Lyapunov functions for stable cascades and applications to global stabilization

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Abstract

This paper generalizes recent Lyapunov constructions for a cascade of two nonlinear systems one of which is stable rather than asymptotically stable. A new a cross term construction in the Lyapunov function allows us to replace earlier growth conditions by a necessary boundedness condition. This method is instrumental in the global stabilization of feedforward systems and new stabilization results are derived from the generalized construction.

KEYWORDS: global stabilization, nonlinear systems, forwarding.

1 Introduction

This paper further contributes to the construction of Lyapunov functions for stable cascade systems and its applications to global stabilization results. The motivations for constructing Lyapunov functions for uncontrolled systems which are stable but not asymptotically stable go back to the work by Jacobson [1] and Jurdjevic and Quinn [3], where the Lyapunov function for the uncontrolled system is then employed to design a "damping" controller, that is, a feedback which adds extra dissipation to achieve asymptotic stability.

New such constructions appeared in the recent works [2, 4, 5] on the stabilization of feedfoward systems using a recursive forwarding design. These Lyapunov techniques followed earlier results by Teel [13, 14] on the stabilization of feedforward systems, based on input-output methods [15] (see also [12] for an application of this approach to the problem of the global asymptotic stabilization by bounded feedback of null controllable linear systems).

Instrumental to the forwarding designs of [2, 4, 5] is the construction of a Lyapunov function for the cascade system

$$(\Sigma_0)\left\{\begin{array}{l}\dot{z}=f(z)+\psi(z,\xi),\quad \psi(z,0)=0\\ \dot{\xi}=a(\xi)\end{array}\right. \quad , \quad (z,\xi)\,\in\, R^{n_z}\times R^{n_\xi}$$

for which the following is assumed:

(A1) the equilibrium $\xi=0$ of $\dot{\xi}=a(\xi)$ is globally asymptotically stable (GAS) and locally exponentially stable (LES). A Lyapunov function $U(\xi)$ is known such that $L_aU(\xi)\le 0$ for all ξ.

(A2) the equilibrium $z=0$ of $\dot{z}=f(z)$ is globally stable (GS) and a positive definite radially unbounded function $W(z)$ is known such that $L_fW(z)\le 0$ for all z.

The stability assumption for $\dot{z}=f(z)$ differs from the *asymptotic* stability assumed in earlier stabilization results for cascade systems (see [6, 9, 10, 11]) where the sum of the Lyapunov functions $W(z)$ and $U(\xi)$ is used as a Lyapunov function for the cascade system. This "composite" Lyapunov approach fails in the present case because $L_fW(z)$ is in general *not* negative definite.

The approach initiated in [2] employs a more general Lyapunov function

$$V_0(z,\xi)\;=\;W(z)+\Psi(z,\xi)+U(\xi) \tag{1.1}$$

where the "cross-term" $\Psi(z,\xi)$ is constructed in such a way that $V_0(z,\xi)$ is nonincreasing along the solutions of (Σ_0). In [2], this construction is accomplished under the following two additional assumptions:

(A3) there exist constants $c_1>0$ and $c_2>0$ such that : $||z||\ge c_1\;\Rightarrow\;\left\|\frac{\partial W}{\partial z}(z)\right\|\,||z||\;\le\;c_2W(z)$.

(A4) $||\psi(z,\xi)||\;\le\;\gamma_1(||\xi||)||z||+\gamma_2(||\xi||)$, with γ_1 and γ_2 differentiable positive functions.

These growth assumptions provide a sufficient condition for the boundedness of all solutions of (Σ_0). They are also used in the construction of the cross-term to ensure that the Lyapunov function $V_0(z,\xi)$ is continuous, positive definite and radially unbounded. However, the assumptions (A3) and (A4) are not *necessary* to ensure the global boundedness of the solutions and the question therefore arises whether a cross-term construction can still be achieved under weaker assumptions.

The main contribution of this paper is to show, in Section 2, that the (sufficient) assumptions (A3) and (A4) can be replaced by the (necessary) assumption

(A3bis) All solutions of (Σ_0) exist for all $t\ge 0$ and no solution goes unbounded as $t\to\infty$.

