A Sudoku Submatrix Study

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Abstract. In our last article ([1]), we gave some properties of Sudoku matrices. We here investigate some properties of the Sudoku submatrices. We first deal with their eigenvalues and try to find an upper bound on them, and then follow exactly the same structure as in [1] to establish a connection between Sudoku matrices' properties and sudoku submatrices' properties.

1 Introduction

In our last article ([1]), we gave some properties of Sudoku matrices. We here investigate some properties of the Sudoku submatrices. We first deal with their eigenvalues and try to find an upper bound on them, and then follow exactly the same structure as in [1] to establish a connection between Sudoku matrices' properties and sudoku submatrices' properties.

2 Subsudoku Eigenvalues

Prof. Steven Leon from the University of Massachussets Dartmouth said in an e-mail conversation we had, that

for each of the 9 submatrices of a (9×9) Sudoku matrix, the absolute value of any (real) eigenvalue of the submatrix is less than or equal to 22.

This statement will not be proved totally, but we will here come close to this result.

Theorem 1 (Submatrices of a Sudoku matrix are not Sudoku matrices anymore). *The submatrices of a given Sudoku matrix are never Sudoku matrices.* □

Proof. Let's take a $n \times n$ Sudoku matrix, n being the square of an integer. Its submatrices will be $\sqrt{n} \times \sqrt{n}$. The number $\sqrt{n}$ is always in $\mathbb{Z}$ for Sudoku rules. These matrices will be filled with elements in $\{1, \cdots, n\}$, thus leading to a non-Sudoku matrix, by definition. □

We now want to give an upper bound on the eigenvalues of Sudoku matrix's submatrices.

Theorem 2 (Perron). *If all of the entries of a matrix are positive, then the matrix has a dominant eigenvalue that is real and has multiplicity 1.* □

The proof of this theorem is not given here.

Theorem 3 (Max eigenvalue of a square and positive matrix). *The dominant eigenvalue of any square and positive matrix where each row and column have the same sum, will equal that sum.* □

Referring to Theorem 3, we would need to find the related sum, if Sudoku submatrices' rows all had the same sums. But rows do not (necessarily) have the same sums. As a result, we will not be able to use Theorem 3 directly as we did for Sudoku matrices. We will however use a property of matrix norms to find an upper bound on Sudoku submatrices' eigenvalues.

We now give a general upper bound on the Sudoku matrices' submatrices. This bound will be used by specific examples to find particular values.

Theorem 4 (Maximum possible eigenvalue of a Sudoku submatrix). *If S is the matrix of a $n \times n$ Sudoku, with n being a square number, the maximum (bounding) eigenvalue of any of its submatrices, say the kth, S'_k, $k \in \{1, \cdots, n\}$, is, with the writing convention that $S' := S'_k$:*

$$
\max \left\{ \left(\underbrace{\max_{i \in \{1, \cdots, \sqrt{n}\}} \sum_{j=1}^{\sqrt{n}} |S'_{i,j}|}_{=\|S'\|_\infty} \right), \left(\underbrace{\max_{j \in \{1, \cdots, \sqrt{n}\}} \sum_{i=1}^{\sqrt{n}} |S'_{i,j}|}_{=\|S'\|_1} \right) \right\}. \tag{1}
$$

This is an upper bound, and there is not necessarily an eigenvalue of these submatrices which equals this value. □

What this theorem means is that, having chosen a submatrix of size $\sqrt{n} \times \sqrt{n}$, S'_k, and having computed its 1-norm and ∞-norm, we can find an upper bound on its maximum possible eigenvalues.

