Oscillators as Systems and Synchrony as a Design Principle*

Rodolphe Sepulchre

Electrical Engineering and Computer Science Institute
Montefiore B28, B-4000 Liège, Belgium
r.sepulchre@ulg.ac.be

Summary. The chapter presents an expository survey of ongoing research by the author on a system theory for oscillators. Oscillators are regarded as open systems that can be interconnected to robustly stabilize ensemble phenomena characterized by a certain level of synchrony. The first part of the chapter provides examples of design (stabilization) problems in which synchrony plays an important role. The second part of the chapter shows that dissipativity theory provides an interconnection theory for oscillators.

1 Introduction

Oscillators are dynamical systems that exhibit stable limit cycle oscillations. The emphasis in this chapter is on oscillators as *open* systems, that is, as systems that can be interconnected to other systems. Synchrony refers to the tendency of interconnected oscillators to produce ensemble phenomena, that is, to phase lock as if an invisible conductor was orchestrating them. The emphasis in this chapter is on synchrony as a design principle, that is, on the use of synchrony to achieve stable oscillations in interconnected systems.

The manifestations of synchrony are numerous both in nature and in engineered devices. The interested reader will find several compelling illustrations in the stimulating recent essay by Strogatz [30]. As narrated in this essay and elsewhere, the accidental discovery by Huygens that two clocks in the same room tend to synchronize was soon regarded as the discovery of an undesirable phenomenon, revealing the sensitivity of clocks to external small perturbations at a time where the challenge was to engineer robust devices that could travel the ocean and provide a precise measure of longitude. Today, the growing interest for synchrony in engineering applications is precisely due

*This chapter presents research results of the Belgian Program on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with its author.

to the robustness of collective phenomena, making an ensemble phenomenon insensitive to individual failures. In this sense, synchrony is a system concept.

In order to become a design principle, synchrony requires an interconnection theory for oscillators. Detailed models of oscillators abound in the literature, most frequently in the form of a set of nonlinear differential equations whose solutions robustly converge to a limit cycle oscillation. Local stability analysis is possible by means of Floquet theory, but global convergence analysis is usually restricted to second-order models and uses phase plane techniques. When analyzing collective phenomena in possibly large ensembles of interconnected oscillators, the dynamical model for each oscillator is usually further simplified, such as in phase models [35] where the state variable of each oscillator is a single phase variable on the circle.

The objective of this chapter is twofold: first, to motivate the use of synchrony as a design principle and the need for an interconnection theory of oscillators; second, to propose an external characterization of oscillators based on dissipativity theory and to examine its implications for the stability and synchrony analysis of interconnected oscillators.

In the first part of the chapter, we describe two stabilization problems in which synchrony plays an important role. Section 2 studies the stabilization of a bounce juggler, illustrative of rhythmic control tasks encountered in multileg robotics. We show how the stabilization of period two orbits (which mimics the shower pattern of a juggler) is best understood as achieving a phase-locking property for two impact oscillators. A distinctive feature of the proposed control is that it uses no feedback (sensorless control), even though the orbit is exponentially unstable in the unactuated system. The phase-locked property of the impact oscillators is induced by suitable oscillatory forcing of their input.

Section 3 describes a collective stabilization problem for N particles that move at unit speed in the plane with steering control. (Relative) equilibria of the model correspond to parallel or circular motions of the group. The orientation of each particle is a phase variable on the circle. Treating the orientation variables of the particles as phase variables of oscillators, parallel motion corresponds to synchronization whereas circular motion can be understood as a form of desynchronization. The synchrony measure is here the velocity of the center of mass of the group. It is maximal in parallel motions and minimal in circular motions. It coincides with a usual measure of synchrony in phase models of oscillators [31].

The two examples illustrate the role of synchrony as a design principle. Their ad hoc treatment also underlines the lack of interconnection theory for oscillators. In Section 4, we further illustrate with models from neurodynamics the persistent gap between physical models of oscillators and abstract models used to study their interconnections. This prompts us to introduce in Section 5 an external characterization of oscillators that fits their description by physical state space models but at the same time has implications for the stability and synchrony analysis of their interconnections. Following the dissipativity approach introduced by Willems [33], the external characterization is in the form of a dissipation inequality, with a new supply rate enabling a limit cycle behavior for the solutions of the isolated oscillator. We examine the implications of this dissipativity characterization for (i) (global) stability analysis of an isolated oscillator, (ii) (global) analysis of interconnections of N identical oscillators, and (iii) (global) synchrony analysis of interconnections of N identical oscillators. The theory covers two basic oscillation mechanisms, illustrated in the simplest way by van der Pol model and by Fitzhugh–Nagumo model, respectively.

