Janvier 1974

Rapport SA-33



# VARIATIONAL PRINCIPLES AND THE PATCH TEST

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## SUMMARY

The patch test is shown to be contained in the variational formulations of the finite element methods at the assembling level, all of which require the vanishing of the virtual work of interface connexion loads. By a systematic introduction of stress generating functions, attention is drawn to the fact that any given finite element model can be assembled in two different ways : either by identification of a set of boundary displacements (leading to the direct stiffness method), or by identification of a set of local stress function values (leading to the dual direct flexibility method). Looking at any conjugate couple (generalized displacement - generalized surface traction) at an interface, one is strongly transmitted, the other weakly. Discretization of the zero virtual work condition at an interface of plate bending models, by means of Legendre polynomial expansions, allows a systematic construction of so-called "non-conforming" elements that pass the patch test. They are in fact identified with weakly conforming, but strongly diffusive, hybrids, and the lowest degree element (quadratic) is in fact the Morley constant-moment element. Examples are given for higher degree displacement fields.
The case of plate stretching elements can be handled by duality, the difficulties being here associated with the requirements for diffusivity. Non-diffusive elements that pass the zero interface virtual work test can be constructed systematically and are identifiable with weakly diffusive, but strongly conforming, hybrids of the type first proposed by T.H.H. PIAN.

## INTRODUCTION

There is a functional that generates all the equations of linear elasticity theory in the form of variational derivatives and natural boundary conditions.
Its original construction [12] followed the method proposed by FRIEDRICHS [13,14] for one-dimensional problems. It can be considered as a very general three-field principle, from which most, if not all, finite element models can be derived through specializing assumptions [6].

With surface tractions $\overline{t}_j$ given at the boundary $\partial E$ of the simply connected domain E of the element, the functional is

$$\int_E \{ W(\varepsilon) + \tau_{ij} \left(\frac{D_i u_j + D_j u_i}{2} - \varepsilon_{ij}\right) - X_j \, u_j \} \, dE$$

$$- \int_{\partial E} \overline{t}_j \, u_j \, dS \qquad \min_u \{ \max_\tau \, (\min_\varepsilon) \} \qquad (1)$$

The strain energy density W is assumed to be a homogeneous, positive definite, quadratic form of the elements $\varepsilon_{ij}$ of the strain tensor.
The variational derivatives are : with respect to the strains

$$\tau_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \qquad (2)$$

which are the linear elastic constitutive equations; with respect to the stresses $\tau_{ij}$

$$\varepsilon_{ij} = \frac{1}{2} (D_i \, u_j + D_j \, u_i) \qquad (3)$$

the strain-displacement relations.
The variations on the displacement field occur both in the interior of E, generating the volume equilibrium equations

$$D_i \, \tau_{ij} + X_j = 0 \qquad (4)$$

and on the boundary $\partial E$, where they produce "natural" surface equilibrium equations

$$n_i \, \tau_{ij} = \overline{t}_j \qquad \text{on} \quad \partial E \qquad (5)$$

involving the direction cosines $n_i$ of the outward normal.
Should we specify the displacements $\overline{u}_j$ on the boundary, instead of the surface tractions, the variations on the displacements become limited to the interior of E but the variations on the stress field do now take place on the boundary as well as in the interior. This appears clearly in the corresponding functional

$$\int_E \{ W(\varepsilon) - \tau_{ij}\,\varepsilon_{ij} - u_j\,(D_i\,\tau_{ij} + X_j) \}\,dE + \int_{\partial E} n_i\,\tau_{ij}\,\bar{u}_j\,dS \qquad (1')$$

$$\min_u \{ \max_\tau (\min_\varepsilon) \}$$

that has the same variational derivatives as the preceding one, but other natural boundary conditions obtained from the variation of $t_j = n_i\,\tau_{ij}$ on $\partial E$

$$u_j = \bar{u}_j \qquad \text{on} \quad \partial E \qquad (5').$$

In many cases the boundary data may be "mixed" in the sense that the boundary is made of the union of partial boundaries $\partial_\alpha E$

$$\partial E = \cup\, \partial_\alpha E$$

on each of which the data, given componentwise, are not necessarily all of the displacement type or all of the surface traction type.
The mixing of boundary data is however limited by the condition that a single functional exists, having such data involved in natural boundary conditions.
Rotational equilibrium is assumed to hold everywhere by virtue of the symmetry $\tau_{ji} = \tau_{ij}$ of the stress tensor. Still more general variational principles are known, wherein rotational equilibrium is contained as a natural variational result [15]. They provide finite element models circumventing the difficulties associated with $C_1$ continuity [16]; for the sake of simplicity they will not be considered here.

## THE CANONICAL PRINCIPLE

Since equations (2) are invertible, they provide the exact minimizing choice for the strains in the three-field principle

$$\varepsilon_{ij} = \frac{\partial \Phi}{\partial \tau_{ij}} \qquad (6)$$

where

$$\phi(\tau) = \varepsilon_{ij}\,\tau_{ij} - W \qquad (7)$$

is the stress energy density, or complementary energy density. Moreover this optimal choice does not impose any constraints on the remaining fields of stresses and displacements and has no influence on the boundary conditions (the strain field is confined to the interior of E).

There is little sense therefore in keeping the strain field for discretization purposes. Following the ideas of FRIEDRICHS, the stresses are first conceived as Lagrangian multipliers liberating the strain-displacement differential constraints, and equations (2) identify those multipliers through the fact that they satisfy the constitutive equations. Elimination of the strains through their minimizing choice (6) leads to the two-field principle called canonical (in the Hamilton-Jacobi sense) or involutory by FRIEDRICHS and later known as the HELLINGER-REISSNER principle. According to the nature of the boundary data it can be given either of the following forms :

$$\int_E (\tau_{ij} D_i u_j - \Phi(\tau) - X_j u_j)\, dE - \int_{\partial E} \bar{t}_j u_j \, dS \tag{8}$$

$$\min_u (\max_\tau)$$

$$- \int_E \{ \Phi(\tau) + u_j (D_i \tau_{ij} + X_j) \}\, dE + \int_{\partial E} n_i \tau_{ij} \bar{u}_j \, dS \tag{8'}$$

with the same mixing possibilities of boundary data.