Proof. If $\lambda_1, \cdots, \lambda_p$ are the eigenvalues of any S'_k submatrix taken from S, it is known (this is property of matrix norms) that

$$
|\lambda_m| \leq \|S'_k\|,
$$

whatever the S'_k (k having the same value in this proof, from the beginning to the end) and the $m \in \{1, \cdots, p\}$, where $\|S'_k\|$ is a norm of S'_k. As, if, to simplify writings, we consider locally $S'_{i,j} = S'_{k_{i,j}}$,

$$
\left(\underbrace{\max_{i \in \{1, \cdots, \sqrt{n}\}} \sum_{j=1}^{\sqrt{n}} |S'_{i,j}|}_{=\|S'\|_\infty} \right) \text{ and } \left(\underbrace{\max_{j \in \{1, \cdots, \sqrt{n}\}} \sum_{i=1}^{\sqrt{n}} |S'_{i,j}|}_{=\|S'\|_1} \right)
$$

are both norms (the first is the maximal sum on a row of S', and the second is the maximal sum on a column of S'), they verify

$$
|\lambda_m| \leq \|S'_k\|.
$$

□

Remark 1. Let us now consider

$$
P := \sum_{i=0}^{\sqrt{n}-1} (n - i). \tag{2}
$$

We want to find a relationship between this sum and the aforementioned norms. If the concerned Sudoku submatrix was filled with elements so that $\|S'_k\|_\infty = \|S'_k\|_1$, we could write that an upper bound on the the eigenvalues of S_k would be given by either norm.

But both norms could not equal P. □

Proof. 1. It is possible to have a Sudoku matrix so that one of its submatrices verifies $\|S'_k\|_\infty = \|S'_k\|_1$. Take for example $n = 9$: $\sqrt{9} = 3$, and we have

$$\begin{pmatrix} 9 & 7 & 5 \\ 8 & 1 & 2 \\ 4 & 3 & 6 \end{pmatrix}$$

whose ∞ and 1-norms are equal. Same goes for

$$\begin{pmatrix} 9 & 7 & 6 \\ 8 & 1 & 2 \\ 5 & 3 & 4 \end{pmatrix}$$

whose both norms amount to 22.

2. It is impossible to have a Sudoku matrix so that one of its submatrices verifies $\|S'_k\|_\infty = \|S'_k\|_1 = P$. That would need to use $\sqrt{n}$ biggest elements amongst $\{1, \cdots, n\}$, twice, which would go against Sudoku rules. Moreover, if one norm amounts to P, the other norm is smaller than P.

□

Corollary 1. *If S is the matrix of a $n \times n$ Sudoku, with n being a square number, a maximum bound on the eigenvalues of one of all its submatrices S'_k, $k \in \{1, \cdots, n\}$, if we consider that $S'_k = S'$, is*

$$\max\left\{\left(\underbrace{\max_{i\in\{1,\cdots,\sqrt{n}\}} \sum_{j=1}^{\sqrt{n}} \left|S'_{i,j}\right|}_{=\|S'\|_\infty}\right), \left(\underbrace{\max_{j\in\{1,\cdots,\sqrt{n}\}} \sum_{i=1}^{\sqrt{n}} \left|S'_{i,j}\right|}_{=\|S'\|_1}\right)\right\}. \tag{3}$$

Specifically,

– *If both norms are equal, an upper bound on the maximum eigenvalue is given by either norm,*
– *If one norm amounts to P, P is an upper bound on the maximum eigenvalue.*

□

Remark 2. We know that if $a \leq b$ and $a \leq c$, $a \leq \min(b, c)$ is true, but a less stringent condition is that $a \leq \max(b, c)$ (which is true too). As a result, one might replace max by min in our preceding upper bounds. Consequently, if one norm amounts to P, knowing that the other norm cannot amount to P (because it is strictly inferior to) gives the following result, as we now take the min between both norms. □

Corollary 2. *If one norm amounts to P, an upper bound on the eigenvalues of the submatrices is strictly inferior to P.* □

Remark 3. We have

$$\sum_{i=0}^{\sqrt{n}-1} (n - i) = \sum_{i=0}^{\sqrt{n}-1} n - \sum_{i=0}^{\sqrt{n}-1} i \tag{4}$$

$$= (\sqrt{n} - 1 + 1)n - \frac{(\sqrt{n} - 1)(\sqrt{n})}{2} \tag{5}$$

$$= \frac{\sqrt{n}}{2}\left(2n - (\sqrt{n} - 1)\right). \tag{6}$$

□

Without reasoning on numerical cases, it is difficult to find the minimum of two norms. As a result, we first illustrate the values on the upper bound

$$\sum_{i=0}^{\sqrt{n}-1} (n-i)$$

of the 9×9 case. We saw before that if one norm equals P, an upper bound on the eigenvalues was strictly inferior to P. We can here quantify the 'strictly inferior' in different examples.

