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QUANTITATIVE CONVERGENCE ANALYSIS OF MULTI-AGENT SYSTEMS

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Abstract: We introduce a characterization of contraction for bounded convex sets. For discrete-time multi-agent systems we provide an explicit upperbound on the rate of convergence to a consensus under the assumptions of contractiveness and (weak) connectedness (across an interval.) Convergence is shown to be exponential when either the system or the function characterizing the contraction is linear.

Keywords: multi-agent systems, consensus, convexity

1. INTRODUCTION

A multi-agent system is a collection of subsystems where each subsystem (called an agent) updates itself in accordance with the information it gathers from some of the other agents, i.e. from its *neighbors*. In general, the neighbors of an agent are subject to change in time which introduces a *switching* behaviour to the dynamics of the system through communication links. It has proved itself important to understand the effect of this varying communication topology on some common task to be accomplished (i.e. reaching a *consensus*) by the agents composing the system. Among related applications are formation control, synchronization of coupled oscillators, and distributed sensor fusion in sensor networks; see, for instance, (Fax and Murray, 2004), (Sepulchre *et al.*, 2004), (Xiao *et al.*, 2005), respectively. We refer the reader to the recent survey (Olfati-Saber *et al.*, 2007) for details and a myriad of references.

Moreau shows in (Moreau, 2005) that states of all agents eventually reach a consensus by converging to a common point if the following assumptions hold:

(a) the state of each agent at the next time step is in the (relative) interior of the convex hull of the set comprising the current state of that agent and its neighbors, and

(b) the graph describing the communication topology is connected uniformly over an interval.

His work is generalized in (Angeli and Bliman, 2006) in which assumption (a) is weakened. In our work, we study the quantitative aspects of convergence to a consensus. For our purpose, we keep assumption (b) and replace (a) with a contractiveness condition (see Definition 2) on the system. In that setting, we provide an upperbound on the decay of the diameter of the set comprising the states of the agents via a class-$\mathcal{KL}$ function which we explicitly express in terms of the number of agents, the length of the interval over which the communication graph is connected, and some class-$\mathcal{K}$ function ω characterizing contraction. We show that whenever ω is linear, the convergence is exponential. We also remark that for multi-agent systems that can be expressed as a switched linear system, ω can be explicitly computed in terms of the system parameters.

Clearly, an important question is *how restrictive is contractiveness condition?* We provide the answer by displaying the equivalence of contractiveness to assumption (a). The contribution of this paper is therefore the following: (i) we introduce the concept of contractiveness which we show is equivalent to assumption (a); and (ii) we provide algorithms to compute bounds on the rate of convergence to a consensus in terms of system parameters and a class-$\mathcal{K}$ function characterizing contractiveness.

Convexity plays a central role not only in our key definition but also in our proof techniques. We list (Boyd and Vanderberghe, 2004) and (Rockafellar and Wets, 1998) as two fine references on the subject.

The rest of the paper is organized as follows. Notation and definitions reside in Section 2. In Section 3 we give the system description. The equivalence of contractiveness and assumption (a) is shown in Section 4. Quantitative convergence analysis is presented in Section 5. In Section 6 we study a linear example. Finally, we conclude.

2. NOTATION AND DEFINITIONS

Nonnegative integers are denoted by $\mathbb{N}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to *class-*$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is zero at zero, continuous, and strictly increasing. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to *class-*$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each fixed t, $\beta(\cdot, t)$ is zero at zero, nondecreasing, and $\lim_{s \to 0^+} \beta(s, t) = 0$; and for each fixed s, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$. Function id is such that $\mathrm{id}(s) = s$ for all $s \geq 0$. For $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $k \in \mathbb{N}$, $\gamma^{k+1}(\cdot) = \gamma(\gamma^k(\cdot))$ where $\gamma^0 = \mathrm{id}$. Given two functions $\alpha_1, \alpha_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we write $\alpha_1 \leq \alpha_2$ to imply $\alpha_1(s) \leq \alpha_2(s)$ for all $s \geq 0$. (Meaning of $\alpha_1 < \alpha_2$ should be obvious.)

Given a set $\mathcal{X} \subset \mathbb{R}^n$, its boundary is denoted by $\mathrm{boun}\,\mathcal{X}$ and its closed convex hull by $\mathrm{conv}\,\mathcal{X}$. The *relative interior* of a convex set in $\mathbb{R}^n$ is the interior taken with respect to its affine hull. The *distance* between two sets $\mathcal{X}$, $\mathcal{Y}$ is defined as $\mathrm{dist}(\mathcal{X}, \mathcal{Y}) := \inf_{x\in\mathcal{X},\, y\in\mathcal{Y}} |x - y|$ where $|\cdot|$ is the Euclidean norm. The *diameter* of $\mathcal{X}$ is defined as $\mathrm{diam}\,\mathcal{X} := \sup_{x,y\in\mathcal{X}} |x - y|$.