This chapter should be regarded as an expository survey of ongoing research. The simplest examples are employed to illustrate the concepts and we refer the reader to more technical papers for the general treatment and for complete statements of the results.

2 Sensorless stabilization of rhythmic tasks

Synchrony plays an essential role in the robust coordination of rhythmic tasks. Neuroscientists have long identified the role of central pattern generators in living organisms as autonomous neural clocks that provide the rhythmic signals necessary to coordinate multileg locomotion such as walking, hopping, or swimming. We describe here a manifestation of synchrony as a design principle in a very contrived but illuminating example: the stabilization of periodic orbits in the bounce juggler model illustrated in Figure 1.

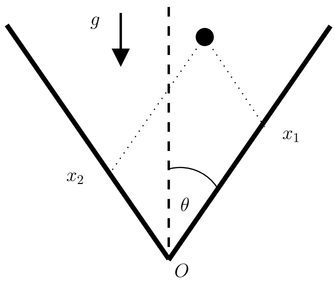

Fig. 1. A bounce juggler model.

This toy stabilization problem captures several important issues of impact control problems and is the subject of ongoing research [23, 3, 20, 21]. Here we only describe the problem in its simplest configuration and underline the phase-locking properties of the stabilized system.

The bounce juggler model describes the dynamics of a point mass (ball) in the plane under the action of a constant gravitational field. The ball undergoes elastic collisions with two intersecting edges, an idealization of the juggler's two arms. The two edges form a fixed angle θ with the direction of gravity. In a coordinate system aligned with the edges, the (nondimensional) equations of motion write

$$\begin{aligned} \ddot{x}_1 &= -1, \; x_1 \geq 0 \\ \ddot{x}_2 &= -1, \; x_2 \geq 0. \end{aligned} \tag{1}$$

A collision occurs when either x_1 or x_2 becomes zero, in which case Newton's rule is applied, that is, the normal velocity is reversed and the tangential velocity is conserved. The solution of (1) is then continued (restarted) with this new initial condition until a new collision occurs. The system is conservative (energy is conserved both during the frictionless flight of the ball and through the elastic collisions), and, except for the collision times, the two degrees of freedom of the system are decoupled. In spite of this apparent simplicity, this Hamiltonian system with collisions exhibits very rich dynamics [13, 36]. In particular, for $\theta > 45\,\mathrm{deg}$, the system possesses an infinite family of periodic orbits, each of which is exponentially unstable. In recent work [20, 21], we have shown that period one and period two orbits can be stabilized by proper oscillatory actuation of the wedge. This sensorless stabilization phenomenon is rather surprising because an exponentially unstable periodic orbit of the unactuated wedge becomes exponentially stable in the actuated wedge in spite of any feedback measurement. The analysis in [20, 21] shows that the phenomenon persists over a broad range of angles θ and, when the collisions are nonelastic, over a broad range of coefficients of restitution. Recent experimental validation of this sensorless stabilization suggests that the phenomenon is also quite robust.

The essence of the just-described sensorless stabilization phenomenon is best understood by considering the model in the special configuration illustrated in Figure 2, that is, for the particular angle $\theta = 45\,\mathrm{deg}$ (orthogonal wedge), and when the actuation of each edge is restricted to the direction orthogonal to the edge.

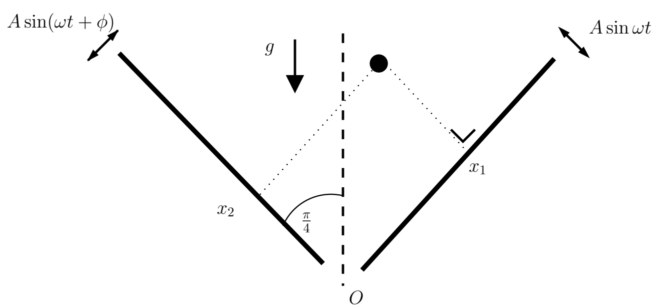

Fig. 2. A cartoon of the orthogonal wedge with orthogonal actuation of the two edges, leading to decoupled closed-loop dynamics.