The replacement of assumptions (A3) and (A4) by (A3bis) requires modification of the cross-term construction performed in [2]. This is illustrated by a simple example in Section 2. The more general construction introduced in the present paper exploits the available flexibility in the choice of the cross-term $\Psi(z,\xi)$.

Our Lyapunov construction is also valid for the more general cascade

$$(\Sigma_0')\left\{\begin{array}{l}\dot{z}=f(z)+\psi(z,\xi),\quad \psi(z,0)=0\\ \dot{\xi}=a(\xi,z),\quad a(0,z)=0\end{array}\right.$$

provided that the equilibrium $\xi=0$ of $\dot{\xi}=a(\xi,z)$ is GAS and LES uniformly in z.

The new assumption (A3bis) is the weakest possible assumption to allow for a Lyapunov construction but it is of practical use only if boundedness of the solutions can be verified independently from the solutions. This requires a boundedness criterion, such as, for instance, the previous assumptions (A3) and (A4). In the second part of the paper, we provide a new boundedness criterion which generalizes assumptions (A3) and (A4). As an application, we obtain a new global stabilization result for the cascade:

$$\begin{cases} \dot{z} = f(z) + \psi(z,\xi) + g(z,\xi)u, & \psi(z,0) = 0 \\ \dot{\xi} = u, \end{cases}$$

in which the growth of $\psi(z,\xi)$ with respect to the variable z is assumed to be polynomial whereas a *linear* growth is imposed in [2].

For completeness, we mention that stability results have been recently obtained in [7, 8] for nonautonomous extensions of (Σ_0) under growth assumptions similar to (A3)(A4).

Notations.

1. A C^0 function $\gamma : I\!R^+ \longrightarrow I\!R^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$.
2. We denote by $(\tilde{z}(s), \tilde{\xi}(s))$ the solution of (Σ_0) at time $t = s$ with initial conditions $(\tilde{z}(0), \tilde{\xi}(0)) = (z,\xi) \in I\!R^{n_z} \times I\!R^{n_\xi}$.
3. State vector x is defined by $x = (z,\xi)$.

2 Lyapunov functions for stable cascades

2.1 Construction of the cross-term

We will construct a Lyapunov function of the form (1.1) for (Σ_0) under the sole assumption that no solution of (Σ_0) is unbounded. The time-derivative of V_0, along the solutions of (Σ_0) is:

$$\dot{V}_0 = L_f W + L_\psi W + \dot{\Psi} + L_a U \tag{2.2}$$

The terms $L_f W$ and $L_a U$ are non positive. This implies that $\dot{V}_0$ is non positive if the cross-term $\Psi(z,\xi)$ satisfies

$$\dot{\Psi} + L_\psi W \leq 0 \tag{2.3}$$

In particular, this inequality holds for all Ψ such that

$$\dot{\Psi}(z,\xi) = -L_\psi W(z,\xi) - \gamma(z,\xi)||\xi|| =: -\phi_\gamma(z,\xi) \quad , \quad \Psi(z,0) = 0 \tag{2.4}$$

where γ is any positive function. We then have

$$\dot{V}_0(z,\xi) \leq L_f W(z) - \gamma(z,\xi)||\xi|| + L_a U(\xi) \leq 0$$

We deduce from (2.4) that Ψ is the line-integral of the function ϕ_γ along the solution of (Σ_0) which starts at (z,ξ) :

$$\Psi(z,\xi) = \Psi(\tilde{z}(0), \tilde{\xi}(0)) := \int_0^\infty \phi_\gamma(\tilde{z}(s), \tilde{\xi}(s))\, ds \tag{2.5}$$

The following theorem shows that the integral is well defined and that the resulting V_0 is a Lyapunov function for (Σ_0).

Theorem 1 *(Lyapunov function with a cross-term)*
Assume that ϕ_γ in (2.4) is continuous. Under the assumptions (A1), (A2) and (A3bis), the following holds:

(i) $\Psi(z,\xi)$ exists and is continuous in $I\!R^{n_z} \times I\!R^{n_\xi}$;

(ii) $V_0(z,\xi)$ is positive definite;

(iii) If the function $\gamma(z,\xi)$ is chosen in such a way that

$$\Psi(z,\xi) \geq -\kappa(||\xi||), \quad \forall z \tag{2.6}$$

where κ is a positive continuous function, then $V_0(z,\xi)$ is radially unbounded. In particular, (2.6) is ensured with the choice

$$\gamma(z,\xi) = (L_\psi W)_+ := \begin{cases} -\dfrac{L_\psi W(z,\xi)}{||\xi||} & \text{if} \quad L_\psi W(z,\xi) \leq 0 \\ 0 & \text{if} \quad L_\psi W(z,\xi) > 0 \end{cases}$$

(iv) $V_0(z,\xi)$ is nonincreasing along the solutions of (Σ_0), i.e. the cascade is globally stable.