2.1 Hyperslices and ω-contraction

A hyperplane in $\mathbb{R}^n$ is a set of the form $\{x : \langle a, x\rangle = b\}$ where $|a| > 0$. For $\mathcal{X} \subset \mathbb{R}^n$, let $\bar{x} \in \mathrm{boun}\,\mathcal{X}$. If $|a| > 0$ satisfies $\langle a, x\rangle \leq \langle a, \bar{x}\rangle$ for all $x \in \mathcal{X}$, then the hyperplane $\{x : \langle a, x\rangle = \langle a, \bar{x}\rangle\}$ is called a *supporting hyperplane* to $\mathcal{X}$ at $\bar{x}$. A hyperplane is a supporting hyperplane to $\mathcal{X}$ if it is a supporting hyperplane to $\mathcal{X}$ at some boundary point of $\mathcal{X}$.

We define a *hyperslice* as the convex hull of the union of two parallel hyperplanes, which are called the *edges* of the hyperslice. Note that the boundary of a hyperslice is its edges. The *thickness* of a hyperslice is the distance between its edges. Note that a hyperplane is a hyperslice with 0 thickness. Given $\mathcal{X} \subset \mathbb{R}^n$, a hyperslice $\mathcal{S}$ is called a *supporting hyperslice* of $\mathcal{X}$ if $\mathcal{X} \subset \mathcal{S}$ and both edges of $\mathcal{S}$ are supporting hyperplanes to $\mathcal{X}$. Given a line $L \subset \mathbb{R}^n$ and a hyperslice $\mathcal{S}$, L is perpendicular to $\mathcal{S}$ if it is perpendicular to the edges of $\mathcal{S}$. We let $\mathcal{L}(\mathbb{R}^n)$ denote the set of lines in $\mathbb{R}^n$.

Let us be given a bounded set $\mathcal{X}$ and a line L, both in $\mathbb{R}^n$, and a class-$\mathcal{K}$ function ω. Let $\mathcal{S}$ be the supporting hyperslice of $\mathcal{X}$ that is perpendicular to L. Let S have edges $\mathcal{H}_1$, $\mathcal{H}_2$ and thickness δ. Also let $\mathcal{H}_{1|2}$ be the hyperplane satisfying $\mathrm{dist}(\mathcal{H}_{1|2}, \mathcal{H}_1) = \mathrm{dist}(\mathcal{H}_{1|2}, \mathcal{H}_2) = \delta/2$. Then the *$\omega$-contraction of $\mathcal{X}$ along* L is defined as

$$\overline{\mathrm{cont}}(\mathcal{X}, \omega, L) := \{x \in \mathrm{conv}\mathcal{X} : \mathrm{dist}(\{x\}, \mathcal{H}_{1|2}) \leq \delta/2 - \omega(\delta)\}\,.$$

We then define (see Fig. 1) *ω-contraction of $\mathcal{X}$* as

$$\mathrm{cont}(\mathcal{X}, \omega) := \bigcap_{L\in\mathcal{L}(\mathbb{R}^n)} \overline{\mathrm{cont}}(\mathcal{X}, \omega, L)\,.$$

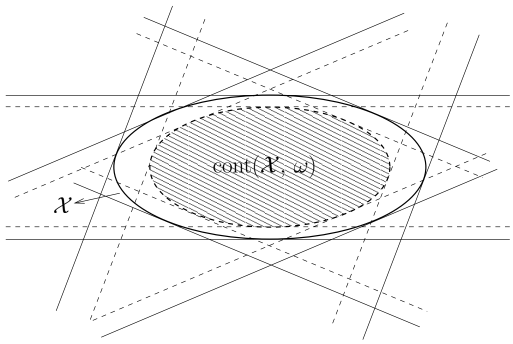

Fig. 1. Contraction of a set with an elliptic border and a few of its supporting hyperslices.

2.2 Projection onto a line

Given a set $\mathcal{X}$ and a line L in $\mathbb{R}^n$, the *projection of $\mathcal{X}$ onto* L is

$$\wp_L(\mathcal{X}) := \{\bar{x} \in L : |x - \bar{x}| = \mathrm{dist}(\{x\}, L),\, x \in \mathcal{X}\}\,.$$

A *geodesic* from $x \in \mathbb{R}^n$ to $y \in \mathbb{R}^n$ is a map c from a closed interval $[0, \ell] \subset \mathbb{R}$ to $\mathbb{R}^n$ such that $c(0) = x$, $c(\ell) = y$ and $|c(t_1) - c(t_2)| = |t_1 - t_2|$ for all $t_1, t_2 \in [0, \ell]$. The image of a geodesic is called a *line segment.*

Lemma 1. Given a bounded set $\mathcal{X} \subset \mathbb{R}^n$ and a class-$\mathcal{K}$ function $\omega \leq \mathrm{id}/2$, the following holds for all $L \in \mathcal{L}(\mathbb{R}^n)$

$$\wp_L(\mathrm{cont}(\mathcal{X}, \omega)) \subset \wp_L(\overline{\mathrm{cont}}(\mathcal{X}, \omega, L))\,.$$

2.3 Directed graphs and connectedness

A *directed graph* is a pair $(\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ is a nonempty finite set (of *nodes*) and $\mathcal{A}$ is a finite collection of pairs (*arcs*) (n_i, n_j) with $n_i, n_j \in \mathcal{N}$. A *path* from n_1 to n_ℓ is a sequence of nodes $\{n_1, n_2, \ldots, n_\ell\}$ such that (n_i, n_{i+1}) is an arc for $i \in \{1, 2, \ldots, \ell-1\}$. A directed graph is *connected* if it has a node to which there exists a path from every other node.

