

$$\delta\alpha_{uv} = \alpha_{up}\,\delta\omega_{vp}$$

$$\left(\frac{\partial r_{mm}}{\partial \alpha_{uv}} + \frac{\partial R}{\partial r_{qn}}\frac{\partial r_{qn}}{\partial \alpha_{uv}}\right)\alpha_{up}\,\delta\omega_{vp} = 0$$

or

$$[t_{vu} + \tfrac{1}{2}(h_{qv}t_{qu} + h_{vn}t_{nu})]\,\alpha_{up}\,\delta\omega_{vp} = 0$$

or

$$t_{nu}(\delta_{nv} + h_{nv})\,\alpha_{up} = t_{nu}(\delta_{np} + h_{np})\,\alpha_{uv},$$

a disguised form of the rotational equilibrium conditions.

7. THE COMPLEMENTARY ENERGY PRINCIPLE BASED ON THE POLAR DECOMPOSITION

It suffices now to integrate by parts in (80) and to assume the equilibrium equations (24) and (47) satisfied to reduce the principle to a complementary energy form

$$\int_V (t_{jj} - r_{mm} - R)\,\mathrm{d}V + \int_{S_1} n_m t_{mj}\bar{u}_j\,\mathrm{d}S \quad \text{stationary.} \tag{81}$$

Because the rotational degrees of freedom are still free and rotational equilibrium is still cared for by the principle itself, satisfaction of the translational equilibrium equations requires only 'first order' stress functions. For instance

$$t_{mj} = e_{mrs}D_r A_{sj} \tag{82}$$

satisfies equations (24) in the absence of body loads and introduces a tensor A_{sj} of functions that generate the Piola stresses by their first derivatives. Then, provided we satisfy *a priori*

$$e_{mrs}n_m D_r A_{sj} = t_j \quad \text{on} \quad S_2, \tag{83}$$

and consider the auxiliary variables (76) to represent

$$r_{qm} = \tfrac{1}{2}(e_{qrs}\,\alpha_{jm} + e_{mrs}\,\alpha_{jq})D_r A_{sj}, \tag{84}$$

we obtain a principle governing the first order stress functions. A satisfactory treatment of the natural boundary conditions requires that (83) be incorporated into (81) by means of Lagrangian multipliers; this transforms (81) into

$$\int_V (t_{jj} - r_{mm} - R)\,\mathrm{d}V + \int_S n_m u_j t_{mj}\,\mathrm{d}S - \int_{S_2} u_j t_j\,\mathrm{d}S \quad \text{stationary} \tag{85}$$

where the u_j are imposed displacements on S_1 but free Lagrangian multipliers, that are of course identified to the surface displacements, on S_2. Preparatory calculations based on (78), (79), (82) and (84), yield

$$\begin{aligned}
\delta t_{jj} &= e_{jrs} D_r\,\delta A_{sj}\\
\delta r_{mm} &= e_{hrs}\,\alpha_{jp} D_r A_{sj}\,\delta\omega_{np} + e_{mrs}\,\alpha_{jm} D_r\,\delta A_{sj}\\
\delta R &= e_{mrs} h_{mn}(\alpha_{jp} D_r A_{sj}\delta\omega_{np} + \alpha_{jn} D_r \delta A_{sj}).
\end{aligned}$$

From them, the variational equations associated with the skew-symmetric variations $\delta\omega_{np}$ (rotational equilibrium equations) are obtained in the form

$$e_{mrs} D_r A_{sj}\{\alpha_{jp}(\delta_{mn} + h_{mn}) - \alpha_{jn}(\delta_{mp} + h_{mp})\} = 0. \tag{86}$$

The variational equations associated to δA_{sj} follow from an integration by parts as

$$e_{mrs} D_r\{\alpha_{jn}(\delta_{mn} + h_{mn})\} = 0 \qquad \text{all} \quad (s,j). \tag{87}$$

They are equivalent to the equations obtained from the polar decomposition (64) on eliminating the displacements by cross-differentiation. Hence they can be considered as the local integrability conditions for a displacement field. The surface terms, containing variations of the stress functions, are collected below:

$$\int_S \{e_{jrs} n_r - e_{mrs} n_r\,\alpha_{jn}(\delta_{mn} + h_{mn})\}\,\delta A_{sj}\,\mathrm{d}S + \int_S e_{mrs} n_m u_j D_r\,\delta A_{sj}\,\mathrm{d}S = 0 \tag{88}$$

and the second integral requires manipulation to remove the derivatives of the variations. To this effect we use Stokes theorem

$$\oint_c \vec{b}\,.\,\mathrm{d}\vec{s} = \int_{S'} \mathrm{rot}\,\vec{b}\,.\,\vec{n}\,\mathrm{d}S,$$