Proof: The proof is a generalization of Theorem 1 in [2] where only the particular case $\gamma = 0$ is treated.
Proof of (i). The existence and the continuity of Ψ are proven as in [2].
Proof of (ii). Evaluating W along the solutions of (Σ_0) yields:

$$\begin{aligned} W(\tilde{z}(t)) &= W(z) + \int_0^t \frac{d}{ds} W(\tilde{z}(s))ds \\ &= W(z) + \int_0^t L_\psi W(\tilde{z}(s),\tilde{\xi}(s))ds + \int_0^t L_f W(\tilde{z}(s))ds \end{aligned} \tag{2.7}$$

It follows that for all $t \geq 0$:

$$\begin{aligned} W(z) + \int_0^t \phi_\gamma(\tilde{z}(s),\tilde{\xi}(s))ds &= W(\tilde{z}(t)) - \int_0^t L_f W(\tilde{z}(s))ds \\ &\quad + \int_0^t \gamma(\tilde{z}(s),\tilde{\xi}(s))||\tilde{\xi}(s)||ds \end{aligned} \tag{2.8}$$

Taking the limit for $t \to \infty$, we obtain

$$\begin{aligned} W(z) + \Psi(z,\xi) &= \lim_{t\to\infty} W(\tilde{z}(t)) - \int_0^\infty L_f W(\tilde{z}(s))ds \\ &\quad + \int_0^\infty \gamma(\tilde{z}(s),\tilde{\xi}(s))||\tilde{\xi}(s)||ds \end{aligned} \tag{2.9}$$

Since each term of the right hand side of (2.9) is positive, the function $W(z) + \Psi(z,\xi)$ is positive. It follows that

$$V_0(z,\xi) \geq U(\xi) \geq 0 \tag{2.10}$$

So if $V_0(z,\xi) = 0$ then $\xi = 0$. But $V_0(z,0) = W(z)$. So if $V_0(z,0) = 0$ then $W(z) = 0$, which in turn implies that $z = 0$. This proves that V_0 is positive definite.
Proof of (iii). By (2.10) we have:

$$\lim_{||\xi||\to+\infty} V_0(z,\xi) = +\infty \tag{2.11}$$

So to check that V_0 is radially unbounded, it is sufficient to show that whenever $\|\xi\|$ is bounded, V_0 tends to infinity as $\|z\| \to \infty$. With (2.6), the result follows, because $V_0(z,\xi) \geq W(z)+U(\xi)-\kappa(\|\xi\|)$ and $W(z)$ is radially unbounded.

Various choices for the function $\gamma(z,\xi)$ in (2.4) ensure the condition (2.6), therefore guaranteeing that V_0 is radially unbounded. The simplest choice is:

$$\gamma(z,\xi) = (L_\psi W)_+ := \begin{cases} -\dfrac{L_\psi W(z,\xi)}{||\xi||} & \text{if} \quad L_\psi W(z,\xi) \leq 0 \\ 0 & \text{if} \quad L_\psi W(z,\xi) > 0 \end{cases}$$

The resulting $\phi_\gamma = (L_\psi W)_+$ is continuous and the cross-term only integrates the positive part of $L_\psi W$, that is

$$\Psi(z,\xi) \;=\; \int_0^\infty (L_\psi W)_+(\tilde{z}(s),\tilde{\xi}(s))ds \geq 0 \tag{2.12}$$

Proof of (iv). $\dot{V}_0(z,\xi) \leq 0$ by construction. Global stability follows from (i) and (ii). □

In [2], only the choice $\gamma = 0$ is considered and Theorem 1 is proven under two additional assumptions (see Section 2) which guarantee global boundedness of the solutions of (Σ_0). Without these extra assumptions, a choice $\gamma \neq 0$ is in general necessary to guarantee radial unboundedness of V_0, as illustrated by the following example.