3. SYSTEM DESCRIPTION

Consider the system of p agents

$$\begin{aligned} \mathbf{x}_1^+ &= \mathbf{f}_1(\mathbf{x}, g) \\ \mathbf{x}_2^+ &= \mathbf{f}_2(\mathbf{x}, g) \\ &\vdots \\ \mathbf{x}_p^+ &= \mathbf{f}_p(\mathbf{x}, g) \end{aligned} \qquad (1)$$

where $\mathbf{x}_i \in \mathbb{X} \subset \mathbb{R}^n$ is the *state of the* ith *agent*, g is a parameter taking its values from some set $\mathcal{G}$, and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \in \mathbb{X}^p$. Notation $\mathbf{x}_i^+$ denotes the value of the ith agent's state at the next time instant. We take $\mathbb{X}$ closed. Notation $\mathbf{g}$ denotes a sequence $\{g_0, g_1, \ldots\}$ in $\mathcal{G}$. The solution of system (1) at time $k \in \mathbb{N}$, having started at the initial condition $\mathbf{x}$, and having evolved under the influence of the sequence $\mathbf{g}$ is denoted by $\Phi(k, \mathbf{x}, \mathbf{g})$. Likewise, $\Phi_i(k, \mathbf{x}, \mathbf{g})$ denotes the solution of the ith agent. For any $\mathbf{y} \in \mathbb{X}^p$, $\{\mathbf{y}\}$ will denote $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$, i.e. a subset of $\mathbb{R}^n$.

Let $\mathcal{N} = \{n_1, n_2, \ldots, n_p\}$ be the set of nodes of some graph $(\mathcal{N}, \mathcal{A})$ where node n_i represents the ith agent. Then, given a set of arcs $\mathcal{A}$, *extended neighbor set* of ith agent is $\mathbf{n}_i(\mathbf{x}, \mathcal{A}) := \{\mathbf{x}_j : (n_i, n_j) \in \mathcal{A}\} \cup \{\mathbf{x}_i\}$. For each $g \in \mathcal{G}$ there is an associated set of arcs $\mathcal{A}_g$ and hence an associated graph $(\mathcal{N}, \mathcal{A}_g)$. For $N \in \mathbb{N}$, let us let $\mathbf{G}_N$ denote the set of sequences $\mathbf{g} = \{g_0, g_1, \ldots\}$ such that for each $k_0 \in \mathbb{N}$, the union $(\mathcal{N}, \cup_{k=k_0}^{k_0+N} \mathcal{A}_{g_k})$ is connected.

Definition 2. Given a set $\mathcal{C} \subset \mathbb{X}$ and a class-$\mathcal{K}$ function $\omega \leq \mathrm{id}/2$, system (1) is said to be *contractive on* $\mathcal{C}$ *with* ω if for all $\{\mathbf{x}\} \in \mathcal{C}$, g, and i

$$\mathbf{f}_i(\mathbf{x}, g) \in \mathrm{cont}(\mathbf{n}_i(\mathbf{x}, \mathcal{A}_g), \omega)\,; \qquad (2)$$

and is said to be *contractive* if for each bounded set $C \subset \mathbb{X}$ there exists a class-$\mathcal{K}$ function $\omega \leq \mathrm{id}/2$ such that system (1) is contractive on C with ω.

Let $\mathbb{E} \subset \mathbb{X}^p$ denote the *set of equilibrium points* which we define as

$$\mathbb{E} := \{\mathbf{x} \in \mathbb{X}^p : \mathrm{diam}\{\mathbf{x}\} = 0\}\,.$$

Definition 3. System (1), with respect to sequence set $\mathbf{G}$ is

1. *stable* if for each $\xi \in \mathbb{E}$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{x} - \xi| \leq \delta$ and $\mathbf{g} \in \mathbf{G}$ imply $|\Phi(k, \mathbf{x}, \mathbf{g}) - \xi| \leq \varepsilon$ for all k.
2. *bounded* if for each $\xi \in \mathbb{E}$, for all $\delta > 0$ there exists $\varepsilon > 0$ such that $|\mathbf{x} - \xi| \leq \delta$ and $\mathbf{g} \in \mathbf{G}$ imply $|\Phi(k, \mathbf{x}, \mathbf{g}) - \xi| \leq \varepsilon$ for all k.
3. *attractive* if for each $\xi \in \mathbb{E}$, for all $\varepsilon, \delta > 0$ there exists $K \in \mathbb{N}$ such that $|\mathbf{x} - \xi| \leq \delta$ and $\mathbf{g} \in \mathbf{G}$ imply the existence of $\eta \in \mathbb{E}$ such that $|\Phi(k, \mathbf{x}, \mathbf{g}) - \eta| \leq \varepsilon$ for all $k \geq K$.
4. *asymptotically stable* if it is stable, bounded, and attractive.