according to which the circulation of vector $\vec{b}$ around a closed contour c is equal to the flux of rot $\vec{b}$ through the surface S' bordered by this contour. Let S'' denote a second surface bordered by the same contour and modify the orientation of the normal on it so that it points away from the volume V comprized between both surfaces. Then

$$\oint_c \vec{b}\,.\,\mathrm{d}\vec{s} = -\int_{S''} \mathrm{rot}\,\vec{b}\,.\,\vec{n}\,\mathrm{d}S$$

and, subtracting both formulas,

$$\int_S \operatorname{rot} \vec{b} \,.\, \vec{n} \, dS = \int_S e_{mrs} n_m D_r b_s \, dS = 0$$

where S is the complete surface bounding the volume. Thus, if in the second integral of (88) we observe that

$$u_j D_r \delta A_{sj} = D_r(u_j \delta A_{sj}) - D_r u_j \,.\, \delta A_{sj},$$

only the second term will contribute. The natural boundary conditions resulting from the free variations δA_{sj} on S are consequently:

$$e_{jrs} n_r - e_{mrs}\{n_r \alpha_{jn}(\delta_{mn} + h_{mn}) + n_m D_r u_j\} = 0. \tag{89}$$

Since

$$e_{mrs} n_m D_r u_j = (\vec{n} \times \operatorname{grad} u_j)_s$$

they involve only surface derivatives of the data $\bar{u}_j$ on S_1.
They are also seen to be identically satisfied by using the polar decomposition

$$\alpha_{jn}(\delta_{mn} + h_{mn}) = D_m \xi_j = \delta_{mj} + D_m u_j$$

giving

$$e_{jrs} n_r - e_{mrs} n_r \delta_{mj} - e_{mrs} n_r D_m u_j - e_{mrs} n_m D_r u_j = 0.$$

The two first terms cancel and, exchanging the dummy subscripts r and m in the third and noting that $e_{rms} = -e_{mrs}$,

$$e_{mrs} \,.\, n_m(D_r u_j - D_r u_j) = 0.$$

8. LINEARIZED VERSION OF THE COMPLEMENTARY ENERGY PRINCIPLE

Assuming very small rotations, we can write

$$\alpha_{im} = \delta_{im} + \omega_{im} \tag{90}$$

where

$$\omega_{im} = -\omega_{mi} \quad \text{and} \quad |\omega_{mi}| \ll 1. \tag{91}$$

The following approximations can then be introduced. To lowest order

$$r_{qm} = \tfrac{1}{2}(t_{qm} + t_{mq}) \tag{92}$$

as arguments of the complementary energy density and

$$r_{mm} - t_{jj} = t_{mj}\omega_{jm}. \tag{93}$$

The linearized version of principle (81) reduces then to

$$-\int_V (t_{mj}\omega_{jm}+R)\,\mathrm{d}V+\int_{S_1} n_m t_{mj}\bar{u}_j\,\mathrm{d}S \text{ stationary} \tag{94}$$

where the approximation

$$t_{qm}=t_{mq} \tag{95}$$

becomes a variational result. Hence, as for the non linear case, we need only first order stress functions in order to free the translational equilibrium equations. In approximate solutions it becomes possible to discretize the rotational equilibrium conditions.

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Résumé—Des principes de variations pour des déplacements élastiques finis ont été formulés en termes de contrainte de Green et de tenseurs de contrainte de Kirchoff—Trefftz. Le premier représente une fonctionnel le dépendant uniquement du champ de déplacement et implique que le potentiel total soit stationnaire. Le second est un principe canonique, dans le sens de Friedrichs (1), comprenant à la fois les contraintes et les déplacements, et généralisant le principe de Reissner[5]. Au contraire de la théorie de l'élasticité linéarisée géométriquement, il ne peut pas être ramené à un principe d'énergie complémentaire comportant seulement les contraintes d'équilibre 9, 10. Cet article étudie les formulations de Levinson [9] et Zubov [11] en termes de gradients de déplacements et du tenseur de contrainte de Piola, lequel, bien qu'il soit intéressant d'un point de vue théorique, n'apparait pas convenir à des applications pratiques. On découvre qu'un nouveau groupe de principes des variations, de type de déplacement, de type canonique ou d'énergie complémentaire, découle de l'utilisation du théorème de la décomposition polaire du Jacobien. Il comprende le tenseur de déformation mécanique et le tenseur de contrainte conjugué doit être considéré comme la rotation matérielle. Cette formulation de l'énergie complémentaire est étudiée en termes de fonctions de contrainte du premier ordre. La présence des degrés de liberté de rotation ouvre la possibilité de donner des valeurs discrètes aux équations d'équilibre de rotation dans des solutions approchées.