Example 1 The system

$$\begin{cases} \dot{z} &= -z^3\xi^2 \\ \dot{\xi} &= -\xi \end{cases}$$

satisfies the assumptions on (Σ_0) with the Lyapunov function $W(z) = \frac{1}{2}z^2$ and $U(\xi) = \frac{1}{2}\xi^2$. The choice $\gamma \equiv 0$ results in

$$\Psi(z,\xi) \;=\; \frac{1}{2}\frac{z^2 - \xi^2 z^4}{1+\xi^2 z^2}$$

and

$$V_0(z,\xi) \;=\; \frac{z^2}{1+\xi^2 z^2} + \xi^2$$

which is not radially unbounded. On the other hand, the choice $\phi_\gamma = (L_\psi W)_+$ results in $\Psi \equiv 0$ and $V_0 = W + U$ is a Lyapunov function for the cascade.

2.2 Smoothness of the cross-term

In [2], C^∞ differentiability of the cross-term Ψ in the case $\gamma = 0$ is established under the following extra assumption on $\dot{z} = f(z)$.
(A5) *The vector field f has the form :*

$$f(z) \;=\; \begin{pmatrix} f_1(z_1) \\ F_2 z_2 + f_2(z_1,z_2) \end{pmatrix}, \quad z \;=\; \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Furthermore, $f_2(0,z_2) = 0$, the equilibrium $z_1 = 0$ of $\dot{z}_1 = f_1(z_1)$ is GAS, and the system $\dot{z}_2 = F_2 z_2$ is Lyapunov stable.

If γ is not identically equal to zero, this differentiability property is unchanged provided that the function ϕ_γ in (2.4) is differentiable. This is not so with the choice $\phi_\gamma = (L_\psi W)_+$, which, in general, is only continuous. A smooth construction is provided by the following proposition.

Proposition 1 *Assume that $W(z)$, $U(\xi)$, and $\psi(z,\xi)$ are smooth functions. Then a function γ such that (2.6) holds can always be selected to be smooth.*

Proof. To ensure the smoothness of ϕ_γ, we choose γ in the form :

$$\gamma(z,\xi) \;=\; \Gamma(2\|x\|^2)\|\xi\| \tag{2.13}$$

where Γ is a positive smooth nondecreasing function to be determined and where x is the full state vector (see Notation). Condition (2.6) then becomes

$$\int_0^\infty L_\psi W(\tilde{x}(s))ds + \int_0^\infty \Gamma(2\|\tilde{x}(s)\|^2)\|\tilde{\xi}(s)\|^2 ds \geq -\kappa(\|\xi\|) \tag{2.14}$$

We prove in Appendix 6 the following fact :

Fact 1 *There exist a smooth function $\gamma_1 \in \mathcal{K}$ and a constant $\lambda > 0$ such that*

$$|L_\psi W(\tilde{x}(s))| \leq \gamma_1(\|x\|)e^{-\lambda s}\|\xi\|$$

Moreover, γ_1 and λ can be determined explicitly, provided that W, U, ψ are known.

By integrating, we obtain the lower bound

$$\int_0^\infty L_\psi W(\tilde{x}(s))ds \geq -\frac{\gamma_1(\|x\|)}{\lambda}\|\xi\| \tag{2.15}$$

To obtain a lower bound for the second term of (2.14), we employ the estimate

$$\|\tilde{x}(s)\|^2 \geq \|x\|^2 e^{-\gamma_2(\|x\|)s} \tag{2.16}$$

where $\gamma_2(.)$ is a smooth strictly positive and increasing function. We prove this inequality in Appendix 7. Using it, we have $\|\tilde{x}(s)\|^2 \geq \frac{\|x\|^2}{2}$ for all $s \in [0, \frac{\ln 2}{\gamma_2(\|x\|)}]$ and hence,

$$\int_0^{\frac{\ln 2}{\gamma_2(\|x\|)}} \Gamma(2\|\tilde{x}(s)\|^2)\|\tilde{\xi}(s)\|^2 ds \geq \Gamma(\|x\|^2)\frac{\ln 2}{4\gamma_2(\|x\|)}\|\xi\|^2 \tag{2.17}$$