## VARIATIONAL DERIVATION OF TRANSITION CONDITIONS

Ideally, at an interface between two adjacent finite elements the displacement field should be continuous

$$(u_j)_+ = (u_j)_- \tag{9}$$

and the surface tractions in equilibrium, writing $t_j = n_i \tau_{ij}$ ,

$$(t_j)_+ + (t_j)_- = 0 \tag{10}$$

Then, provided the coefficients of the constitutive equations (elastic moduli) are themselves continuous at the interface, the transition conditions (9) and (10) will induce continuity of the stress and strain fields. The exact transition conditions (9) and (10) are obtained variationally as follows.

Consider the elements E+ and E−, whose partial boundaries F+ and F− are the faces of an interface F. If the elasticity problem for E is solved with $\bar{t}_{j+}$ as boundary data on $F_+$ and similarly the elasticity problem for $E_-$ with $\bar{t}_{j-}$ on $F_-$, we must assume the data to be coherent; which means, in this case, in equilibrium on F

$$(\bar{t}_j)_+ + (\bar{t}_j)_- = 0$$

In fact, opportunity can be taken of the existence of the interface to introduce a prescribed external interface loading $\hat{t}_j$, in which case the equilibrium condition is non homogeneous

$$(\bar{t}_j)_+ + (\bar{t}_j)_- = \hat{t}_j$$

It can be solved by setting

$$(\bar{t}_j)_+ = \frac{1}{2}\hat{t}_j + t_j \qquad\qquad (\bar{t}_j)_- = \frac{1}{2}\hat{t}_j - t_j \qquad (11)$$

While at each element level $(\bar{t}_j)_+$ and $(\bar{t}_j)_-$ are fixed values, at the assembled level $t_j$ is an unknown internal force. Thus, if the elasticity problem is to be solved by a variational principle, the functional being now extended to the union of the domains $E_+$ and $E_-$, the variation of the terms depending on $t_j$ with respect to $t_j$ must disappear :

$$\delta\int_{F_+} (\bar{t}_j\, u_j)_+\, dS + \delta\int_{F_-} (\bar{t}_j\, u_j)_-\, dS = \int_F \delta t_j (u_{j+} - u_{j-})\, dS = 0 \qquad (12)$$

If $t_j$ is unconstrained (not discretized) this results in the requirement of <u>strong conformity</u> (9).

Conversely, assume the data on $F_+$ and $F_-$ to be respectively $\bar{u}_{j+}$ and $\bar{u}_{j-}$.
If coherent they must satisfy continuity $\bar{u}_{j+} = \bar{u}_{j-}$.
More generally, if opportunity is taken to introduce a prescribed dislocation

$$\bar{u}_{j+} - \bar{u}_{j-} = \Delta u_j \qquad ,$$

this can be solved by setting

$$\bar{u}_{j+} = \frac{1}{2}\Delta u_j + u_j \qquad\qquad \bar{u}_j = -\frac{1}{2}\Delta u_j + u_j \qquad (13)$$

with $u_j$ an internal unknown at the assembled level. Then we must have

$$\delta \int_{F_+} (t_j \bar{u}_j)_+ \, dS + \delta \int_{F_-} (t_j \bar{u}_j)_- \, dS = \int_F \delta u_j \, (t_{j+} + t_{j-}) \, dS = 0 \tag{14}$$

and, for unconstrained $u_j$, this results in the requirement of strong diffusitivity (10).

Note that in formulas (12) and (14) the repeated subscript j does not necessarily mean application of the summation convention as used in the formulation of the variational principles.

In the case of mixed data the "connectors" at an interface may well be $\bar{t}_j$ for one particular value of j and $\bar{u}_j$ for another.

As a final observation, the fact that there are two faces at an interface, allows the imposition of both dislocations and external loading. It is however necessary in this case to add potentials to the functional considered at the assembled level. In the case of $\bar{t}_j$ as connector we add to the functional the dislocation potential $+\int_F t_j \, \Delta u_j \, dS$. Then, collecting the terms depending on $t_j$

$$-\int_{F_+} (\bar{t}_j \, u_j)_+ \, dS - \int_{F_-} (\bar{t}_j \, u_j)_- \, dS + \int_F t_j \, \Delta u_j \, dS$$

substituting (11) and taking variations on $t_j$, we obtain

$$u_{j+} - u_{j-} = \Delta u_j$$

In the case of $\bar{u}_j$ as connector we add the load potential

$$-\int_F \hat{t}_j \, \frac{\bar{u}_{j+} + \bar{u}_{j-}}{2} \, dS \quad .$$

Then, collecting the terms depending on $u_j$

$$\int_{F_+} (t_j \, \bar{u}_j)_+ \, dS + \int_{F_-} (t_j \, \bar{u}_j)_- \, dS - \int_F \hat{t}_j \, \frac{\bar{u}_{j+} + \bar{u}_{j-}}{2} \, dS$$

substituting (13) and taking variations on $u_j$

$$t_{j+} + t_{j-} = \hat{t}_j$$

In the sequel we consider only unloaded and undislocated interfaces.

## THE PURE MODELS

If we use only unconstrained $\bar{t}_j$ connectors for a finite element, the relevant variational principle is given by (8) and, as was just established, there is a requirement for strong conformity (9) at the boundaries.
Assume that this requirement can be satisfied by a discretization of the displacement field of the type

$$u_j(x) = q_m \, Q_{mj}(x) + b_n \, B_{nj}(x) \qquad (15)$$

where the set of independent generalized boundary displacements $q_m$ describes completely the boundary displacement field through the shaping functions $Q_{mj}(x)$ and an additional set of bubble coordinates $b_n$ is eventually needed to complete the description of the interior field. By definition, the bubble functions are such that

$$B_{nj}(x) = 0 \qquad \text{for} \qquad x \in \partial E \quad .$$

Each partial boundary $\partial_\alpha E$ that is the face of an interface has its displacements governed by a subset $\{ q_m \mid m \in m_\alpha \}$ of the $q_m$, which are usually, through not necessarily, local boundary displacements. Necessary and sufficient conditions for strong conformity are then that

$$\left.\begin{array}{l} q_{m+} = q_{m-} \\ Q_{mj+}(x) = Q_{mj-}(x) \qquad \text{when} \qquad x \in \partial_\alpha E_+ = \partial_\alpha E_- \end{array}\right\} \quad m \in m_\alpha$$