Zusammenfassung—Variationsprinzipien für endliche elastische Verdrängungen wurden in Ausdrücken von Green'schem Beanspruchungstensor und Kirchhoff-Trefftz'schem Spannungstensor formuliert. Der erstere ist nur ein Funktional des Verdrängungsfeldes und schliesst das Gleichbleiben des Totalpotentials ein. Der zweite ist ein kanonisches Prinzip im Sinne von Friedrichs [1], wobei sowohl Spannungen als auch Verdrängungen beteiligt sind und Reissner's Prinzip [5] verallgemeinert wird. Zum Unterschied von der geometrisch linearisierten Elastizitätstheorie kann es nicht auf ein komplementäres Energieprinzip reduziert werden, das nur Gleichgewichtsspannungen einbezieht [9, 10]. Die Arbeit bespricht die Formulierung von Levinson [9] und Zubov [11] in Ausdrücken von Verdrängungsgradienten und des Piola'schen Spannungstensors, was, wie interessant es von einem theoretischen Standpunkt sein möge, für eine praktische Anwendung nicht geeignet scheint. Es wird gefunden, dass ein neuer Satz von Variationsprinzipien, von Verdrängung, kanonischen oder komplementären Energietypen, sich von der Verwendung des polaren Zersetzungstheorems der

Jakobischen Determinante ableitet. Es schliesst den technischen Verzerrungstensor ein und sein konjugierter Spannungstensor muss als eine Funktion des Piola'schen Tensors und der Materialrotation betrachtet werden. Die komplementäre Energieformulierung wird in Ausdrücken von Spannungsfunktionen erster Ordnung besprochen. Die Anwesenheit der Drehungsfreiheitsgrade ergibt die Möglichkeit, die Drehungsgleichgewichtsgleichungen in Näherungslösungen zu diskretisieren.

Sommario—Si sono formulati principi variazionali per spostamenti elastici finiti in termini di sollecitazione di Green e tensori di sollecitazione di Kirchhoff–Trefits. Il primo è una funzionale del solo campo di spostamento e comporta stazionarietà del potenziale totale. Il secondo è un principio canonico, secondo Friedrichs [1], che interessa sia sollecitazioni che spostamenti e generalizza il principio di Reissner [5]. In contrasto con la teoria di elasticità geometricamente linearizzata, non può venire ridotto a un principio di energia complementare interessante le sollecitazioni di equilibrio solamente [9, 10]. Nell'articolo si discute la formulazione di Levinson [9] e Zubov [11] in termini di curve di spostamento oltre che il tensore di sollecitazione di Piola che, benché interessante da un punto di vista teorico, non appare indicato per applicazioni pratiche. Si scopre come una nuova serie di principi variazionali, di tipo di spostamento, canonico o di energia complementare, derivi dall'uso del teorema di scomposizione polare del giacobismo. Interessa il tensore di sollecitazione d'ingegneria e il suo tensore di sollecitazione coniugato va inteso come una funzione del tensore di Piola e della rotazione materiale. Si discute la formulazione d'energia complementare in termini di funzioni di sollecitazione di primo grado. La presenza dei gradi rotazionali di libertà apre la possibilità di discretizzare le equazioni di equilibrio rotazionale in soluzioni approssimative.

Абстракт — Формулированы вариационные принципы конечных эластичных смещений, выраженные через тензоры натяжения Грина и тензоры напряжения Кирхгофа-Треффца. Первый принцип является функционалом только от поля смещений, означая стационарность общего потенциала. Второй принцип является каноническим в смысле Фридрихса [1], что связано с напряжениями и с смещениями, а также обобщает принцип Рейсснера [5]. В отличии от геометрически линеаризованной теории упругости приведение его к дополнительному энергетическому принципу, связанному только с равновесными напряжениями, не возможно [9, 10]. Обсуждена формуляция Левинсона [9] и Зубова [11], выраженная через градиенты смещения и тензор напряжения Пиола, которая как бы ни интересна от теоретической точки зрения не оказывается пригодной для практических применений. Установлено новое множество вариационных принципов типов смещения, канонического или дополнительной энергии, которое выводится от использования теоремы полярного разложения якобиана. Оно связано с ниженерным тензором натяжения, а его сопряженный тензор напряжения следует считать функцией от тензора Пиола и от материального вращения. Обсуждена формуляция дополнительной энергии, выраженная через функции напряжения первого порядка. Наличие ротационных ступеней свободы дает возможность дискретизации ротационных равновесных уравнений в приближенных решениях.