Using the lower bounds (2.15) and (2.17), we replace (2.14) by

$$\Gamma(\|x\|^2)\frac{\ln 2}{4\gamma_2(\|x\|)}\|\xi\|^2 - \frac{\gamma_1(\|x\|^2)}{\lambda}\|\xi\| + \kappa(\|\xi\|) \geq 0$$

This inequality is satisfied if we choose $\Gamma(.)$ and $\kappa(.)$ such that

$$(\frac{\gamma_1(\|x\|)}{\lambda})^2 - \Gamma(\|x\|^2)\frac{\ln 2}{\gamma_2(\|x\|)}\kappa(\|\xi\|) \leq 0$$

which holds if we choose $\kappa(\|\xi\|) = \bar{\kappa} > 0$ constant and $\Gamma(.)$ a smooth function which satisfies

$$\Gamma(\|x\|^2) \geq (\frac{\gamma_1(\|x\|)}{\lambda})^2 \frac{\gamma_2(\|x\|)}{(\ln 2)\bar{\kappa}}$$

This choice in (2.13) ensures (2.6), which ends the proof. □

3 Growth conditions for global stability

3.1 A particular class of stable cascade systems

In [2], global stability of (Σ_0) is proved under the assumptions (A3) and (A4). Considering the more general cascade

$$\begin{cases} \dot{z} = f(z) + \psi(z,\xi), & \psi(z,0) = 0 \\ \dot{\xi} = a(\xi, z), & a(0,z) = 0 \end{cases} \tag{3.18}$$

we now relax the linear growth assumption (A4) as follows:

(A4') *The function $\psi(z,\xi)$ satisfies a polynomial growth assumption*

$$\exists k \geq 1 : \quad \|\psi(z,\xi)\| \leq \Gamma_1(\|\xi\|)\|z\|^k + \Gamma_2(\|\xi\|) \tag{3.19}$$

with Γ_1 and Γ_2 differentiable functions of class $\mathcal{K}$. Moreover, there exist $c_1 \geq 1$ and a C^1 function $U_2(\xi)$ positive definite and radially unbounded such that

$$\|(z,\xi)\| \ \geq \ c_1 \ \Rightarrow \ L_a U_2(z,\xi) \ \leq \ -\gamma_3(\|\xi\|) \ - \ \gamma_4(\|\xi\|)\|z\|^{k-1} \tag{3.20}$$

where γ_3 and γ_4 are positive definite continuous functions. Note that Assumption (A4') reduces to (A4) if $k = 1$.

Proposition 2 *Under Assumptions (A1), (A2), (A3), and (A4'), the cascade (3.18) is globally stable. Moreover, the function*

$$V_0(z,\xi) \ = \ W(z) + \Psi(z,\xi) + U(\xi) \tag{3.21}$$

is radially unbounded regardless of the choice of γ in (2.4).

Proof. Because $V_0(z,\xi) \geq U(\xi)$ and $U(.)$ is radially unbounded, it is sufficient to prove that for all ξ

$$\lim_{\|z\| \to +\infty} (W(z) + \Psi(z,\xi)) \ = \ +\infty \tag{3.22}$$

or, using (2.8), that :

$$\lim_{\|z\| \to +\infty} \left(\lim_{t \to +\infty} \left[W(\tilde{z}(t)) \ - \ \int_0^t L_f W(\tilde{z}(s)) ds \right] \right) \ = \ +\infty \tag{3.23}$$

Assumptions (A3) and (A4') imply the existence of positive constants q_1, q_2, and q_3 such that

$$\begin{aligned} \dot{W} - L_f W = L_\psi W \ &\geq \ -\left\| \frac{\partial W}{\partial z} \right\| \|\psi\| \\ &\geq \ -\left\| \frac{\partial W}{\partial z} \right\| \left(\Gamma_1(\|\tilde{\xi}\|)\|\tilde{z}\|^k + \Gamma_2(\|\tilde{\xi}\|) \right) \\ &\geq \ -\left(\left\| \frac{\partial W}{\partial z} \right\| \|\tilde{z}\| + q_1 \right) \left(\Gamma_1(\|\tilde{\xi}\|)\|\tilde{z}\|^{k-1} + \Gamma_2(\|\tilde{\xi}\|) \right) \\ &\geq \ -(q_2 W(\tilde{z}) + q_3) \left(\Gamma_1(\|\tilde{\xi}\|)\|\tilde{z}\|^{k-1} + \Gamma_2(\|\tilde{\xi}\|) \right) \end{aligned}$$

