# Stability margins of Nonlinear Receding Horizon Control via Inverse Optimality*

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**Abstract**

Using the nonlinear analog of the Fake Riccati equation developed for linear systems, we derive an inverse optimality result for several receding horizon control schemes. This inverse optimality result unifies stability proofs and shows that receding horizon control possesses the stability margins of optimal control laws.

**Keywords:** Receding-Horizon Control, Nonlinear Control, Optimal Control, HJB equation, Robustness

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# 1 Introduction

Several receding horizon control schemes for nonlinear systems have been recently proposed which guarantee stability of the closed-loop system [9, 10, 12, 3, 4, 2]. In this note, we show that all these control laws, although based on the (open-loop) solution of a *finite* horizon optimal control problem, also yield the (feedback) solution of an associate *infinite* horizon optimal control problem. This inverse optimal result is based on the derivation of a stationary Hamilton-Jacobi-Belmann (HJB) equation which is the nonlinear analog of the Fake Riccati equation developed for linear systems [5, 13]. The inverse optimality result unifies the stability proofs of different receding horizon control schemes. More significantly, it establishes an important robustness property of receding horizon control since the control laws are shown to possess the stability margins of optimal control laws [6, 7, 14]. From a practical point of view, this result underlines that, with some caution, highly desirable features of control laws which solve an *infinite* horizon optimal control problem are in fact retained with a limited horizon.

Section 2 reviews several receding horizon nonlinear control schemes. The Fake HJB equation is derived in Section 3. A stability proof for the closed-loop system and the stability margins of receding horizon control are derived at once from the inverse optimality result in Section 4.

# 2 Receding horizon control

In receding horizon control, a static state feedback control law is constructed by solving on-line at time $t$ an optimal control problem over the finite horizon $[t, t+L]$. We consider a nonlinear system

$$\dot{x} = F(x,u) = f(x) + g(x)u, \quad f(0) = 0 \tag{1}$$

where $x \in I\!R^n$, $u \in I\!R^m$ and $F$ is at least twice continuously differentiable. The *Finite-Horizon Optimal Control Problem (FHOCP)* is defined by the minimization of the cost function

$$J(x_0, u(\cdot), L) = \int_0^L q(x(t), u(t))dt + m(x(L)) \tag{2}$$

subject to (1) and with $x(0) = x_0$. The choice $t = 0$ for the initial time is irrelevant since the problem is time-invariant.

Throughout this paper, we assume that the cost has the particular form

$$q(x,u) = l(x) + u'R(x)u,\ l(x) \geq 0,\ l(0) = 0,\ \ R(x) = \text{diag}\,\{r_i(x)\} > 0$$

If $l(x)$ is only positive semidefinite, we will also assume the "zero-state detectability in the cost", that is, that all the solutions of $\dot{x} = f(x)$ which satisfy the constraint $l(x(t)) \equiv 0$ converge to $x = 0$ as $t \to \infty$.

For a given initial condition $x_0 \in I\!R^n$, we denote by $u^*(t, x_0)$, $0 \leq t < L$, the optimal solution of the $FHOCP$, that is, a function $u$ in $L^m_\infty[0, L]$ which minimizes the cost (2). In the following, the optimal value of the $FHOCP$ will be denoted by $V(x, L)$ , that is $V(x, L) = J(x, u^*(\cdot, x), L)$. In receding horizon control, the value of the feedback control at state $x$ is obtained by solving the $FHOCP$ and setting

$$\gamma(x) = u^*(0, x) \tag{3}$$

Repeating this on-line calculation continuously along the solutions yields a feedback control as opposed to the open-loop control $u^*(t, x_0)$. This strategy is conceptually simple and attractive to determine a feedback control law for nonlinear systems, in particular in the presence of constraints (no constraints are considered in this paper). However, in contrast to infinite-horizon optimization which, under mild assumptions, ensures closed-loop stability, the receding horizon control scheme does not guarantee stability unless the final state penalty $m(\cdot)$ is chosen appropriately.

To ensure closed-loop stability, different alternatives have been proposed in the literature, which we summarize hereafter:

(i) $m(x(L)) = \int_L^\infty q(x(t), Kx(t))dt$. The final state penalty is the cost incurred over $[L, \infty)$ by applying the linear control law $u(t) = Kx(t)$ from time $t = L$. In this case, the solvability of the finite horizon control problem is guaranteed if the linear control law $u = Kx$ ensures local exponential stability of the equilibrium $x = 0$, with a region of attraction that can be reached from $x_0$ within the time interval $[0, L]$. Under such assumptions closed-loop stability of the receding horizon control scheme is established in [3, 4] (the discrete-time case is considered in these papers).