The common values of $q_m$ across the interface are known as the "nodal displacements" and are usually chosen as the basic unknowns of the global discretized elasticity problem. For the stress field, it was already observed, and it appears clearly by looking at the functional (8), that its choice is confined to the interior domain of each element.
Unconstrained variations of the $\tau_{ij}$ produce the Euler equations

$$\frac{\partial \Phi}{\partial \tau_{ij}} = \frac{1}{2} \left( D_i \, u_j + D_j \, u_i \right) \qquad (16)$$

that can be solved for the stresses and constitute in fact their optimal (maximizing) choice. Since this places no restrictions on the selection of the displacement field, then is little sense in making other choices, in particular in constructing mixed models with a separately discretized stress field. Thus models using $\bar{t}_j$ as unconstrained connectors may advantageously be set up by the simpler principle of variation of displacements :

$$\int_E \{ W(D\,u) - X_j\, u_j \}\, d\,E - \int_{\partial E} \bar{t}_j\, u_j\, dS \qquad \min_u \qquad (17)$$

to which (8) reduces on the acceptance of (16). The notation W(D u) means obviously that the arguments of the strain energy density must be calculated from the displacement field through equations (3). This result was presented earlier [6] in the form of a "limitation principle" in the use of two-field functionals.

The elements constructed in this fashion may be referred to as "kinematical and strongly conforming".

Strong diffusitivity would occur by virtue of the equilibrium properties (11) postulated from the connectors, if the internal stress field were related to the connectors by (5). This equation, however, holds only for unconstrained displacement fields at the boundary. Because of the discretization (15), the equation is replaced by its weak form

$$\int_{\partial E} Q_{mj}(x)\, \{ n_i\, \tau_{ij} - \bar{t}_j \}\, dS = 0 \qquad (18)$$

As the stress field will ultimately be known through (16), (18) shows that numerical values will be received for the weak connectors only

$$\bar{g}_m = \int_{\partial E} Q_{mj}(x)\, \bar{t}_j\, dS \qquad (19).$$

No local (strong) information will ever be obtained for the $\bar{t}_j$ .

The weak connectors are the generalized boundary loads conjugate, in the sense of virtual work, to the generalized boundary displacements $q_m$. From these considerations it appears that kinematical, strongly conforming models of finite elements are only weakly diffusive.

Take now the converse case in which use is made of unconstrained $\bar{u}_j$ connectors for all boundaries; the relevant variational principle being now given by (8').

It was seen that this poses a requirement of strong diffusitivity (10).

To implement this, the stress field must be discretized in such a way that it has a representation of the form

$$\tau_{ij} = g_m T_{mij}(x) + h_n B_{nij}(x) \qquad (20)$$

The $h_n$ are bubble coordinates, the stress-bubble functions being by definition such that they generate no surface tractions on the boundaries :

$$n_i B_{nij}(x) = 0 \qquad \text{for} \qquad x \in \partial E \quad .$$

The surface tractions are completely described by the generalized boundary loads $g_m$

$$t_j = g_m T_{mj}(x) \qquad \text{with} \qquad T_{mj}(x) = n_i T_{mij}(x) \qquad (21).$$

For any partial boundary that is the face of an interface, they depend on a subset of the $g_m$ and diffusitivity follows from the conditions

$$\left.\begin{array}{l} g_{m+} + g_{m-} = 0 \\ T_{mj+}(x) = T_{mj-}(x) \qquad \text{for} \qquad x \in \partial_\alpha E \end{array}\right. \qquad m \in m_\alpha$$

Looking now at the behavior of such models from the viewpoint of conformity we meet with a completely different situation. It is clear from the functional (8') that, by contrast with the former case, it is the displacement field whose choice is now confined to the interior domain. But if we accept an unconstrained choice we do this time impose restrictions on the choice of the stress field. It must satisfy the volume equilibrium equations (4). Assuming that it remains possible to organize strong diffusitivity under such restrictions we obtain an element that can be referred to as "statically admissible and strongly diffusive".
Because $u_j$ disappears completely from the functional in case (4) is satisfied, the relevant variational principle becomes in fact the so-called complementary energy principle, and there is no proper relationship between the connectors and an internal field. There is however a weak information available on displacements.
On the one hand weak displacement connectors

$$\bar{q}_m = \int_{\partial E} T_{mj}(x) \, \bar{u}_j \, dS$$

can be defined, conjugate in the virtual work sense to the $g_m$, conforming ($\bar{q}_{m+} = \bar{q}_{m-}$) by virtue of the property postulated from the $\bar{u}_j$ , and for which numerical values will be received. As a matter of fact they can be used as nodal displacement unknowns in the same direct stiffness method of approach as for kinematical models [6]. On the other hand numerical values can also be obtained for the weak generalized displacements conjugate to a given body load. For those reasons the statically admissible and strongly diffusive models can be considered as weakly conforming. The strain energy bounds that can be obtained from the two types of models described in this section are well known [6,10]. Used in alternate discretizations of the same problem, the dual analysis principle [17], they provide useful assessments of the discretization errors by a quantitative estimate of the energy convergence.

## THE HYBRID MODELS

Cases are known for which discretization by complete polynomials leads to difficulties in organizing strong conformity (e.g. Kirchhoff plate bending) or strong diffusitivity with equilibrated stress fields (e.g. plate stretching). One way to overcome those difficulties has been in the assembling of superelements [2,7,8,9,17]. Another has been the use of hyperconforming elements available for high degree polynomials [2]. Not only are such elements expensive in number of degrees of freedom, but the fact that they make use of unnecessary continuity requirements at their interfaces restricts the application of a standard connexion software whenever the elements are not coplanar, the interface is a discontinuity surface of material properties, or subject to an interface load.
If one is prepared to accept the loss of the bounding properties of the pure elements, hybrid models may provide less expensive solutions.