Using inequality (3.20) of Assumption (A4'), let $U_3(\xi) := \int_0^{U_2(\xi)} \chi_1(s) ds$ be a positive definite and radially unbounded function such that

$$\|(z,\xi)\| \geq c_1 \Rightarrow \dot{U}_3 = \chi_1(U_2(\tilde{\xi})) L_a U_2(\tilde{\xi}) \leq -\left(\Gamma_1(\|\tilde{\xi}\|)\|\tilde{z}\|^{k-1} + \Gamma_2(\|\tilde{\xi}\|) \right)$$

Then we have

$$\begin{aligned} \dot{W} - L_f W \ &\geq \ -\left[\Gamma_1(\|\tilde{\xi}\|) c_1^{k-1} + \Gamma_2(\|\tilde{\xi}\|) \right] [q_2 W(\tilde{z}) + q_3] \\ &\quad + \chi_1(U_2(\tilde{\xi})) L_a U_2(\tilde{z},\tilde{\xi}) [q_2 W(\tilde{z}) + q_3] \end{aligned}$$

So, with $\Gamma_3 = q_2 \left[\Gamma_1 c_1^{k-1} + \Gamma_2 \right]$ and $q_4 = \frac{q_3}{q_2}$, we obtain :

$$\dot{W} - L_f W \ \geq \ \left[-\Gamma_3(\|\tilde{\xi}\|) + q_2 \dot{U}_3(\tilde{\xi}) \right] [W(\tilde{z}) + q_4]$$

Defining

$$p(t) := -\Gamma_3(\|\tilde{\xi}(t)\|) + q_2 \dot{U}_3$$

we have $p(t) \leq 0$ for all $t \geq 0$ and

$$\int_0^\infty p(t)dt = -\int_0^\infty \Gamma_3(\|\tilde{\xi}(t)\|)dt - q_2 U_3(\xi) \geq -\Gamma_4(\|\xi\|) \tag{3.24}$$

for some $\Gamma_4 \in \mathcal{K}$. We now integrate the inequality

$$\dot{W} \geq p(t)(W + q_4) + L_f W$$

to obtain

$$W(\tilde{z}(t)) + q_4 \geq e^{\int_0^t p(s)ds}(W(z) + q_4) + \int_0^t e^{\int_s^t p(\tau)d\tau} L_f W(\tilde{z}(s))ds$$

Because $L_f W \leq 0$ and $\exp(\int_s^t p(\tau)d\tau) \leq 1$ for all $s \leq t$, we have

$$W(\tilde{z}(t)) + q_4 \geq e^{\int_0^t p(s)ds}(W(z) + q_4) + \int_0^t L_f W(\tilde{z}(s))ds$$

and hence

$$W(\tilde{z}(t)) - \int_0^t L_f W(\tilde{z}(s))ds \geq e^{\int_0^t p(s)ds}(W(z) + q_4) - q_4$$

This inequality holds for all $t \geq 0$. Taking the limit as $t \to \infty$, we use (3.24) to conclude

$$\lim_{t\to\infty} W(\tilde{z}(t)) - \int_0^\infty L_f W(\tilde{z}(s))ds \geq e^{-\Gamma_4(\|\xi\|)}(W(z) + q_4) - q_4$$

Because $W(z)$ is radially unbounded, the right-hand side is unbounded for each ξ as $\|z\| \to \infty$. This concludes the proof. □

3.2 A stabilization result

A stabilization result will now be deduced from Proposition 2 for the system

$$\begin{cases} \dot{z} = f(z) + h(z,\xi) + g(z,\xi)u, \quad h(z,0) = 0 \\ \dot{\xi} = u, \end{cases} \tag{3.25}$$

for which we assume in addition to (A2), (A3) and (A5), that $h(z,\xi)$ satisfies the polynomial growth property (3.19) and $g(z,\xi)$ satisfies a linear growth property:

$$\|g(z,\xi)\| \leq \kappa_1(\|\xi\|)\|z\| + \kappa_2(\|\xi\|)$$

where the κ_i are positive functions (not necessarily zero at zero).