Example 1 (9×9 Sudoku matrices). If we take a 9×9 Sudoku matrix, having 3×3 submatrices (which are not Sudoku), the maximal sum one can encounter on a column or a row of a submatrix is

$$\underbrace{n}_{=9} + \underbrace{(n-1)}_{=8} + \underbrace{n - (\sqrt{n} - 1)}_{=9-(\sqrt{9}-1)=9-(2)=7} = 24.$$

Consequently, the maximal eigenvalue one can find in a S'_k, $k \in \{1, \cdots, 9\}$ is 24. However, this is not a low upper bound, as considering

$$\begin{pmatrix} 9 & 7 & 6 \\ 8 & 1 & 2 \\ 5 & 3 & 4 \end{pmatrix} \tag{7}$$

gives two norms of 22, which is, according to the way the matrix was created, a lower upper bound on any $\sqrt{9} \times \sqrt{9}$ Sudoku submatrix. □

Example 2 (16×16 Sudoku matrices). In the case of 16×16 Sudoku matrices, 4×4 Sudoku submatrices could be like

$$\begin{pmatrix} 16 & 14 & 13 & 11 \\ 15 & & & \\ 12 & & & \\ 10 & & & \end{pmatrix} \tag{8}$$

which gives $\|S'_k\|_\infty = 54$ and $\|S'_k\|_1 = 53$, thus an upper bound of 53 (because we take the minimum between both norms). □

Example 3 (25×25 Sudoku matrices). A same method can be applied for the case of a 25×25 Sudoku matrix, giving 104 as an upper bound. □

Remark 4. The graphical representation of the couples $((3,4,5); (22,53,104))$ which were given at Corollary 5, gives a piecewise line. The related function might be refined by experiments on higher n values. A first plot is given at Figure 1. □

Theorem 5 (Maximum eigenvalue of a Sudoku submatrix). *If S is the matrix of a $n \times n$ Sudoku, with n being a square number, the maximum (bounding) eigenvalue of any of its submatrices is,*

- *if $n = 9$, 22,*
- *if $n = 16$, 53,*
- *if $n = 25$, 104.*

□

Proof. The Sudoku submatrices given at Eqs. 7, 8 and Example 3 are, for each matrix format (3×3, 4×4 and 5×5) the 'most stringent' matrices.
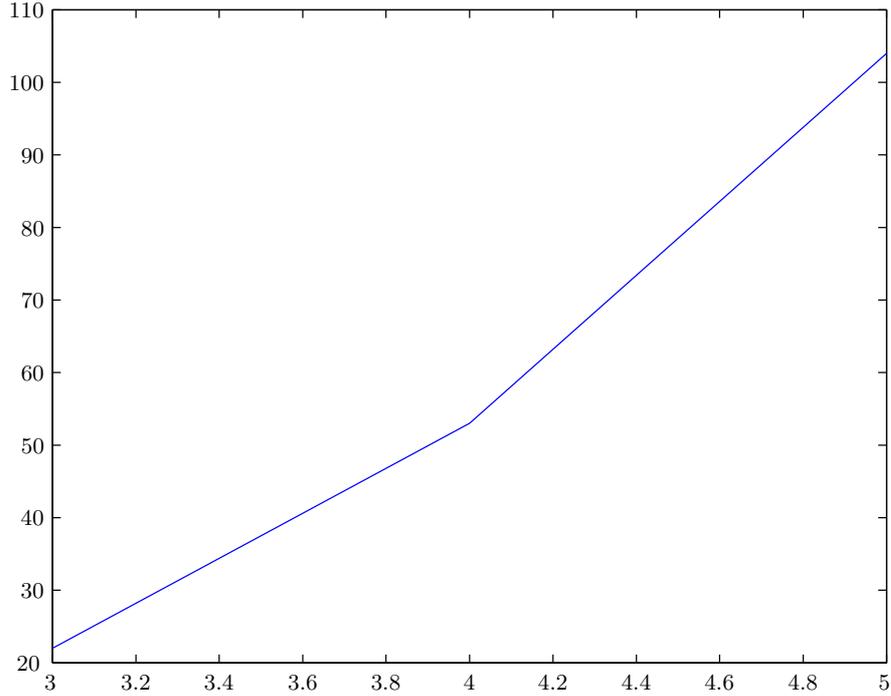
Fig. 1. Evolution of lower upper bounds (for eigenvalues) on Sudoku $\sqrt{n} \times \sqrt{n}$ submatrices, depending on $\sqrt{n}$.