In this special configuration, the dynamics of the vibrating wedge write

$$\begin{aligned} \ddot{x}_1 &= -1, \quad x_1 \geq A\sin(\omega t) \\ \ddot{x}_2 &= -1, \quad x_2 \geq A\sin(\omega t + \phi), \end{aligned} \tag{2}$$

where A and ω are the amplitude and the vibrating frequency of each edge and where ϕ is the phase shift between the vibration of the two edges.

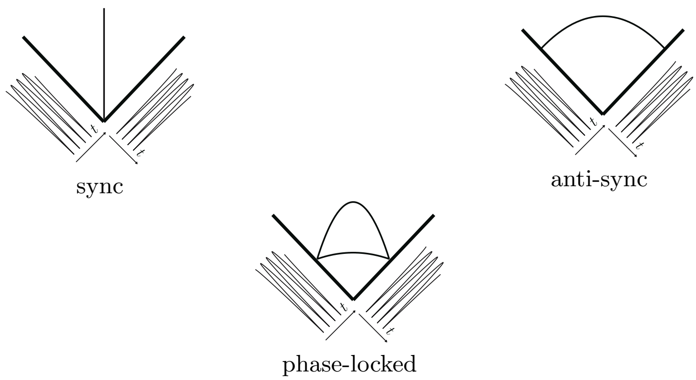

Fig. 3. The equivalence between period two orbits of the orthogonal wedge and two phase-locked impact oscillators (bouncing balls).

Because the two edges vibrate orthogonally to each other, the collision rules reduce to the very simple expression:

$$\begin{aligned} t: &\quad x_1(t) = A\sin\omega t \Rightarrow \dot{x}_1(t^+) = -e\,\dot{x}_1(t^-) \\ t: &\quad x_2(t) = A\sin(\omega t + \phi) \Rightarrow \dot{x}_2(t^+) = -e\,\dot{x}_2(t^-),\ 0 < e < 1, \end{aligned} \tag{3}$$

which means that the two degree-of-freedom dynamics of the bounce juggler decouple into two one degree-of-freedom dynamics. The dynamics of each one degree-of-freedom subsystem are the bouncing ball dynamics first studied by Holmes [6]. It is well known that the bouncing ball dynamics exhibit a stable period one orbit in a suitable parameter range of A (or ω), within which the period between two successive collisions locks with a multiple of the forcing period $T = \frac{2\pi}{\omega}$. In the same parameter range, the orthogonally vibrating bounce juggler exhibits a stable period two orbit. The additional parameter ϕ determines the phase shift between the collisions with each edge, as illustrated in Figure 3.

The period two orbit of the orthogonally vibrating bounce juggler is thus equivalent to the period one motion of two phase-locked bouncing balls or impact oscillators. Because of the coupling, this transparent description of the dynamics is lost when the edges are not orthogonal to each other and when the actuation of the wedge is an oscillatory motion around the fixed vertex instead of an axial vibration of each edge separately. The analysis in [20, 21] nevertheless shows that the exponential stabilization of period two orbits persists over a broad range of parameters, even in this generalized situation.

Stable and unstable periodic orbits have been recently described in models of insect locomotion [7]. These models provide practical and relevant examples of Hamiltonian systems in which the different degrees of freedom are coupled only through collisions. Our current work investigates whether periodic forcing of some parameters can act as a (sensorless) stabilization mechanism in these models similarly to the bounce juggler example described in this chapter. To the best of our knowledge, the stability analysis in all reported examples in the literature is based on (tedious) calculations of Poincare maps that can be determined analytically only in overly simplified situations. A general interconnection theory for such rhythmic oscillators is lacking at the present time.

3 Collective stabilization

Another illustration of synchrony as a design principle is the task of stabilizing a large collection of identical control systems (agents) around a collective motion. This problem has received considerable attention over the last years (see, e.g., [11] and the references therein) and includes numerous engineering applications in unmanned sensor platforms. For example, autonomous underwater vehicles (AUVs) are used to collect oceanographic measurements in network formations that maximize the information intake, see, e.g., [15]. In ongoing work [25, 17, 26], we study a continuous-time kinematic model of N identical, self-propelled particles subject to planar steering controls, first considered in [9, 10]. In complex notation, this model is given by

$$\dot{r}_k = e^{i\theta_k} \tag{4}$$

$$\dot{\theta}_k = u_k, \tag{5}$$

where $r_k \in \mathbb{R}^2$ and $\theta_k \in S^1$ are the position and heading of the kth particle. Unless otherwise indicated, $k = 1, \ldots, N$. The steering control law is denoted by u_k. If we define the relative position and orientation variables, $r_{jk} = r_j - r_k$ and $\theta_{jk} = \theta_j - \theta_k$, then the control, u_k, can be decomposed into relative spacing and alignment terms, i.e.,

$$u_k = u_k^{spac}(r_{jk}, \theta_{jk}) + u_k^{align}(\theta_{jk}). \tag{6}$$

The alignment control is a function of the relative orientation, θ_{jk}, whereas the spacing control is function of both the relative position r_{jk}, and orientation θ_{jk}.