The preliminary feedback

$$u \;=\; \alpha(z,\xi) \;=\; -(1 + ||z||^{k-1})\xi \tag{3.26}$$

transforms (3.25) into the cascade

$$\begin{cases} \dot{z} = f(z) + h(z,\xi) + g(z,\xi)\alpha(z,\xi) \\ \dot{\xi} = -(1 + ||z||^{k-1})\xi \end{cases} \tag{3.27}$$

and global stability of the equilibrium $(z,\xi) = (0,0)$ follows from Proposition 2. ((A1) and (A4') are satisfied with $U(\xi) = \frac{1}{2}\|\xi\|^2$). By Theorem 1 and Proposition 2, the Lyapunov function $V_0(z,\xi)$ is differentiable, positive definite, radially unbounded, and nonincreasing along the solutions of (3.27).

To achieve global *asymptotic* stability of the equilibrium $(z,\xi) = (0,0)$, we enhance the negativity of $\dot{V}_0$ by augmenting the control law (3.26) in the form

$$u \;=\; \alpha(z,\xi) \,+\, v \;=\; \alpha(z,\xi) \;-\; \frac{\partial V_0}{\partial \xi}(z,\xi) \;-\; \frac{\partial V_0}{\partial z}(z,\xi)g(z,\xi)$$

to obtain

$$\dot{V}_0 \;\leq\; -(1+\|z\|^{k-1})\|\xi\|^2 - \left(\frac{\partial V_0}{\partial \xi}(z,\xi) \,+\, \frac{\partial V_0}{\partial z}(z,\xi)g(z,\xi)\right)^2 \;\leq\; 0$$

This inequality and the LaSalle Invariance Principle imply that $(z,\xi) = (0,0)$ is globally asymptotically stable when $\tilde{z} = 0$ is the unique solution of

$$\begin{aligned} \dot{z} \;&=\; f(z) \\ \tfrac{\partial V_0}{\partial \xi}(z,0) \;-\; \tfrac{\partial V_0}{\partial z}(z,0)g(z,0) \;&=\; 0 \end{aligned}$$

We illustrated the above stabilization procedure in the following example which violates the linear growth assumption (A4) required in [2].

Example 2 The system

$$\left\{ \begin{array}{rcl} \dot{z} &=& (1+z^4)\xi + \theta(1+z^4)^{\frac{3}{4}}\xi^2 + \xi z u \\ \dot{\xi} &=& u \end{array} \right. \tag{3.28}$$

where $\theta \in [-1,1]$ is an unknown parameter satisfies Assumptions (A2) and (A5) (the z-subsystem is $\dot{z} = 0$) but the growth of $(1+z^4)\xi$ is polynomial rather than linear, causing the finite escape time to infinity of some solutions when $u = 0$. To satisfy the growth assumption (A4'), we apply the preliminary feedback

$$u \;=\; -(1+z^4)\xi + v \tag{3.29}$$

and obtain the cascade

$$\left\{ \begin{array}{rcl} \dot{z} &=& (1+z^4)(\xi - \xi^2 z) + \theta(1+z^4)^{\frac{3}{4}}\xi^2 + \xi z v \\ \dot{\xi} &=& -(1+z^4)\xi + v \end{array} \right. \tag{3.30}$$

The choice z^4 rather than $|z^3|$ in (3.29) is adopted to obtain an explicit solution for the cross-term. When $v = 0$, the cascade (3.30) satisfies Assumptions (A1) to (A4') with the Lyapunov functions

$$W(z) \;=\; \frac{1}{2}z^2\,, \quad U(\xi) \;=\; \frac{1}{2}\xi^2 \tag{3.31}$$

Using the fact that

$$L_\psi W \;=\; z(1+z^4)\xi - (1+z^4)\xi^2 z^2 + \theta z(1+z^4)^{\frac{3}{4}}\xi^2 \leq z(1+z^4)\xi + (1+z^4)\xi^2$$

we define the cross-term by

$$\dot{\Psi}(z,\xi) \;=\; -\,\phi_\gamma(z,\xi) \;=\; -\,z(1+z^4)\xi \;-\; (1+z^4)\xi^2 \tag{3.32}$$

Letting $\Psi(z,\xi) = F_1(\xi)z + F_2(\xi)$, we obtain by time differentiation :

$$\dot{\Psi}(z,\xi) \;=\; (1+z^4)\left[-F_1'(\xi)z + F_1(\xi)(1-\xi z) - F_2'(\xi)\right]\xi$$