Consider again the case of $\bar{t}_j$ connectors on all boundaries together with the maximizing choice (16) of the internal stress field, so that the relevant variational principle is given by (17). This time however the $\bar{t}_j$ are constrained by a discrete representation

$$\bar{t}_j = \bar{g}_m \, T_{mj}(x) \qquad x \in \partial E \qquad (22).$$

This procedure follows in fact the pattern (20) and (21) except that the surface tractions are only defined on the boundary. There is no difficulty therefore in taking $T_{mj+}(x) = T_{mj-}(x)$ on the interfaces and strong diffusitivity of the surface tractions will follow from $g_{m+} + g_{m-} = 0$.
As the variations on the connectors are now constrained, strong conformity of the internal displacement fields will be replaced by the weak conformity

$$\int_F T_{mj}(x)\ (u_{j+} - u_{j-})\ dS = 0 \qquad (23)$$

Accordingly, the internal displacement fields need no more be discretized in the form (15) but their unknown coefficients are determined, partly, by the boundary displacements $q_m$ conjugate to the $\bar{g}_m$ of (22)

$$q_m = \int_{\partial E} u_j(x)\ T_{mj}(x)\ dS \qquad (24).$$

Coefficients not determined by (24) are considered as internal degrees of freedom eliminated by direct minimization of the energy at the element level. Such elements may be termed "kinematical and weekly conforming".

Another category of hybrid elements, in fact the first introduced into the litterature by T.H.H. PIAN, is that of "statically admissible and weakly diffusive" elements. By a procedure completely analogous to the preceeding one, the complementary energy principle is used with discretized $\bar{u}_j$ connectors :

$$\bar{u}_j = \bar{q}_m\ Q_{mj}(x) \qquad x \in \partial E \qquad (25)$$

Because the shaping functions are only defined on the boundary, there is no difficulty in taking $Q_{mj+}(x) = Q_{mj-}(x)$ on the interfaces and strong conformity of the boundary displacements follows from $\bar{q}_{m+} = \bar{q}_{m-}$.
As the variations on the connectors are now constrained, strong diffusitivity is traded for the weak diffusitivity

$$\int_F Q_{mj}(x)\ (t_{j+} + t_{j-})\ dS = 0 \qquad (26)$$

The internal equilibrated stress field is determined partly by the weak boundary loads conjugate to $\bar{q}_m$

$$g_m = \int_{\partial E} Q_{mj}(x)\ n_i\ \tau_{ij}\ dS \qquad (27).$$

## MIXED MODELS

Mixed elements arise most naturally in the presence of mixed connectors. To give an example from plate bending theory, the connectors may be split into a displacement type $\frac{\partial}{\partial n}\bar{w}$ for the normal slope of the plate deflexion and $\bar{K}_n$ the Kirchhoff shear load together with the corner loads $\bar{Z}$.
The first, if undiscretized will require the existence of an internal bending moment field with strong diffusitivity. The second the existence of an internal transverse displacement field w with strong conformity. The difficulties associated with $C_1$ continuity, that is the strong conformity of both w and $\frac{\partial}{\partial n}w$ will be avoided. The relevant variational principle will then be the proper combination of (8) and (8'). Most of the mixed elements presented in the litterature are of this, or a similar, nature [4,18,19].

Such mixed models share with the kinematical models the advantage of dealing simply with the problem of arbitrary body loadings. Those are automatically converted by virtual work into weak conjugates to the displacement degrees of freedom. The plate bending case is a good showcase because of the many different transverse pressure distributions that must be cared for in practice. By contrast the statically admissible elements require a search for special solutions to the non homogeneous equilibrium equations that are not always easy to find.

## DIRECT STIFFNESS AND DIRECT FLEXIBILITY

It appears that there are finally two methods for assembling any type of finite element model. The best known is that using the nodal displacement identification procedure

$$q_{m+} = q_{m-} = \text{a nodal displacement} \tag{28}$$

It was used from the beginning for the kinematical and strongly conforming models, where the $q_m$ are strong, and later for the statically admissible and strongly diffusive models [6], where the $q_m$ are weak. It applies as well to the hybrid and mixed models. It produces a direct method for assembling the elemental stiffness matrices and is known therefore as the direct stiffness method.
Each $q_m$ having its conjugate $g_m$, weak when $q_m$ is strong and vice-versa, another assembling method would consist in using the relation

$$g_{m+} + g_{m-} = 0 \qquad (29)$$

There are two drawbacks to this. One is lack of symmetry, the other is that (29) is valid only for externally unloaded interfaces. To this second drawback one may object that (28) in turn is only valid for undislocated interfaces, although the possibility of loading an interface is more useful in practice than the prescription of a dislocation. The drawback of symmetry can be overcome by the use of stress functions. It is easily shown that $C_1$ continuity of the Airy function and $C_o$ continuity of the Southwell stress functions are sufficient conditions of strong diffusitivity in the respective cases of plate stretching and plate bending [10]. This allows to assemble models with strongly diffusive connectors by the rule

$$c_{m+} = c_{m-} = \text{nodal stress function value} \qquad (30)$$

and corresponds to a direct method for assembling elemental flexibility matrices, producing a particularly simple version of the Force method known as the direct flexibility method. P. BECKERS [11] was the first to point out that this method applies to all models with suitable definitions of weak stress function values and established the relations between the $g_m$ and the stress functions.
There are limitations to this method : the presence of body loads, interface loading and multiple connectivity of the assembled structure require dislocations in the nodal stress function values.

## INTERFACE VIRTUAL WORK

Consider two finite elements with interface F. Wathever be the particular variational principle and associated type of element used, the terms containing the connectors are all of type

$$T_j = \int_{F_+} (u_j\ t_j)_+\ dS + \int_{F_-} (u_j\ t_j)_-\ dS \qquad \text{(j not summed)}$$

where the bar indicating the connector is either on $u_j$ or on $t_j$.
If $t_j$ is a discretized connector :

$$\bar{t}_{j+} = g_m\ T_{mj}(x)\ ,\ \bar{t}_{j-} = -\ g_m\ T_{mj}(x)\ ,\ x \in F\ ,\ m \in m_F\ ,$$

$$T_j = g_m \int_F T_{mj}(x) \{ u_{j+}(x) - u_{j-}(x) \} dS = 0$$

because each integral vanishes as a result of variations on the independent $g_m$. Similarly, if $u_j$ is a discretized connector :

$$\bar{u}_{j+} = \bar{u}_{j-} = q_m U_{mj}(x) , x \in F , m \in m_F ,$$

$$T_j = q_m \int_F U_{mj}(x) \{ t_{j+}(x) + t_{j-}(x) \} dS = 0$$

because each integral vanishes as a result of variations on the independent $q_m$. The result $T_j = 0$ holds a fortiori when the connectors are undiscretized because of the strong diffusitivity or conformity which are then required. The vanishing of each of its terms causes the total interface virtual work to vanish :

$$IVW = \int_{F_+} (t_j u_j)_+ dS + \int_{F_-} (t_j u_j)_- dS = 0 \qquad \text{(j summed)} \quad (31)$$

As we now consider the two assembled elements to constitute a substructure and add a new element to it, the virtual work at the new interface will vanish by the same arguments. Thus, using a frontal assembling method, it turns out that, wathever be the type of elements, the virtual work vanishes at each interface of the assembled structure.