(ii) $m(x(L)) = ax(L)^T Px(L)$ for some $a \in I\!R$, $a > 0$ and for some positive definite symmetric matrix $P \in I\!R^{n*n}$ [12, 2]. Choices of $a$ and $P$ which ensure closed-loop stability are discussed in [12, 2].

(iii) The terminal state constraint $x(L) = 0$ is imposed, see [9, 10] where the corresponding stability results are derived. This corresponds to a terminal constraint $m(x(L)) = \mu x(L)$ where $\mu$ is a costate of the optimal control problem, see [10] for details. In this case, the feasibility of the optimal control problem requires that the equilibrium $x = 0$ can be reached from $x_0$ in a finite time $L$.

We now introduce the following regularity assumption:

**Blanket Hypohesis:** *The control law (3) is continuously differentiable and the value function $V(x, L)$ is twice continuously differentiable in $\mathbb{R}^n$.*

The regularity of the value function of an optimal control problem is of course a strong requirement. Sufficient conditions are given in [10] for the case (iii), that is with a terminal state constraint $x(L) = 0$. The regularity issue is not discussed in [12, 2] nor in the (discrete-time) treatment of [3].

# 3 A Fake HJB equation

In the literature, the stability analyses of the three receding horizon control schemes recalled in Section 1, employ the value function $V(x, L)$ as a Lyapunov function to establish asymptotic stability of the equilibrium $x = 0$ of the closed-loop system. Hereafter, we show that the value function $V(x, L)$ also satisfies a "Fake" Hamilton-Jacobi-Bellman equation.

**Theorem 1** Under the Blanket Hypothesis of Section 2, the value function $V(x, L)$ is a $C^2$ semipositive definite function. Defining the modified state penalty

$$\bar{l}(x) = l(x) - \frac{\partial}{\partial L} V(x, L) \tag{4}$$

the value function $V(x, L)$ satisfies the HJB equation

$$0 = \bar{l}(x) + V_x(x, L) f(x) - \frac{1}{4} V_x(x, L) g(x) R^{-1} g(x)' V_x(x, L)' \tag{5}$$

with the boundary condition $V(0, L) = 0$.

*Proof:* From standard optimal control results (see for instance [1]), the value function $V$ satisfies the HJB equation

$$-\tfrac{\partial}{\partial t}V(x_0, L-t) =$$
$$min_{u(t)}\{l(x(t)+u(t)'R(x(t))u(t)+V_x(x(t), L-t)'[f(x(t))+g(x(t))u(t)]\}, \quad t \in [0, L]$$

and the minimizing input $u^*(t, x_0)$ is

$$u^*(t, x_0) = -\frac{1}{2}R(x(t))^{-1}g(x(t))'V_x(x(t), L-t)'$$

In particular, we have for $t = 0$

$$-\tfrac{\partial}{\partial t}V(x_0, L-t)|_{t=0} =$$
$$l(x_0) + V_x(x_0, L)f(x_0) - \tfrac{1}{4}V_x(x_0, L)g(x_0)R(x_0)^{-1}g(x_0)'V_x(x_0, L)'$$

To establish (5), we note that

$$V(x, L-(t+\Delta t)) = V(x, (L-\Delta t)-t)$$

which implies

$$-\frac{\partial}{\partial t}V(x_0, L-t)|_{t=0} = \frac{\partial}{\partial L}V(x_0, L)$$

Finally $V(0, L) = 0$ because $J(0, 0, L) = 0$ □

*Remark:* Note that Theorem 1 is true even if the terminal penalty $m(x(L))$ is substituted with a zero-state terminal constraint $x(L) = 0$.

The (stationary) HJB equation (5) is the nonlinear analog of the Fake Riccati equation derived in [5, 13].

Stability and robustness of the receding horizon control scheme will be derived from the corresponding optimality result provided that the modified state penalty (4) is nonnegative and satisfies $\bar{l}(0) = 0$, which follows from the following proposition.