By identifying this expression with (3.32), we obtain that F_1 and F_2 are solutions of :

$$F_1' = -\xi F_1 + 1,\ F_1(0) = 0$$
$$F_2' = F_1 + \xi^2,\ F_2(0) = 0$$

which yields

$$F_1(\xi) = \int_0^\xi \exp[\frac{1}{2}(s^2 - \xi^2)]ds,\ F_2(\xi) = \int_0^\xi \left[\exp(-\frac{1}{2}s^2)\int_0^s \exp(\frac{1}{2}\tau^2)d\tau\right] ds + \frac{1}{3}\xi^3$$

By construction, when $v = 0$, the time-derivative of V_0 along the solutions of (3.30) is $\dot{V}_0 = -(1 + z^4)\xi^2 z^2 - (1 + z^4)\xi^2 \leq 0$. To enhance its negativity, we employ the additional feedback

$$v = \alpha(z, \xi) = -\frac{\partial V_0}{\partial \xi} - z\xi\frac{\partial V_0}{\partial z} = -\xi - \xi^2 - z - F_2'(\xi) - z^2\xi$$

and obtain $\dot{V}_0 = -(1 + z^4)\xi^2 z^2 - (1 + z^4)\xi^2 - \alpha^2(z, \xi) \leq 0$. Because $\dot{V}_0 = 0$ implies that $\xi = 0$ and $\alpha(z, 0) = -z$, the function $\dot{V}_0$ is negative definite, which proves the global asymptotic stability of the equilibrium point $(z, \xi) = (0, 0)$.

4 Conclusion

We have generalized the cross-term constructions proposed in [2] in order to replace the sufficient growth assumptions (A3)(A4) by the necessary assumption (A3bis). This generalization provides a stabilization procedure in situations not covered by previous results.

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6 Proof of Fact 1

Since $L_\psi W(z,0) = 0$ and both W and ψ are smooth, there exists Ω_1 of class $\mathcal{K}$ such that :

$$|L_\psi W(z,\xi)| \;\leq\; \Omega_1(||x||)||\xi|| \tag{6.33}$$

Since $\dot{\xi} = a(\xi)$ is GAS/LES, there exist Ω_2 of class $\mathcal{K}$ and $\lambda > 0$ such that :

$$||\tilde{\xi}(s)|| \;\leq\; \Omega_2(||\xi||)e^{-\lambda s} \quad , \quad \forall\, s \geq 0 \tag{6.34}$$

Since Σ_0 is globally stable, there exists Ω_3 of class $\mathcal{K}$ such that :

$$||\tilde{x}(s)|| \;\leq\; \Omega_3(||x||) \quad , \quad \forall\, s \geq 0 \tag{6.35}$$

Eventually, by combining (6.33), (6.34), (6.35), we deduce that :

$$|L_\psi W(\tilde{x}(s))| \;\leq\; \gamma_1(||x||)e^{-\lambda s}||\xi|| \quad , \quad \forall\, s \geq 0 \tag{6.36}$$

with $\gamma_1(r) \;=\; \Omega_1(\Omega_3(r))\Omega_2(r)$
Remark : The scalar functions Ω_i used in our proof can be explicitly determined from the data ψ, W, U.

7 Proof of inequality (2.16)

Since the functions f, ψ, a are smooth and zero at the origin, we may determine an explicit expression of a strictly positive function ω_1 such that :

$$\frac{d\|\tilde{x}(s)\|^2}{ds} \geq -\omega_1(\|\tilde{x}(s)\|^2)\|\tilde{x}(s)\|^2 \tag{7.37}$$

Using V_0, we may determine an explicit formula of a strictly positive function ω_2 such that :

$$\|\tilde{x}(s)\|^2 \;\leq\; \omega_2(||x||) \;,\; \forall s \geq 0$$

So, we get :

$$\frac{d\|\tilde{x}(s)\|^2}{ds} \;\geq\; -\omega_1(\omega_2(||x||))\|\tilde{x}(s)\|^2 \tag{7.38}$$

By integrating this inequality, we can prove that (2.16) is satisfied with $\gamma_2(s) \;=\; \omega_1(\omega_2(s))$.