This result, that satisfies engineering intuition, is a key ingredient in the verification of the patch test and is in fact equivalent to the statement that there is no strain energy gained or lost at the interfaces, a variational viewpoint of the patch test adopted by STRANG [3] .

## THE ZERO INTERFACE VIRTUAL WORK CONDITION FOR PLATE BENDING ELEMENTS

Conjunction of the zero interface virtual work condition and the dual connector systems (displacements or stress functions) allows a systematic construction of elements that will pass the patch test and possess the associated convergence characteristics.
We take an example where the difficulties in obtaining strong conformity are noteworthy : the Kirchhoff plate bending problem.
For one face of an interface (here an arc between two angular points 1 and 2) the virtual work is

$$\int_1^2 \left(w\, Q_n - \frac{\partial w}{\partial n} M_n - \frac{\partial w}{\partial s} M_{nt}\right) ds \qquad (32)$$

where w is the transverse plate deflection representing the displacement field, $\partial w/\partial n$ the slope in the direction of the local outward normal ($\vec{n}$ makes an angle $\theta$ with a fixed axis Ox), $\partial w/\partial s$ the slope in the direction of the local tangent, oriented from 1 to 2 . $Q_n$ the shear load, $M_n$ the normal bending moment, $M_{nt}$ the twisting moment are representative of the surface tractions. We represent them [10] by means of the Southwell vector stress function (U,V).
If there is an internal self-equilibrated stress field, it is given by

$$M_{xx} = \frac{\partial V}{\partial y} \qquad M_{xy} = -\frac{1}{2}\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) \qquad M_{yy} = \frac{\partial U}{\partial x} \qquad (33)$$

as bending moments field,

$$Q_x = \frac{\partial \Omega}{\partial y} \qquad Q_y = -\frac{\partial \Omega}{\partial x} \qquad \Omega = \frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) \qquad (34)$$

as shear loads field. On the boundary, where the vector is locally resolved in its normal component $U_n$ and tangential component $U_t$ , there results

$$M_n = \frac{\partial U_t}{\partial s} + \dot{\theta}\, U_n \qquad M_{nt} = -\Omega - \frac{\partial U_n}{\partial s} + \dot{\theta}\, U_t \qquad (35)$$

$$\Omega = \frac{1}{2}\left(\frac{\partial U_t}{\partial n} - \frac{\partial U_n}{\partial s} + \dot{\theta}\, U_t\right) \qquad Q_n = \frac{\partial \Omega}{\partial s} \qquad (36)$$

and the Kirchhoff shear load

$$K_n = Q_n + \frac{\partial M_{nt}}{\partial s} = 2\,\frac{\partial M_{nt}}{\partial s} + \frac{\partial M_n}{\partial n} + \dot{\theta}\,(M_n - M_t) = -\frac{\partial}{\partial s}\left(\frac{\partial U_n}{\partial s} - \dot{\theta}\, U_t\right)$$

(37)

In those formulas $\dot{\theta} = d\theta/ds$ measures the curvature of the boundary. Equations (35, 36 and 37) continne to hold if the surface tractions are only defined on the boundary. Formula (32) takes now one of the following forms, derived from each other by integration by parts :

$$\int_1^2 \left\{ w \frac{\partial \Omega}{\partial s} - \frac{\partial w}{\partial n}\left(\frac{\partial U_t}{\partial s} + \dot{\theta}\, U_n\right) + \frac{\partial w}{\partial s}\left(\Omega + \frac{\partial U_n}{\partial s} - \dot{\theta}\, U_t\right) \right\} ds \qquad (38)$$

$$= w\,\Omega \Big|_1^2 + \int_1^2 \{ -\frac{\partial w}{\partial n} (\frac{\partial U_t}{\partial s} + \dot{\theta}\, U_n) + \frac{\partial w}{\partial s} (\frac{\partial U_n}{\partial s} - \dot{\theta}\, U_t) \}\, ds \qquad (39)$$

$$= w\,(\Omega + \frac{\partial U_n}{\partial s} - \dot{\theta}\, U_t) \Big|_1^2 + \int_1^2 \{ -\frac{\partial w}{\partial n} (\frac{\partial U_t}{\partial s} + \dot{\theta}\, U_n) - w \frac{\partial}{\partial s} (\frac{\partial U_n}{\partial s} - \dot{\theta}\, U_t) \}\, ds \qquad (40)$$

$$= w\,\Omega + U_n \frac{\partial w}{\partial s} - U_t \frac{\partial w}{\partial n} \Big|_1^2 + \int_1^2 \{ U_t (\frac{\partial^2 w}{\partial s \partial n} - \dot{\theta} \frac{\partial w}{\partial s}) - U_n (\frac{\partial^2 w}{\partial s^2} + \dot{\theta} \frac{\partial w}{\partial n}) \}\, ds \qquad (41)$$

Form (40) would be the one normally associated with the treatment of boundary or transition conditions from the viewpoint of variation of displacements. Under the integral sign, $w$ and $\partial w/\partial n$ are respectively conjugate to the Kirchhoff shear load $K_n$ and the normal bending moment $M_n$ to be either equated to prescribed boundary values or transmitted to adjacent elements, while the first term is a reminder that the distribution of the twisting moment $M_{nt}$ is equivalent to its contribution $\partial M_{nt}/\partial s$ to $K_n$ plus the end values it takes in 1 and 2.

Form (41) is the one normally associated to the complementary energy principle. Under the integral sign the conjugates to $U_t$ and $U_n$ are boundary deformations, twist for $U_t$ and transverse curvature for $U_n$ ; they are either equated to prescribed values or transmitted to adjacent elements. If U and V are known along all boundaries, $\Omega$ is known at the angular points and the first term in (41) will be the one ensuring single valuedness at angular points of w and its first derivatives.