We borrow from (Moreau, 2005) the following assumption and Theorem 5.

Assumption 4. **(Moreau)** For system (1), for each $\mathbf{x}$, $\mathcal{A}$, and i there is a compact set $\mathbf{e}_i(\mathbf{x}, \mathcal{A}) \subset \mathbb{X}$ satisfying:

1. $\mathbf{f}_i(\mathbf{x}, g) \in \mathbf{e}_i(\mathbf{x}, \mathcal{A}_g)$ for all g;
2. $\mathbf{e}_i(\mathbf{x}, \mathcal{A}) = \{\mathbf{x}_i\}$ whenever $\mathbf{n}_i(\mathbf{x}, \mathcal{A})$ is a singleton;
3. $\mathbf{e}_i(\mathbf{x}, \mathcal{A})$ is contained in the relative interior of $\mathrm{conv}\,\mathbf{n}_i(\mathbf{x}, \mathcal{A})$ whenever $\mathbf{n}_i(\mathbf{x}, \mathcal{A})$ is not a singleton;
4. $\mathbf{e}_i(\cdot, \mathcal{A})$ is a continuous set valued mapping.

Theorem 5. Under Assumption 4, for all N, system (1) is asymptotically stable with respect to set of sequences $\mathbf{G}_N$.

4. EQUIVALENCE OF ASSUMPTION 4 AND CONTRACTIVENESS

Theorem 6. Following are equivalent:

- Assumption 4 holds.
- System (1) is contractive.

PROOF. Suppose that Assumption 4 holds. Let us be given a compact set $\mathcal{C} \subset \mathbb{X}$. Let $d := \mathrm{diam}\,\mathcal{C}$ which is finite due to compactness of $\mathcal{C}$. If $d = 0$ the result trivially follows. Suppose $d > 0$. For each closed $\mathcal{Y} \subset \mathcal{C}$ let $\mathbb{S}_k(\mathcal{Y})$ for $k \in \mathbb{N}$ denote the set of supporting hyperslices (in $\mathbb{R}^n$) of $\mathcal{Y}$ with thickness in the interval $[2^{-k-1}d, 2^{-k}d]$. We define $\mathcal{X}_k^{(i,\mathcal{A})} \subset \mathcal{C}$ as the union of all $\{\mathbf{x}\}$ satisfying $\mathbb{S}_k(\mathbf{n}_i(\mathbf{x}, \mathcal{A})) \neq \emptyset$. Note that there exists a pair $(i, \mathcal{A})$ such that $\mathcal{X}_0^{(i,\mathcal{A})}$ is nonempty. For $\mathcal{X}_k^{(i,\mathcal{A})} \neq \emptyset$ we define the function $\rho_k^{(i,\mathcal{A})} : \mathcal{X}_k^{(i,\mathcal{A})} \to \mathbb{R}_{\geq 0}$ as

$$\rho_k^{(i,\mathcal{A})}(\mathbf{x}) := \inf_{\mathcal{S} \in \mathbb{S}_k(\mathbf{n}_i(\mathbf{x}, \mathcal{A}))} \mathrm{dist}(\mathrm{boun}\,\mathcal{S}, \mathbf{e}_i(\mathbf{x}, \mathcal{A}))\,.$$

For each $\{\mathbf{x}\} \subset \mathcal{X}_k^{(i,\mathcal{A})} \neq \emptyset$, $\rho_k^{(i,\mathcal{A})}(\mathbf{x}) > 0$ and $\rho_k^{(i,\mathcal{A})}$ is continuous due to the continuity of

$\mathbf{e}_i(\cdot, \mathcal{A})$. We observe also that $\mathcal{X}_k^{(i,\mathcal{A})}$ is compact for all i, $\mathcal{A}$, and k. Therefore for $\mathcal{X}_k^{(i,\mathcal{A})}$ nonempty, $\rho_k^{(i,\mathcal{A})}$ attains a positive minimum on $\mathcal{X}_k^{(i,\mathcal{A})}$ which we let $\delta_k^{(i,\mathcal{A})}$ denote. For k which $\mathcal{X}_k^{(i,\mathcal{A})} = \emptyset$, we let $\delta_k^{(i,\mathcal{A})} = \infty$. Let $I_k := [2^{-k-1}d,\, 2^{-k}d)$. Now we define $\omega^{(i,\mathcal{A})} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ as

$$\omega^{(i,\mathcal{A})}(s) := \begin{cases} \delta_0^{(i,\mathcal{A})} & \text{for } s \in [2^{-1}d,\, \infty) \\ \min\{\delta_k^{(i,\mathcal{A})},\, \delta_{k-1}^{(i,\mathcal{A})}/2\} & \text{for } s \in I_k,\ k \geq 1 \\ 0 & \text{for } s = 0 \end{cases}$$

Finally let ω be a class-$\mathcal{K}$ function satisfying

$$\omega(s) \leq \min_{(i,\mathcal{A})} \omega^{(i,\mathcal{A})}(s)$$

for all $s \geq 0$. Such ω exists since there are finite many pairs $(i, \mathcal{A})$. By construction, ω is such that system (1) satisfies (2) for all $\{\mathbf{x}\} \subset \mathcal{C}$.