3 Subsudoku Determinants

Theorem 6 (Determinant of a sudoku submatrix can be 0 or differ from 0). *The determinant of a sudoku submatrix can equal 0 or not.* $\qquad\square$

Proof. Consider

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0 \qquad \text{and} \qquad \begin{vmatrix} 4 & 3 & 2 \\ 5 & 6 & 1 \\ 8 & 7 & 9 \end{vmatrix} = 51.$$

$\qquad\square$

Theorem 7. $\det(\widetilde{S'_k}) = \det(S'_k)$, *where S'_k is a kth submatrix (of size $\sqrt{n} \times \sqrt{n}$) taken from a Sudoku matrix S of size $n \times n$, with $1 \leq k \leq n$.* $\qquad\square$

4 Subsudoku Ranks

Conjecture 1. The rank ρ of a Sudoku submatrix of size $\sqrt{n} \times \sqrt{n}$ verifies

$$\rho > 1. \tag{9}$$

$\qquad\square$

Suggestion 1 *The n elements composing the $\sqrt{n} \times \sqrt{n}$ Sudoku submatrix need to lie in $\{1, \cdots, n\}$. As a result, they need to be integers. One might first try to create the matrix*

$$\begin{pmatrix} 1 \ 2 \ 3 \end{pmatrix}$$

where the second and third rows' elements could be integer multiples of the first row's elements, but that would lead to a same element appearing at least twice in the matrix (consider a 2 coefficient: $1 \times 2 = 2$).

Another solution would be to begin with

$$\begin{pmatrix} 1 \ 3 \ 4 \\ 2 \ 6 \ 8 \end{pmatrix}$$

and then fill the last row, but there is no combination (using elements 5, 7, 9) which would yield to a rank 1-matrix. □

Corollary 3. *The rank ρ of a Sudoku submatrix of size $\sqrt{n} \times \sqrt{n}$ verifies*

$$1 \overset{?}{<} \rho \leq \sqrt{n}.$$

□

Proof. – Example where $\rho = \sqrt{n} - 1$. Using the submatrix

$$\begin{vmatrix} 1 \ 2 \ 3 \\ 4 \ 5 \ 6 \\ 7 \ 8 \ 9 \end{vmatrix} = 0$$

one sees that such a submatrix can also be non-full-rank. But we have e.g.

$$\begin{vmatrix} 1 \ 2 \\ 4 \ 5 \end{vmatrix} = 5 - 8 \neq 0.$$

Thus, the submatrix is of rank 2.

– Example where $\rho = \sqrt{n}$. Using

$$\begin{vmatrix} 4 \ 3 \ 2 \\ 5 \ 6 \ 1 \\ 8 \ 7 \ 9 \end{vmatrix} \neq 0$$

one sees that such a submatrix is full rank.

5   Transpose

5.1   General Considerations

Theorem 8 (Block transposition can loose Sudoku character). *Consider a Sudoku matrix S with submatrices S'_k. Transposition of each block can lead to an erroneous (in the Sudoku sense) Sudoku matrix, i.e. submatrices' transpositions can lead to an erroneous (in the Sudoku sense) Sudoku matrix.* □

Proof. Consider a part of a Sudoku matrix such as

$$\begin{pmatrix} 6 \ 1 \ 5 \ 8 \ 4 \ 9 \\ 3 \ 8 \ 7 \ 2 \ 5 \ 1 \\ 2 \ 9 \ 4 \ 3 \ 7 \ 6 \end{pmatrix}.$$

3×3 block transposition gives

$$\begin{pmatrix} 6 \ 3 \ 2 \ 8 \ {\color{red}2} \ 3 \\ 1 \ 8 \ 9 \ 4 \ 5 \ 7 \\ 5 \ 7 \ 4 \ 9 \ 1 \ 6 \end{pmatrix}$$

which would, if it aimed at constituting a Sudoku matrix, create an erroneous Sudoku matrix.