The condition of zero interface virtual work will now be written by adding the contributions of $F_+$ and $F_-$ . If, in doeing so, we adopt a common orientation for the normal and the tangent, say that on $F_+$ , we must simply make the difference between the formulas (38), (39), (40) or (41) for $F_+$ and $F_-$ . This difference will be denoted by $\Delta$ and, using (39), the condition is, assuming a straight boundary $\dot{\theta} = 0$ ,

$$\Delta \Big[ w\,\Omega \Big|_1^2 + \int_1^2 \{ -\frac{\partial w}{\partial n}\frac{\partial U_t}{\partial s} + \frac{\partial w}{\partial s}\frac{\partial U_n}{\partial s} \}\, ds \Big] = 0 \qquad (42)$$

Denote by 2a the length of the interface and introduce the reduced variables

$$\sigma = \frac{s}{a} \qquad \nu = \frac{n}{a} \qquad \Omega^* = a\,\Omega \qquad (43)$$

so that (42) becomes

$$\Delta \left[ w\, \Omega^{*} \Big|_{-1}^{+1} + \int_{-1}^{+1} \{ - \frac{\partial w}{\partial \nu} \frac{\partial U_t}{\partial \sigma} + \frac{\partial w}{\partial \sigma} \frac{\partial U_n}{\partial \sigma} \} \, d\sigma \right] = 0 \qquad (44)$$

Assuming a discretization by polynomials, we now introduce the expansions in Legendre polynomials

$$P_m(\sigma) = \frac{1}{2^m\, m!} \frac{d^m}{d\sigma^2} (\sigma^2 - 1)^m \qquad (45)$$

The first Legendre polynomials are

$$P_o(\sigma) = 1 \qquad P_2(\sigma) = \frac{3\sigma^2 - 1}{2} \qquad P_4(\sigma) = \frac{35\sigma^4 - 30\sigma^2 + 3}{8}$$

$$P_1(\sigma) = \sigma \qquad P_3(\sigma) = \frac{5\sigma^3 - 3\sigma}{2} \qquad (46)$$

Take

$$\frac{\partial w}{\partial \sigma} = \sum_o p_m P_m(\sigma) \qquad \frac{\partial w}{\partial \nu} = \sum_o q_m P_m(\sigma)$$

$$\frac{\partial U_n}{\partial \sigma} = \sum_o u_m P_m(\sigma) \qquad \frac{\partial U_t}{\partial \sigma} = \sum_o v_m P_m(\sigma) \qquad (47)$$

As Legendre polynomials satisfy the differential equation

$$\frac{d}{d\sigma} \{ (\sigma^2 - 1) \frac{d P_m}{d\sigma} \} = m(m+1)\, P_m \qquad (48)$$

we can adopt as first integrals

$$Q_{m+1}(\sigma) = \int P_m(\sigma)\, d\sigma$$

those, which for $m > 0$, vanish at the ends of the interval

$$Q_{m+1}(\sigma) = \frac{\sigma^2 - 1}{m(m+1)} \frac{dP_m}{d\sigma} \qquad m > 0 \qquad (49),$$

while for $m = 1$ we make the particular choice

$$Q_1(\sigma) = \sigma \qquad (50).$$

It is useful to recall that the Legendre polynomials themselves take the following end values :

$$P_m(\underline{+}1) = 1 \quad \text{for } m \text{ even}$$
$$= \underline{+} 1 \quad \text{for } m \text{ odd} \qquad (51)$$

Integration of the expansions furnishes consequently

$$w = p_{-1} + p_o\,\sigma + \sum_1 p_m\, Q_{m+1}(\sigma)$$
$$U_n = u_{-1} + u_o\,\sigma + \sum_1 u_m\, Q_{m+1}(\sigma) \qquad (52)$$
$$U_t = v_{-1} + v_o\,\sigma + \sum_1 v_m\, Q_{m+1}(\sigma)$$

Taking into account the orthogonality property

$$\int_{-1}^{1} P_m(\sigma)\; P_n(\sigma)\; d\sigma = \frac{2}{2m+1}\,\delta_{mn} \qquad (53)$$

the zero interface virtual work conditions (44) takes the form

$$\Delta\,\Big[\, p_{-1}(\Omega^*(1)-\Omega^*(-1)) + p_o(\Omega^*(1) + \Omega^*(-1) + 2u_o) - 2q_o\, v_o$$
$$+ \sum_1 \frac{2}{2m+1}\,(p_m\, u_m - q_m\, v_m)\,\Big] = 0 \qquad (54)$$

It becomes simpler yet if we recognize in the conjugates to $p_{-1}$, $p_o$ and $q_o$ reduction elements of the resultant of the total load transmitted across the interface. Computed on $F_+$, the total bending moment is

$$B = \int_{-a}^{a} M_n\, ds = \int_{-1}^{+1} \frac{\partial U_t}{\partial\sigma}\, d\sigma = U_t\,\Big|_{-1}^{+1} = 2\, v_o \qquad (55);$$

the total vertical load

$$V = -\,M_{nt}\,\Big|_{-a}^{a} + \int_{-a}^{a} K_n\, ds = -\,M_{nt}\,\Big|_{-1}^{1} - \int_{-1}^{1} \frac{\partial}{\partial\sigma}\Big(\frac{\partial U_n}{\partial s}\Big)\, d\sigma = -\,\Big(M_{nt} + \frac{\partial U_n}{\partial s}\Big)\,\Big|_{-1}^{1}$$
$$= \Omega\,\Big|_{-1}^{1} = \frac{1}{a}\,\{\,\Omega^*(1) - \Omega^*(-1)\,\} \qquad (56);$$

the total twisting moment about the mid-point of the interface

$$T = - s M_{nt} \Big|_{-a}^{a} + \int_{-a}^{a} s K_n \, ds = - s M_{nt} \Big|_{-a}^{a} - \int_{-a}^{a} s \frac{\partial^2 U_n}{\partial s^2} ds$$

$$= - s \left(M_{nt} + \frac{\partial U_n}{\partial s}\right) \Big|_{-a}^{a} + \int_{-a}^{a} \frac{\partial U_n}{\partial s} ds = s\Omega + U_n \Big|_{-a}^{a}$$

$$= \Omega^*(1) + \Omega^*(-1) + 2u_o \qquad (57).$$

Computed on $F_-$ with the orientation of normal and tangent prevailing on $F_+$, these formulas undergo a change of sign.
Finally (54) becomes