**Proposition 1** For the three receding horizon control schemes recalled in Section 2, the following holds:

$$\frac{\partial}{\partial L}V(x, L) \leq 0, \quad \frac{\partial}{\partial L}V(0, L) = 0$$

*Proof:* Consider algorithm (i). Let $u^*(t, x_0)$ be the optimal control over the finite horizon $[0, L]$. Suppose that we apply the (open-loop) control $u^*$ over the finite interval $[0, L]$ and the (feedback) control $u = Kx$ for $t \geq L$, that is

$$\tilde{u} = \begin{cases} u^*(t, x_0) & for \quad t \in [0, L] \\ Kx(t) & for \quad t \geq L \end{cases} \tag{6}$$

By construction, the cost of this suboptimal strategy is $V(x_0, L)$ for any horizon larger than $L$:

$$J(x_0, \tilde{u}(\cdot), L + \Delta L) = V(x_0, L) \geq \min_{u(\cdot)} J(x_0, u(\cdot), L + \Delta L) = V(x_0, L + \Delta L) \tag{7}$$

Since $V$ is continuously differentiable in $L$, it follows that $\frac{\partial}{\partial L} V(x_0, L) \leq 0$, $\forall L > 0$.
Finally, $V(0, L) = 0$ for all $L$ and then $\frac{\partial}{\partial L} V(0, L) = 0$.

The proof for the algorithm (ii) is carried out in a similar way in [12]. For (iii), the control $\tilde{u}$ guaranteeing (7) is

$$\tilde{u} = \begin{cases} u^*(t, x_0) & for \quad t \in [0, L] \\ 0 & for \quad t \in [L, L + \Delta L] \end{cases}$$

# 4 Inverse optimality and stability margins of receding horizon control

Equation (5) is the stationary HJB equation associated to an infinite horizon optimal control problem, namely the minimization of the modified cost

$$J_{IH}(x_0, u(\cdot)) = \int_0^\infty \bar{l}(x(t)) + u(t)' R(x) u(t) dt \tag{8}$$

The following inverse optimality result directly follows (see for instance [14]):

**Theorem 2** For the three receding horizon control schemes discussed in Section 2, under the Blanket Hypothesis, the control law $\gamma(x)$ is the *optimal stabilizing control* which minimizes the cost $J_{IH}(x_0, u(\cdot))$ over all $u$ guaranteeing $lim_{t \to \infty} x(t) = 0$. Moreover, $V^*(x) := V(x, L)$ is the optimal value function and the control law is of the form

$$\gamma(x) = -\frac{1}{2} R^{-1}(x) g(x)' V_x^*(x)'$$

If $V^*(x)$ is radially unbounded (i.e. $V^*(x_0) \to \infty$ as $\|x_0\| \to \infty$), all these properties are global.

The fact that $\gamma(x)$ achieves asymptotic stability of $x = 0$ is standard if $l(x)$ is positive definite because $V(x, L)$ is then positive definite and $\dot{V} \leq -\bar{l}(x) \leq -l(x)$ is negative definite. If $l(x)$ is only positive semidefinite, $V(x, L)$ and $\dot{V}(x, L)$ are only semidefinite, but asymptotic stability of $x = 0$ can still be concluded from the detectability in the cost (see [14] for details).

Radial unboundedness of the value function $V^*(x)$ can be guaranteed by a growth assumption on $f$.

**Proposition 2** Suppose that there exists a constant $K_f$ and radius $r > 0$ such that

$$\|(x, u)\| \geq r \Rightarrow \|f(x, u)\| \leq K_f \; q(x, u) \tag{9}$$

Then $V^*(x)$ is radially unbounded.

*Proof:* a proof for algorithm (iii) is given in [10]. We give a proof for algorithm (i), which is easily extended to (ii).

Let $x(t)$ be the solution obtained for the initial condition $x_0$ and the control $\tilde{u}(\cdot)$ defined by (6). If $\|(x(0), \tilde{u}(0))\| > r$, then, because $x(t)$ (exponentially) converges to zero and $\tilde{u}(t) = Kx(t)$ for $t \geq L$, there exists a time $T_r \geq 0$ such that $\|(x(t), \tilde{u}(t))\| > r$, for $t < T_r$ and $\|(x(T_r), \tilde{u}(T_r))\| = r$. The distance between the solution at time $t = 0$ and $t = T_r$ satisfies

$$\|x_0 - x(T_r)\| \;=\; \|\textstyle\int_0^{T_r} \dot{x}(t)dt\| = \|\int_0^{T_r} f(x(t), \tilde{u}(t))dt\| \leq \int_0^{T_r} \|f(x(t), \tilde{u}(t))\|dt$$