Theorem 9. *We have $S'_k \neq \widetilde{S'_k}$.* □

Proof. S'_k would have to be symmetrical, or every value from 1 to n needs to be placed in a S'_k, once.

5.2 Trace

Theorem 10. *We have $\mathrm{Tr}(\widetilde{S'_k}) = \mathrm{Tr}(S'_k)$.* □

6 Non-Hermitianity

Theorem 11. *No sudoku submatrix can be Hermitian.* □

Proof. Trivial using Theorem 9.

7 Non-Normality

Theorem 12. *Sudoku submatrices are not normal.* □

Proof. We here use partly the proof of [1].

Let's consider a Sudoku matrix S. If we have

$$S = \begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,n} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n,1} & S_{n,2} & \cdots & S_{n,n} \end{pmatrix},$$

we can take from it a $\sqrt{n} \times \sqrt{n}$ block, say the upper-left one, S'_k. We have normality of S'_k iff $S'_k \widetilde{S'_k} = \widetilde{S'_k} S'_k$. That means that we need

$$\begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,\sqrt{n}} \\ S_{2,1} & \ddots & \cdots & S_{2,\sqrt{n}} \\ \vdots & \cdots & \ddots & \vdots \\ S_{\sqrt{n},1} & \cdots & \cdots & S_{\sqrt{n},\sqrt{n}} \end{pmatrix} \begin{pmatrix} S_{1,1} & S_{2,1} & \cdots & S_{\sqrt{n},1} \\ S_{1,2} & \ddots & \cdots & S_{\sqrt{n},2} \\ \vdots & \cdots & \ddots & \vdots \\ S_{1,\sqrt{n}} & \cdots & \cdots & S_{\sqrt{n},\sqrt{n}} \end{pmatrix} = \begin{pmatrix} S_{1,1} & S_{2,1} & \cdots & S_{\sqrt{n},1} \\ S_{1,2} & \ddots & \cdots & S_{\sqrt{n},2} \\ \vdots & \cdots & \ddots & \vdots \\ S_{1,\sqrt{n}} & \cdots & \cdots & S_{\sqrt{n},\sqrt{n}} \end{pmatrix} \begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,\sqrt{n}} \\ S_{2,1} & \ddots & \cdots & S_{2,\sqrt{n}} \\ \vdots & \cdots & \ddots & \vdots \\ S_{\sqrt{n},1} & \cdots & \cdots & S_{\sqrt{n},\sqrt{n}} \end{pmatrix}.$$

To respect these equalities, the first equality to establish is $S_{1,\sqrt{n}} = S_{\sqrt{n},1}$, which would transgress the rules of Sudoku if it was verified. □

8 Orthogonality Discussion

A matrix C is *orthogonal* if and only if

$$\widetilde{C}C = I_n.$$

Theorem 13 (Non-orthogonality of S'_k if $\det(S'_k) = 0$). *If $\det(S'_k) = 0$, S'_k cannot be orthogonal.* □

Proof. We have

$$\det(\widetilde{S'_k} \cdot S'_k) = \det(\widetilde{S'_k})\det(S'_k),$$

which equals 0 if $\det(S'_k) = 0$, and $\det(I_n) = 1$. □

Conjecture 2 (Non-orthogonality of S'_k if $\det(S'_k) \neq 0$). If $\det(S'_k) \neq 0$, S'_k cannot be orthogonal. □