$$\Delta \left[ p_{-1} aV + p_o T - q_o B + \sum_1 \frac{2}{2m+1} (p_m u_m - q_m v_m) \right] = 0 \qquad (58)$$

and we can discuss its application to several types of plate bending models.
We note that (58) is a sum of products of generalized conjugate displacements and surface tractions. Denoting by q c the generic term, the conformity condition is

$$\Delta q = 0 \qquad \text{or} \qquad q_+ = q_- = q$$

and the diffusitivity condition

$$\Delta c = 0 \qquad \text{or} \qquad c_+ = c_- = c$$

showing, by comparison with (29), the restoration of symmetry accompanying the use of the stress functions. In any type of model one of the factors in the product will be a connector. If q is the connector, $\Delta q = 0$ a priori and

$$\Delta(qc) = q\Delta c$$

Then, provided q is an independent discretization parameter of the element, its variation will result in $\Delta c = 0$. The converse implication $\Delta c = 0 \rightarrow \Delta q = 0$ will follow is c is an independent discretization parameter of the element. However, since any discretization is limited as regards the degree of the polynomials involved, the infinite sum in (58) is always truncated so that either conformity or diffusivity, or even both, are weak.

They are in fact the simplest models. In the case of triangular shapes complete polynomials of degree n are used for the vector stress function (U,V) defining the self-equilibrating internal stress field. Strong diffusivity follows from $C_o$ continuity of U and V at the interfaces and this requirement is known, by analogy with the kinematical models of plate stretching, to be easily enforced by a suitable selection of local values on the boundary. Bubble stress modes appear for $n \geqslant 3$ (their exact number is $(n-1)(n-2)$) which, being unconnected, are to be eliminated by energy minimization of the isolated element. The local stress function values on the boundary are independent and their variations produce the weak conformity properties

$$\Delta p_{-1} = 0 \quad \text{and} \quad \Delta p_m = 0 \quad , \quad \Delta q_m = 0 \quad \text{for} \quad 0 \leqslant m \leqslant n-1 \qquad (59)$$

where, by virtue of the expansions,

$$p_m = \frac{2m+1}{2} \int_{-1}^{1} \frac{\partial w}{\partial \sigma} P_m \, d\sigma = \frac{2m+1}{2} \left[ w \, P_m \Big|_{-1}^{1} - \int_{-1}^{1} w \frac{dP_m}{d\sigma} d\sigma \right] \qquad (60)$$

$$q_m = \frac{2m+1}{2} \int_{-1}^{1} \frac{\partial w}{\partial \nu} P_m \, d\sigma \qquad (61)$$

A remark is needed here concerning the vanishing a piori of $\Delta\Omega$ at both ends of the interface, which is necessary for the results $\Delta p_{-1} = 0$ and $\Delta p_o = 0$. The $C_o$ continuity of U and V along the interface is not sufficient to this effect; from (35) we obtain at an end point of the interface

$$(M_{nt})_+ = -\Omega_+ - \left(\frac{\partial U_n}{\partial s}\right)_+$$

and, keeping on $F_-$ the same orientation of normal and tangent

$$(M_{nt})_- = +\Omega_- + \left(\frac{\partial U_n}{\partial s}\right)_- \quad .$$

Whence the reciprocity condition

$$(M_{nt})_+ + (M_{nt})_- = -\Delta\Omega - \Delta \frac{\partial U_n}{\partial s} = 0$$

And the result follows since the second term vanishes already by the continuity of $U_n$.

The singlevaluedness of $\Omega$ at a corner is moreover necessary in order that, at the assembled level, the jump in $M_{nt}$ remains equal to the externaly applied concentrated transverse load.
The lowest value of the polynomial degree in the stress functions is $n = 1$ and Fig. 1a shows the choice of local values ensuring $C_o$ continuity. The weak conformity properties are reduced to $\Delta p_{-1} = 0$ , $\Delta p_o = 0$ , $\Delta q_o = 0$.
From (52) and the fact that $Q_{m+1}(\pm 1) = 0$ for $m \geqslant 1$, it follows that $\Delta p_{-1} = 0$ and $\Delta p_o = 0$ require the transverse deflections to be transmitted at the corner points. From (61) with $m = 0$, it appears that $\Delta q_o = 0$ requires transmission of the average normal slope. This is represented symbolically on Fig. 1b.
It will be noticed that the connexion is just sufficient to transmit the resultant of the boundary loads across. In fact the element is somewhat deficient because $K_n \equiv 0$ and no additional solution can be found to account even for a uniform transverse pressure distribution without altering the existing boundary loading modes. The external load can however be replaced in a sense of best statical equivalence by concentrated loads at the nodal points of the mesh.

The case $n = 2$ is better known and operational [20,21].
Fig. 2a. shows the choice of local stress functions value leading to $C_o$ continuity and use of a direct flexibility solution at the assembled level.
We now have the additional weak conformity properties
$\Delta p_1 = 0$ and $\Delta q_1 = 0$ expressing the continuity of

$$p_1 = \frac{3}{2}\left(w(1) + w(-1) - \int_{-1}^{1} w \, d\sigma\right)$$

$$q_1 = \frac{3}{2}\int_{-1}^{1} \sigma \frac{\partial w}{\partial \nu} \, d\sigma$$

Considering that the corner deflections are already continuous, the first implies simply the additional continuity of the average interface deflection. The second can be used as such or be combined with the $\Delta q_o = 0$ condition in the equivalent continuity of the pair [21]

$$\int_{-1}^{1} (1+\sigma) \frac{\partial w}{\partial \nu} \, d\sigma \qquad\qquad \int_{-1}^{1} (1-\sigma) \frac{\partial w}{\partial \nu} \, d\sigma$$

Fig. 2b. shows symbolically this weak conformity property, by which the elements of this type can be assembled by the direct stifness method.
Here an additional solution can be found that does not disturb the boundary loading modes and equilibrates a uniform transverse pressure [21].

This special solution is worked out in local coordinates and involves a second degree bending moments field. As pointed out by L.S.D. MORLEY in a private communication, it would be logical to remove the dependency of the particular solution on the choice of local coordinates by adding the two cubic bubble stress modes in U and V (that also generate a second degree bending moment field), their contribution being settled by energy minimization.