Now we show the other direction. Suppose that system (1) is contractive. For $k \in \mathbb{N}$ we define

$$\mathcal{B}_k := \{\mathbf{x} \in \mathbb{R}^{np} : |\mathbf{x}| \in [2k,\, 2k+2]\}$$
$$\mathcal{B}_k^- := \{\mathbf{x} \in \mathbb{R}^{np} : |\mathbf{x}| \in [2k,\, 2k+1]\}$$
$$\mathcal{B}_k^+ := \{\mathbf{x} \in \mathbb{R}^{np} : |\mathbf{x}| \in [2k+1,\, 2k+2]\}\,.$$

Note that $\mathcal{B}_k \cap \mathbb{X}$ is compact for each k. Hence for each k there exists $\omega_k \in \mathcal{K}$ such that $\mathbf{f}_i(\mathbf{x}, g) \in \mathrm{cont}(\mathbf{n}_i(\mathbf{x}, \mathcal{A}_g), \omega_k)$ for all $\mathbf{x} \in \mathcal{B}_k \cap \mathbb{X}$, i, and g. Without loss of generality we can take $\omega_{k+1} \leq \omega_k$. Let $d_k(\mathbf{x})$ denote the distance of point $\mathbf{x}$ to the set $\mathcal{B}_k$. Then we define $\alpha : \mathbb{X}^p \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as

$$\alpha(\mathbf{x}, \cdot) := \begin{cases} \omega_k & \text{for } \mathbf{x} \in \mathcal{B}_k^- \\ d_{k+1}(\mathbf{x})\omega_k + (1 - d_{k+1}(\mathbf{x}))\omega_{k+1} & \text{for } \mathbf{x} \in \mathcal{B}_k^+ \end{cases}$$

Finally, for each i and $\mathcal{A}$ let

$$\mathbf{e}_i(\mathbf{x}, \mathcal{A}) := \mathrm{cont}(\mathbf{n}_i(\mathbf{x}, \mathcal{A}), \alpha(\mathbf{x}, \cdot))\,.$$

Note that $\mathbf{e}_i(\cdot, \mathcal{A})$ is continuous and satisfies other conditions of Assumption 4 by construction. ■

The following result is a direct consequence of Theorem 5 and Theorem 6.

Theorem 7. If system (1) is contractive, then, for all N, it is asymptotically stable with respect to set of sequences $\mathbf{G}_N$.

For the sake of completeness, we give the following result.

Theorem 8. If system (1) is contractive, then $\{\Phi(k, \mathbf{x}, \mathbf{g})\} \subset \mathrm{conv}\{\mathbf{x}\}$ for all $\mathbf{x}$, $\mathbf{g}$, and k.

5. QUANTITATIVE CONVERGENCE ANALYSIS

The main advantage of reformulating Assumption 4 as a contractiveness condition is to quantify the convergence result of Theorem 7 with a class-$\mathcal{KL}$ function β. In this section, we provide an algorithm to construct such β and show convergence to a consensus with respect to it.

Algorithm 1. Given a triple (ω, p, N) where $\omega \leq \mathrm{id}/2$ is a class-$\mathcal{K}$ function, $p \in \mathbb{N}_{\geq 1}$, and $N \in \mathbb{N}$, construct $\beta_{(\omega,p,N)} \in \mathcal{KL}$ through the following steps.

(1) Let $\gamma_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be such that
$$\omega(s - \gamma_1(s)) = \gamma_1(s) \quad \forall s \geq 0\,. \tag{3}$$
(Note that $\gamma_1 \in \mathcal{K}$ and $\gamma_1 \leq \mathrm{id}/3$.)

(2) Define $\gamma_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as
$$\gamma_2(s) := s - \gamma_1^{(p-1)(N+1)}(\gamma_1(s)/2) \quad \forall s \geq 0\,.$$
(Note that $5\mathrm{id}/6 \leq \gamma_2 < \mathrm{id}$.)

(3) If γ_2 is nondecreasing, define $\gamma_3(\cdot) := \gamma_2(\cdot)$; else pick some $\gamma_3 \in \mathcal{K}$ satisfying $\gamma_2 \leq \gamma_3 < \mathrm{id}$.

(4) Finally,
$$\beta_{(\omega,p,N)}(s,\, t) := \gamma_3^l(s) \quad \forall s \geq 0$$
for $t \in [(p-1)(N+1)l,\, (p-1)(N+1)(l+1)-1]$ and for $l \in \mathbb{N}$.