The particle model is a specialization of the Frenet-Serret equations in $SE(2)$, the group of planar rigid motions, restricted to a constant velocity, see, e.g., [22]. The Lie group structure of the state space has important implications. If the control law (6) depends only on the relative orientations and positions of the particles, then the system is invariant under the action of the group $SE(2)$ (i.e., there is a planar rotation and translation symmetry). Under this assumption, the configuration space of the particles can be described on a reduced shape space. Justh and Krishnaprasad [9] show that fixed points in the shape space, which correspond to relative equilibria, occur for

$$u_1 = u_2 = \ldots = u_N. \tag{7}$$

In particular, the relative equilibrium with $u_1 = u_2 = \ldots = u_N = 0$ results in parallel trajectories of the group; the relative equilibrium with $u_1 = u_2 = \ldots = u_N \neq 0$ results in all the vehicles orbiting the same point at the same (constant) radius. The control problem is to design a feedback (6) that stabilizes a particular relative equilibrium of the model, that necessarily correspond either to a parallel motion or a circular motion for the group.

A key parameter for the stabilization of the group is the velocity of the center of mass

$$v = |\dot{R}| = |\frac{1}{N}\sum_k \dot{r}_k| = |\frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}|. \tag{8}$$

The velocity v is maximal ($v = 1$) for parallel motion whereas it is minimal ($v = 0$) for circular motion around the (fixed) center of mass. This suggests to control the potential

$$U = \frac{N}{2}v^2 = \frac{1}{2N}\sum_{k=1}^{N}\sum_{j=1}^{N} \cos\theta_{kj}.$$

Gradient dynamics with respect to U yield

$$\dot{\theta}_k = K\frac{\partial U}{\partial \theta_k} = -\frac{K}{N}\sum_{j=1}^{N} \sin(\theta_k - \theta_j). \tag{9}$$

The only critical points of U are its minima, corresponding to $v = 0$, that is, motions around a fixed center of mass, and its maxima, corresponding to $v = 1$, that is, parallel motion.

The parameter v thus provides a good measure of synchrony for the group. Its interpretation in connection with the literature of phase models of coupled oscillators will be discussed in the next section. It prompts us to choose the orientation control

$$u_k^{align}(\theta_{jk}) = K\frac{\partial U}{\partial \theta_k}. \tag{10}$$

When K is positive, the orientation control (10) stabilizes parallel motions of the group. The spacing control acts as an additive correction to control the shape of the group formation, e.g., the relative distance between particles.

When K is negative, the orientation control (10) stabilizes group motions around a fixed center of mass. The spacing control acts as an additive correction to control the shape of the group formation, e.g., the distance to the center of mass in order to stabilize a circular motion.

The reader is referred to [25, 17] for further details on how to stabilize particular relative equilibria. We provide one illustration below to show that the design of the spacing control becomes somewhat decentralized once the group parameter v has been stabilized. In [10], (a variant of) the following control law is proposed to stabilize circular motion of particle k around a fixed beacon R_0:

$$u_k = -f(\rho_k) < \frac{\tilde{r}_k}{\rho_k}, ie^{i\theta_k} > - < \frac{\tilde{r}_k}{\rho_k}, e^{i\theta_k} >, \tag{11}$$

with $\tilde{r}_k = r_k - R_0$ and $\rho_k = \| \tilde{r}_k \|$. The second term of the control law (11) stabilizes circular motions: it vanishes when the velocity vector is orthogonal to the relative position vector. The function $f(\cdot)$ in (11) provides attraction to the beacon when the distance ρ_k exceeds the equilibrium distance d_0 and repulsion otherwise. (The choice $f(\rho_k) = 1 - (d_0/\rho_k)^2$ is proposed in [10].)