Obviously there are no difficulties in raising arbitrarily the value of n for this class of elements and the weak connexion properties required for application of the standard direct stiffness procedure follow automatically from working out (60) and (61) up to $m = n-1$.

Figs 3a and b show the case $n = 3$.

## KINEMATICAL, WEAKLY CONFORMING PLATE BENDING MODELS

With kinematical models we enter the area of difficulties in obtaining strong conformity. The converse of the procedure followed in the preceeding section fails, essentially because the boundary displacements required for strong conformity cannot be made independent. Their number exceeds the number of independent coefficients available in a complete polynomial of specified degree.

By discretizing the stress functions introduced as connectors at the boundary we construct hybrids of the kinematical, weakly conforming variety.

Take first for w in a triangle a complete polynomial of degree $n = 2$ and linear variations for U and V along the boundary. The summation in (58) is then truncated to $m < 2$ for both reasoms, but all conjugates are present up to $m = 1$. Fig. 4a shows the same organization of $C_o$ continuity of the stress functions as in Fig. 1a but they need not (although they can in this case) be defined in the interior domain. The weak conformity properties are the same as before and may in fact be used as illustrated on Fig. 1b. However, and this is the origin of the concept of non-conforming [2,3] or delinquent [1] elements, the change in context allows a reinterpretation of the weak displacement connectors in terms of local values. Indeed, while in the former case the displacement coefficients $p_m$ and $q_m$ were simply unknown for $m > 1$, they are now a priori set equal to zero because of the existence of the internal displacement field and its limitation to a second degree polynomial. Thus the average normal slope can here be identified with its constant value along the interface, in particular the mid-point value (Fig. 4b). The element becomes in fact one proposed by L.S.D. MORLEY [5], already recognized as passing the patch test [2], and exhibiting convergence in practice.

The case where w is a complete cubic produces new elements.
The truncated expansions of w and its normal slope are

$$w = p_{-1} + p_o\sigma + p_1 \frac{\sigma^2-1}{2} + p_2 \frac{\sigma(\sigma^2-1)}{2} \quad (62)$$

$$\frac{\partial w}{\partial \nu} = q_o + q_1\sigma + q_2 \frac{3\sigma^2-1}{2} \quad (63)$$

There are two choices for the stress functions on the boundary that allow the 10 coefficients of the displacement field to be expressed in terms of generalized boundary displacements without violating the zero interface virtual work condition.
In the first, the tangential component of the stress function is only allowed linear variations but the normal component has quadratic variations.
Hence in (58) $v_1 = 0$ but not $u_1$. The resultant weak conformity requirements are :

$\Delta p_{-1} = 0$ $\qquad \Delta p_o = 0$ $\qquad$ unplying as we know the continuity of w at corner points,

$\Delta q_o = 0$ $\qquad$ implying continuity of $\int_{-1}^{1} \frac{\partial w}{\partial \nu} d\sigma$

$\Delta p_1 = 0$ $\qquad$ implying continuity of $\int_{-1}^{1} w \, d\sigma$

Inspection of (62) reveals that, since the $p_2$ term vanishes for $\sigma = 0$, the last condition can be replaced by the continuity of w at mid-point.
On the other hand (63) does not reveal any local slope that would be related to $q_o$ only. Hence it is not always possible (nor is it necessary) to obtain a weak connexion interpretation in terms of local displacements only. Figs. 5 a and b illustrate the two connexion schemes. As we have only nine boundary displacements we must add a pseudo-bubble shaping function (by this we understand a shaping function in which the nine displacements are set equal to zero) generated, for example, by the transverse displacement of the barycenter. This unconnected degree of freedom is to be eliminated by minimization of the energy of the element.
It remains to be verified (e.g. by using areal coordinates) that the transformation between the 10 coefficients of the cubic w field and our 10 degrees of freedom is non singular.
In the second choice (Figs. 6 a and b) the normal component of the vector stress function is restricted to linear variations and the tangential one to quadratic,

whereby $u_1 = 0$ but not $v_1$.
The continuity requirement for $p_1$ is traded for the continuity of $q_1$ or $\int_{-1}^{1} \sigma \frac{\partial w}{\partial \nu} d\sigma$ .

Inspection of (63) reveals that, since the $q_2$ term vanishes at the Gauss points $\sigma = \pm \frac{1}{\sqrt{3}}$ , continuity of the normal slopes at those points is equivalent to both

$$\Delta q_o = 0 \quad \text{and} \quad \Delta q_1 = 0.$$

Again a pseudo bubble degree of freedom, connected with the barycenter, must be energy eliminated.

Suppose now that each of the connected elements is in a state of uniform strain, so that $p_2 = 0$ and $q_2 = 0$. Then for the first of our new cubic displacement models, $p_{-1}$ , $p_o$, $p_1$ and $q_o$ are continuous by virtue of the connexions and there will be exact conformity provided $q_1$ is also continuous. This will be the case if the two elements have the same state of uniform strain, because the twist $q_1 = \frac{\partial^2 w}{\partial \nu \partial \sigma}$ is one of the strain measures.
Similarly for the second model where $p_{-1}$ , $p_o$ , $q_o$ and $q_1$ are already continuous by virtue of the connexions, exact conformity will follow from the condition of common uniform strain because $p_1 = \frac{\partial^2 w}{\partial \sigma^2}$ will be a common curvature.
This illustrates the verification of the patch test as it was originally conceived by IRONS [1] . Non conforming elements pass the patch test when they become exactly conforming under a common state of uniform strain.

## CONCLUSIONS

The patch test is most easily verified by expressing the zero interface virtual work condition with the help of stress functions and is in fact identical to the construction of hybrid models. Clearly as the degree of polynomial approximation is raised, elements can be constructed that pass a higher order patch test, e.g. they become exactly conforming under a state of uniform strain variation and this property must be related to a higher rate of convergence.

Examples were only given for Kirchhoff plate bending and triangular elements, because this problem has received much attention in the past. The extension to rectangular elements is straightforward. Moreover the method followed is directly applicable to the plate stretching case, using the Airy stress function. The hybrid models of PIAN type, or non-diffusive elements, constructed in this way may prove to be useful substitutes for the superelements required to enforce strong diffusivity.

The three-dimensional case is decidedly more difficult because of the complexity and lack of uniqueness of the stress function tensor.

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