Algorithm 2. Given a triple (w, p, N) where $0 < w \leq 1/2$, $p \in \mathbb{N}_{\geq 1}$, and $N \in \mathbb{N}$, obtain the pair (M, σ) as

$$M := \left(1 - \frac{1}{2}\left(\frac{w}{1+w}\right)^{(p-1)(N+1)+1}\right)^{-1}$$
$$\sigma := \left(1 - \frac{1}{2}\left(\frac{w}{1+w}\right)^{(p-1)(N+1)+1}\right)^{\frac{1}{(p-1)(N+1)}}\,.$$

Lemma 9. Let $\ell \geq 0$, $c : [0,\, \ell] \to \mathbb{X}$ be a geodesic, $L \in \mathcal{L}(\mathbb{R}^n)$ be a line such that $c([0,\, \ell]) \subset L$ and $\omega \leq \mathrm{id}/2$ be a class-$\mathcal{K}$ function. Let $\gamma_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfy (3). Then, given a line segment S in $c([0,\, \ell])$ satisfying $S \supset c([a,\, b])$ where $0 \leq a \leq \gamma_1(\ell)/2$ and $\ell - \gamma_1(\ell)/2 \leq b \leq \ell$ we have

$$\overline{\mathrm{cont}}(S,\, \omega,\, L) \subset c([\gamma_1(\ell),\, \ell - \gamma_1(\ell)])\,.$$

Lemma 10. Let $\ell \geq 0$, $c : [0,\, \ell] \to \mathbb{X}$ be a geodesic, $L \in \mathcal{L}(\mathbb{R}^n)$ be a line such that $c([0,\, \ell]) \subset L$ and $\omega \leq \mathrm{id}/2$ be a class-$\mathcal{K}$ function. Let $\gamma_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfy (3). Given $t \in [0,\, \ell]$ and a line segment $S \subset c([0,\, \ell])$ satisfying $c(t) \in S$ we can write

$$\overline{\mathrm{cont}}(S,\, \omega,\, L) \subset c([\gamma_1(h),\, \ell - \gamma_1(h)])$$

where $h = \min\{t,\, \ell - t\}$.

Lemma 11. Let C be a convex subset of $\mathbb{X}$, $\omega \leq \mathrm{id}/2$ be a class-$\mathcal{K}$ function, and $N \in \mathbb{N}$. Suppose that system (1) is contractive on C with ω. Then for all $\{\mathbf{x}\} \subset C$ and $\mathbf{g} \in \mathbf{G}_N$ we have

$$\operatorname{diam} \wp_L(\{\Phi(k,\, \mathbf{x},\, \mathbf{g})\}) \leq \beta_{(\omega,p,N)}(\operatorname{diam} \wp_L(\{\mathbf{x}\}),\, k)$$

for all $L \in \mathcal{L}(\mathbb{R}^n)$ and $k \in \mathbb{N}$, where function $\beta_{(\omega,p,N)}$ is constructed according to Algorithm 1.

PROOF. First we point out that, due to Theorem 8, the solution of system (1) stays in C at all times if the initial condition lies in C. Let us be given $L \in \mathcal{L}(\mathbb{R}^n)$, $\{\mathbf{x}\} \subset C$, and $\mathbf{g} = \{g_0,\, g_1,\, \ldots\} \in \mathbf{G}_N$. Then, for economic purposes, let us let $\phi_i^k := \Phi_i(k,\, \mathbf{x},\, \mathbf{g})$ for $i \in \{1,\, 2,\, \ldots,\, p\}$ and $\phi^k := \Phi(k,\, \mathbf{x},\, \mathbf{g})$. Note that $\{\phi^k\} \subset C$ for all $k \in \mathbb{N}$. Since the system is contractive on C with ω, we can write

$$\begin{aligned}\wp_L(\phi_i^{k+1}) &= \wp_L(\mathbf{f}_i(\phi^k,\, g_k))\\ &\in \wp_L(\mathrm{cont}(\mathrm{conv}\, \mathbf{n}_i(\phi^k,\, \mathcal{A}_{g_k}),\, \omega))\\ &\subset \overline{\mathrm{cont}}(\wp_L(\mathrm{conv}\, \mathbf{n}_i(\phi^k,\, \mathcal{A}_{g_k})),\, \omega,\, L)\\ &= \overline{\mathrm{cont}}(\mathrm{conv}\, \wp_L(\mathbf{n}_i(\phi^k,\, \mathcal{A}_{g_k})),\, \omega,\, L) \quad (4)\end{aligned}$$

for all i and k. An implication of (4) is that $\wp_L(\phi_i^k) \in \mathrm{conv}\, \wp_L(\{\phi^0\})$ for all i and k. Hence, due to time invariance we can write

$$\wp_L(\{\phi^{k+1}\}) \in \mathrm{conv}\, \wp_L(\{\phi^k\})\,. \qquad (5)$$

Let $\ell = \operatorname{diam} \wp_L(\{\phi^0\})$ and $c : [0,\, \ell] \to \mathbb{X}$ be the geodesic associated to line segment $\mathrm{conv}\, \wp_L(\{\phi^0\})$. Note that $\wp_L(\{\phi^k\}) \subset c([0,\, \ell])$ for all $k \in \mathbb{N}$. We now claim that there exist some $i_1 \in \{1,\, 2,\, \ldots,\, p\}$ and $k \in \{0,\, 1,\, \ldots,\, N+1\}$ such that