Suggestion 2 *The only way to make S'_k orthogonal is to have $\widetilde{S'_k} = (S'_k)^{-1}$. Or this equality should be impossible: as*

$$(S'_k)^{-1} = \widetilde{\mathrm{cofact}(S'_k)} \cdot (\det(S'_k))^{-1},$$

it would be equivalent to ask

$$\widetilde{S'_k} = \widetilde{\mathrm{cofact}(S'_k)} \cdot (\det(S'_k))^{-1},$$

which should never be true for a S'_k Sudoku submatrix. □

9 Condition Number

Theorem 14 (No well-conditioning for S'_k can arise). *The system $S'_k x = b$ can be else than well-conditioned (assuming $b \neq \mathbf{0}_n$).* □

Proof. Consider a Sudoku matrix S whose kth submatrix is

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Its condition number for the infinity norm, i.e. κ_∞ is approximatively 8.6469×10^{17}. This is justified, because its determinant equals 0. □

10 Trace

Theorem 15 (The trace of a Subsudoku matrix is not constant). *The trace of a Subsudoku matrix is not constant.* □

Proof. Consider

$$\begin{pmatrix} \boxed{6} & 7 & 3 & 5 & 1 & 4 & 9 & 8 & 2 \\ 9 & \boxed{2} & 1 & 3 & 6 & 8 & 7 & 5 & 4 \\ 5 & 8 & \boxed{4} & 7 & 9 & 2 & 1 & 3 & 6 \\ 8 & 9 & 6 & 2 & 3 & 7 & 4 & 1 & 5 \\ 2 & 1 & 5 & 9 & 4 & 6 & 3 & 7 & 8 \\ 3 & 4 & 7 & 1 & 8 & 5 & 2 & 6 & 9 \\ 4 & 5 & 2 & 8 & 7 & 1 & 6 & 9 & 3 \\ 1 & 6 & 9 & 4 & 5 & 3 & 8 & 2 & 7 \\ 7 & 3 & 8 & 6 & 2 & 9 & 5 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \boxed{9} & 4 & 7 & 2 & 5 & 8 & 1 & 3 & 6 \\ 1 & \boxed{2} & 3 & 4 & 6 & 7 & 9 & 8 & 5 \\ 6 & 5 & \boxed{8} & 1 & 9 & 3 & 7 & 2 & 4 \\ 8 & 9 & 5 & 6 & 4 & 2 & 3 & 7 & 1 \\ 7 & 6 & 4 & 9 & 3 & 1 & 2 & 5 & 8 \\ 3 & 1 & 2 & 8 & 7 & 5 & 4 & 6 & 9 \\ 4 & 8 & 9 & 7 & 2 & 6 & 5 & 1 & 3 \\ 2 & 3 & 6 & 5 & 1 & 9 & 8 & 4 & 7 \\ 5 & 7 & 1 & 3 & 8 & 4 & 6 & 9 & 2 \end{pmatrix},$$

whose 3×3 respective first Sudoku submatrices (whose diagonal elements are boxed)' traces evaluate to 12 and 19, respectively. □

11 Comparison Between Sudokus and Subsudokus

We here compare the interesting properties that can be compared, between Sudokus and Subsudokus. Here, 'YES' means that the property is always verified, 'NO' that it is never verified, and 'NN' that it is not necessarily verified. Comparisons are shown at Table 1.

Property	Sudoku	Subsudoku
Sudoku	YES	NO
det can also equal 0	YES	YES
(Entire) divisibility of det by $\frac{n(n+1)}{2}$	YES	NO
Transpose is (valid) Sudoku	YES	NO
Rank ρ	[2]	$1 \overset{?}{<} \rho \leq \sqrt{n}$
Hermitian	NO	NO
Normal	NO	NO
Orthogonal (det $S \neq 0$)	?	?
$\kappa_\infty \gg$	NN	NN
Trace is not constant	YES	YES

Table 1. Comparison table between Sudokus and Subsodokus properties.

References

1. L. MERCIADRI, *A Sudoku Matrix Study: Determinant, Golden Ratio, Eigenvalues, Transpose, Non-Hermitianity, Non-Normality, Orthogonality Discussion, Order, Condition Number.* 2009.
2. L. MERCIADRI AND C. EASTHAM, *Sudokus' Ranks.* 2010.