The control law (11) is a single particle control law: u_k only depends on the state (r_k, θ_k) of particle k. But stabilization of the center of mass by means of the orientation control (10) suggests that the beacon control law (11) may serve as a spacing control law if the beacon R_0 is replaced by the center of mass $R = \frac{1}{N}\sum_{k=1}^{N} r_k$ in the definition of $\tilde{r}_k$. One then obtains a composite control law

$$u_k = \frac{1}{N}\sum_{j=1}^{N}\left(-K\sin(\theta_k - \theta_j) - f(\rho_k) < \frac{r_{kj}}{\rho_k}, ie^{i\theta_k} > - < \frac{r_{kj}}{\rho_k}, e^{i\theta_k} >\right) \tag{12}$$

to stabilize circular motions of the group on a unique circle centered at the (fixed) center of mass. Convergence analysis is provided in [25] for large negative values of the parameter K, based on a time-scale separation between the fast convergence of the group center of mass and the slow(er) convergence of each particle relative to the center of mass.

In ongoing work, we have obtained a Lyapunov analysis showing almost global convergence for any value of $K < 0$ for a variant of the control law (12); we are also studying extensions of this basic design to more specific collective motions such as the splay state formation, which is a circular relative equilibrium characterized by N particles equally spaced on the circle [26]. Central to all these designs is the decoupling between a prescribed level of synchrony for the group (achieved by the orientation control) and the relative spacing of individual particles relative to the center of mass (achieved by a decentralized spacing control).

4 Oscillators, accumulators (storages), and phasors

Models of oscillators abound in physics and biology. In this section we briefly review common ways of simplifying these models when it comes to studying their stability properties or the stability properties of their interconnections. We illustrate our point with models from neurodynamics, which have been studied extensively in the literature and appear with many variants depending on the context. For the sake of illustration, we briefly review a biophysical model of the action potential, its two-dimensional simplification used for stability analysis, and two distinct one-dimensional simplifications used for interconnection analysis. The situation is exemplative of most dynamical models of oscillators in physics and biology (see, for instance, [4] for several models of biochemical oscillators).

Nerve cells (neurons) *fire*, that is, a current stimulus above threshold at their input triggers a succession of short electric pulses (action potentials) at their output. Even though action potentials do not persist forever, they are conveniently modeled as a sustained limit cycle oscillation of the electric potential across the cell membrane. The physical basis for this oscillatory mechanism is provided by the celebrated model of Hodgkin and Huxley [5]. The membrane is modeled as a capacitive circuit and the membrane potential depends on several ionic currents (mainly sodium and potassium) flowing through the membrane. Ion channels regulate the flow of each ion across the membrane. A central feature of the model is that ion channels are voltage dependent. The voltage dependence is such that sodium and potassium currents vary out of phase, creating a sustained switch between positive potential (when sodium channels are open and potassium channels are closed) and negative potential (in the opposite situation). The original model, not recalled here, consists of a state space model of dimension 4: one variable to describe the membrane potential, and three additional variables to describe the voltage dependent opening of the ion channels. More detailed models of action potentials take into account the effect of further ionic currents, increasing the dimension of the model up to 10 or 15 state variables.

The Hodgkin–Huxley model and its many variants exhibit sustained oscillations in numerical simulations, in good (quantitative) agreement with experimental data. Rigorous stability analysis of the limit cycle is usually restricted to two-dimensional simplifications of the model, such as the Fitzhugh–Nagumo model

$$\begin{aligned} \dot{V} &= kV - V^3 + R \\ \tau \dot{R} &= -R - V, \end{aligned} \tag{13}$$

which qualitatively describes the limit cycle oscillation of the potential V with a single adaptation variable R to model the voltage dependence of the ion channels. We will come back on this example in the next section.

In order to study the dynamical behavior of large networks of interconnected neurons, the dynamics of each neuron are usually further simplified.

Two important models extensively studied in the literature are the models of Hopfield [8] and Kuramoto [12].