$$\wp_L(\phi_{i_1}^k) \in c([\gamma_1(\ell)/2,\, \ell - \gamma_1(\ell)/2]) \qquad (6)$$

where $\gamma_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we borrow from Algorithm 1. Now suppose that our claim is false. That implies that there exist two scalars a, b satisfying $0 \leq a < \gamma_1(\ell)/2$ and $\ell - \gamma_1(\ell)/2 < b \leq \ell$ and for all $k \in \{0,\, 1,\, \ldots,\, N\}$ we have $\wp_L(\phi_i^k) \in c([0,\, a]) \cup c([b,\, \ell])$ for all i. Therefore, since $\mathbf{g} \in \mathbf{G}_N$, at some time $k \in \{0,\, 1,\, \ldots,\, N\}$ there must exist a pair $(i,\, j)$ such that $\phi_i^k \in \mathbf{n}_j(\phi^k,\, \mathcal{A}_{g_k})$ and one of the following holds:

- $\wp_L(\phi_i^k) \in c([0,\, a])$ and $\wp_L(\phi_j^k) \in c([b,\, \ell])$, or
- $\wp_L(\phi_j^k) \in c([0,\, a])$ and $\wp_L(\phi_i^k) \in c([b,\, \ell])$.

That implies $\mathrm{conv}\, \wp_L(\mathbf{n}_j(\phi^k,\, \mathcal{A}_{g_k})) \supset c([a,\, b])$. Thence it follows by Lemma 9 and (4) that $\wp_L(\phi_j^{k+1}) \in c([\gamma_1(\ell),\, \ell - \gamma_1(\ell)])$, which poses a contradiction.

Now, recall that $\wp_L(\phi_i^k) \in \mathrm{conv}\, \wp_L(\mathbf{n}_i(\phi^k,\, \mathcal{A}_{g_k}))$ for all i and k. Therefore, if we combine (6) and Lemma 10, we can write

$$\wp_L(\phi_{i_1}^k) \in c([\gamma_1^k(\gamma_1(\ell)/2),\, \ell - \gamma_1^k(\gamma_1(\ell)/2)]) \quad (7)$$

for $k \in \mathbb{N}_{\geq N+1}$. For compactness, let $h_k := \gamma_1^k(\gamma_1(\ell)/2)$. We can now make our second claim:

$$\operatorname{diam} \wp_L(\{\phi^{(p-1)(N+1)}\}) \leq \ell - h_{(p-1)(N+1)}\,. \quad (8)$$

Suppose not. Then there exist i, j such that $\wp_L(\phi_i^k) \in c([0,\, h_k))$ and $\wp_L(\phi_j^k) \in c((\ell - h_k,\, \ell])$ for all $k \in \{0,\, 1,\, \ldots,\, (p-1)(N+1)\}$. Observe that, due to Lemma 10, if some agent l satisfies $\wp_L(\phi_l^{k_0}) \in c([h_{k_0},\, \ell - h_{k_0}])$ for some k_0 then $\wp_L(\phi_l^k) \in c([h_k,\, \ell - h_k])$ for all $k \in \mathbb{N}_{\geq k_0}$. Due to connectedness, there must be an agent $i_2 \neq i_1$ such that $\mathrm{conv}\, \wp_L(\mathbf{n}_{i_2}(\phi^k,\, \mathcal{A}_{g_k})) \ni \wp_L(\phi_{i_1}^k)$ for some $k \in \{N+1,\, N+2,\, \ldots,\, 2N+1\}$. (Note that this does not necessarily imply $\mathbf{n}_{i_2}(\phi^k,\, \mathcal{A}_{g_k}) \ni \phi_{i_1}^k$.) Therefore, by (7) and Lemma 10 we can write

$$\wp_L(\phi_{i_2}^k) \in c([h_k,\, \ell - h_k])$$

for $k \in \mathbb{N}_{\geq 2N+2}$. That is to say for all $k \in \mathbb{N}_{\geq 2N+2}$ there will be at least two agents whose projections fall in $c([h_k,\, \ell - h_k])$. The generalization is straightforward and lets us assert that for all $k \in \mathbb{N}_{\geq q(N+1)}$ there will be at least q agents whose projections fall in $c([h_k,\, \ell - h_k])$. When $q = p - 1$ we have a contradiction. Therefore (8) holds. Hence

$$\begin{aligned}\operatorname{diam} \wp_L(\{\phi^{(p-1)(N+1)}\}) &\leq \ell - \gamma_1^{(p-1)(N+1)}(\gamma_1(\ell)/2)\\ &\leq \gamma_3(\operatorname{diam} \wp_L(\{\phi^0\}))\,.\end{aligned}$$

Going one step further we can write

$$\operatorname{diam} \wp_L(\{\phi^{m(p-1)(N+1)}\}) \leq \gamma_3^m(\operatorname{diam} \wp_L(\{\phi^0\}))(9)$$

for all $m \in \mathbb{N}$. All that is left is to combine (5) with (9). ■

Below is the main result of this section.