Using the growth assumption (9), we obtain

$$\|x_0 - x(T_r)\| \leq K_f \int_0^{T_r} q(x(t), \tilde{u}(t))dt \leq K_f \int_0^{\infty} q(x(t), \tilde{u}(t))dt = K_f V(x_0, L) \tag{10}$$

Because $\|x(T_r)\| \leq r$, the left hand side grows unbounded as $x_0 \to \infty$, which shows that $V(x_0, L) \equiv V^*(x_0)$ is radially unbounded. $\square$

The main engineering significance of the inverse optimality result of Theorem 2 is in the stability margins that it guarantees for the closed-loop system. This important indicator of robustness refers to uncertainties $\Delta$ that can be tolerated at the input, see Figure 1, without causing the loss of stability.

A classical property of optimal feedback systems is that stabilizing control laws which minimize a cost of the form (8), achieve a *sector margin* $(\frac{1}{2}, \infty)$, see [6]. This means that the closed-loop system in Figure 1 remains asymptotically stable if $\Delta$ is any static nonlinearity $u = \phi(\cdot)$ in the sector $(\frac{1}{2}, \infty)$, that is, $\frac{1}{2}s's < s'\phi(s) < \infty$ for all $s$ in $I\!R^m$.

If $R(x) = I$ in (8), the cost becomes

$$J_{IH}(x_0, u(\cdot)) = \int_0^{\infty} \bar{l}(x(t)) + u(t)'u(t)dt \tag{11}$$

and the closed-loop system also tolerates a class of *dynamic* uncertainties at the input. This is because the optimality property of $u = \gamma(x)$ is in this case equivalent to a passivity property for the system

$$(H) \qquad \dot{x} = f(x) + g(x)u, \;\; y = -\gamma(x) \tag{12}$$

Connections between optimality and passivity were established in [8] for the linear case and generalized to the nonlinear case in [11]. If $u = \gamma(x)$ is optimal stabilizing for (11), then the input-output system (12) has a shortage of passivity characterized by the fact that it is rendered passive by the output feedback transformation $u = -\frac{1}{2}y + v$. In the linear case, this shortage of passivity translates into the fact that the Nyquist plot of the input-output transfer function of (12) does not enter the circle of radius one and centered at $(-1, 0)$. By analogy, the shortage of passivity of (12) is referred to as a *disk margin* property.

To guarantee the stability of the feedback interconnection in Figure 1, the shortage of passivity of (12) must be compensated for by a sufficient excess of passivity of the uncertainty $\Delta$. This excess of passivity characterizes the class of uncertainties that can be tolerated at the input if the system possess a disk margin. These uncertainties include static sector nonlinearities but also all the linear dynamic uncertainties whose Nyquist plot lies to the right of the vertical axis $s = \frac{1}{2}$.

The above connections between optimality and passivity and their engineering significance for the stability margins of optimal feedback systems are reviewed in the recent monograph [14].

# 5 Conclusions

In this paper we have shown that, under regularity assumptions, stabilizing nonlinear receding control laws which are based on the on-line solution of a *finite* horizon optimal control problem are (inverse) optimal with respect to a modified *infinite* horizon optimal control problem. This inverse optimal result has been obtained by showing that the value function of the finite horizon problem is solution of a stationary HJB equation, the nonlinear analog of the Fake Riccati equation derived for linear systems [5, 13]. The main significance of the inverse optimal property of receding horizon control laws is in the stability margins that it guarantees for the closed-loop system.

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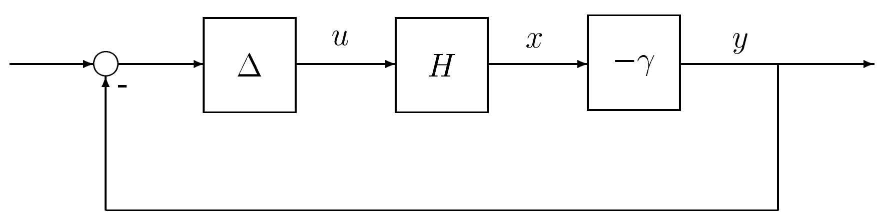


Figure 1: Nonlinear feedback loop with the control law $\gamma(x)$ and input uncertainty $\Delta$.