Theorem 12. Let C be a convex subset of $\mathbb{X}$, $\omega \leq \mathrm{id}/2$ be a class-$\mathcal{K}$ function, and $N \in \mathbb{N}$. Suppose that system (1) is contractive on C with ω. Then for all $\{\mathbf{x}\} \subset C$ and $\mathbf{g} \in \mathbf{G}_N$ we have

$$\operatorname{diam}\{\Phi(k,\, \mathbf{x},\, \mathbf{g})\} \leq \beta_{(\omega,p,N)}(\operatorname{diam}\{\mathbf{x}\},\, k)$$

for all $k \in \mathbb{N}$, where function $\beta_{(\omega,p,N)}$ is constructed according to Algorithm 1.

PROOF. Let us be given $\{\mathbf{x}\} \subset C$, $\mathbf{g} \in \mathbf{G}_N$, and $k \in \mathbb{N}$. Let $x, y \in \{\Phi(k,\, \mathbf{x},\, \mathbf{g})\}$ be such that $|x - y| = \operatorname{diam}\{\Phi(k,\, \mathbf{x},\, \mathbf{g})\}$. Then let $L \in$

$\mathcal{L}(\mathbb{R}^n)$ be such that $x, y \in L$. By Lemma 11 and remembering that $\beta_{(\omega,p,N)}$ is a class-$\mathcal{KL}$ function, we can write

$$\begin{aligned}\operatorname{diam}\{\Phi(k, \mathbf{x}, \mathbf{g})\} &= \operatorname{diam} \wp_L(\{\Phi(k, \mathbf{x}, \mathbf{g})\}) \\ &\le \beta_{(\omega,p,N)}(\operatorname{diam} \wp_L(\{\mathbf{x}\}), k) \\ &\le \beta_{(\omega,p,N)}(\operatorname{diam}\{\mathbf{x}\}, k)\,.\end{aligned}$$

Hence the result. ■

The convergence shown in Theorem 12 becomes exponential when the contraction is characterized by a linear function. The following corollary formalizes this.

Corollary 13. Let C be a convex subset of $\mathbb{X}$, $w \in (0, 1/2]$, and $N \in \mathbb{N}$. Suppose that system (1) is contractive on C with $w \cdot \mathrm{id}$. Then for all $\{\mathbf{x}\} \subset C$ and $\mathbf{g} \in \mathbf{G}_N$ we have

$$\operatorname{diam}\{\Phi(k, \mathbf{x}, \mathbf{g})\} \le M\sigma^k \operatorname{diam}\{\mathbf{x}\}$$

for all $k \in \mathbb{N}$, where pair (M, σ) is obtained from Algorithm 2.

6. LINEAR EXAMPLE

In this section we study a linear system (cf. (Blondel *et al.*, 2005)) and assert that it is contractive uniformly with a single linear class-$\mathcal{K}$ function which can be explicitly computed.

Let $w_{ij} : \mathcal{G} \to \mathbb{R}_{\ge 0}$, defined for $i, j \in \{1, 2, \ldots, p\}$, be a *weight map* satisfying the following conditions:

(1) $w_{ij}(g) = 0$ if $j \notin \{l : \mathbf{x}_l \in \mathbf{n}_i(\mathbf{x}, \mathcal{A}_g)\}$,
(2) $w_{ij}(g) \ge w_{\min} > 0$ if $j \in \{l : \mathbf{x}_l \in \mathbf{n}_i(\mathbf{x}, \mathcal{A}_g)\}$,
(3) $\sum_j w_{ij}(g) = 1$.

Proposition 14. Consider system (1). Let $\mathbb{X} = \mathbb{R}^n$ and the righthand side obeys

$$\mathbf{f}_i(\mathbf{x}, g) = \sum_j w_{ij}(g) \cdot \mathbf{x}_j$$

for all $i \in \{1, 2, \ldots, p\}$. Then system (1) is contractive on $\mathbb{R}^n$ with $w_{\min} \cdot \mathrm{id}$.

Corollary 15. Consider system (1). Let $\mathbb{X} = \mathbb{R}^n$ and for all $i \in \{1, 2, \ldots, p\}$

$$\mathbf{f}_i(\mathbf{x}, g) = \sum_j w_{ij}(g) \cdot \mathbf{x}_j\,.$$

Then for all $\{\mathbf{x}\} \subset \mathbb{R}^n$ and $\mathbf{g} \in \mathbf{G}_N$

$$\operatorname{diam}\{\Phi(k, \mathbf{x}, \mathbf{g})\} \le M\sigma^k \operatorname{diam}\{\mathbf{x}\}$$

for all $k \in \mathbb{N}$, where pair (M, σ) is obtained from Algorithm 2 with $w = w_{\min}$.

7. CONCLUSION

We have presented a contraction characterization for (convex) sets in terms of a class-$\mathcal{K}$ function, call it ω. The characterization provides explicit rates of convergence to a consensus for multi-agent systems. In that respect, our work quantifies the qualitative convergence result in (Moreau, 2005). An interesting problem not yet tackled is to find out the relation of ω with the eigenvalues of the Laplacian matrix describing the communication topology for linear multi-agent systems.

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