In Hopfield model, the dynamics of neuron k are described by a single variable x_k and the first-order equation

$$\tau \dot{x}_k = -x_k + S(u_k), \tag{14}$$

where $S(\cdot)$ is monotone and usually has a finite range (classical descriptions of this static nonlinearity include the sigmoid function [in computer science] or the Michaelis–Menton function [in reaction networks]). The state x_k has no correspondence with the physical variables of the Hodgkin–Huxley model but models an average activity of the neuron (it is often thought of as the averaging firing rate of the neuron). Hopfield studied the dynamics of N interconnected neurons with the interconnection determined by the linear coupling

$$u_k = \sum_{k=1}^{N} \Gamma_{kj} x_j .$$

The matrix Γ thus determines the network topology and affects the dynamical behavior of the network. Hopfield showed that symmetric network topologies $\Gamma = \Gamma^T$ result in gradient dynamics, in which case all solutions converge to critical points of a scalar potential. Hopfield models abound in neuroscience and have been used to describe the dynamics of a number of computational tasks (see, for instance, [34] for several illustrations in vision). In these examples, the oscillatory behavior of the neuron is unimportant. The state x_k only models the storage capacity of the neuron.

Storage models of oscillators neglect the phase variable of periodic solutions. As a consequence, they are inadequate for synchrony analysis. In contrast, phase models of oscillators disregard the dynamical behavior of the oscillator away from its limit cycle solution. The dynamics of neuron k are described by a single phase variable θ_k and the first-order equation

$$\dot{\theta}_k = \omega_k + u_k, \tag{15}$$

which is a state space equation on the circle S^1. In the absence of input, the phase variable travels on the circle at uniform speed ω_k. Kuramoto studied the dynamics of N interconnected phasors with the interconnection determined by the all-to-all coupling

$$u_k = -\frac{K}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j), \quad K > 0.$$

If all the oscillators are identical ($\omega_k = \omega$ for all k), the dynamics of the interconnected system are

$$\dot{\theta}_k = \omega - \frac{K}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j).$$

In a coordinate frame rotating at uniform speed ω, these are the gradient dynamics (9) discussed in Section 3 and the synchronized state is a stable equilibrium. The convergence analysis is much more involved when the oscillators are not identical (see [31] for a recent review), but a stable phase-locked equilibrium exists if the coupling is large enough. The centroid $\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$ of the oscillators plays an important role in the analysis of the Kuramoto model as a measure of synchrony. It coincides with the linear momentum of the group of particles in Section 3.

Several authors have studied how to reduce general models of oscillators to phase models of the type (15) in the limit of weak coupling, that is, when the coupling between the oscillators does not affect the convergence of each oscillator to a limit cycle solution. For more details, we refer the reader to the recent paper [2] and to references therein.

5 A dissipativity theory for oscillators

Dissipativity theory, introduced by Willems [33], is an interconnection theory for open systems described by state space models:

$$\begin{cases} \dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \\ y = h(x), \quad y \in \mathbb{R}^m. \end{cases} \tag{16}$$

We assume that the vector fields f and g and the function h are smooth, and that the origin $x = 0$ is an equilibrium point of the zero-input system, that is, $f(0) = 0$.

Lyapunov stability of the equilibrium $x = 0$ of the closed system $\dot{x} = f(x)$ is often characterized by a dissipation inequality $\dot{V} \leq 0$ where the scalar Lyapunov function $V(x) > 0$ has a strict minimum at $x = 0$. Dissipativity generalizes this concept to the open system (16): the system is dissipative if it satisfies a dissipation inequality

$$\dot{S} \leq w(u, y), \tag{17}$$

where the scalar *storage* function $S(x) \geq 0$ is analogous to the Lyapunov function of a closed system (with the physical interpretation of an internal energy). The scalar function $w(u, y)$ is called the supply rate. The dissipation inequality expresses that the rate of change of the internal energy (storage) of the system is bounded by the supply rate, that is, the rate at which the system can exchange energy with the external world through its external variables.

Dissipativity theory is fundamental to the stability analysis of interconnections. We here restrict the discussion to passivity theory, which is dissipativity theory for

$$w(u, y) = u^T y - d(y), \quad d(y) \geq 0, \tag{18}$$

a supply rate that prevails in physical models. The usual terminology is *passivity* when $d(y) = 0$ and *strict (output) passivity* when $d(y) > 0$ for $y \neq 0$. The (negative) feedback interconnection of two systems (16), labeled 1 and 2, respectively, defined by the interconnection rule $u_1 = u - y_2$, $u_2 = y_1 = y$, yields a new system (16) with external variables (u, y). The fundamental passivity theorem says that if both system 1 and system 2 are passive, then their feedback interconnection is also passive. Indeed, the storage function $S = S_1 + S_2$ satisfies the dissipation inequality $\dot{S} \leq u_1^T y_1 + u_2^T y_2 - d_1(y_1) - d_2(y_2) \leq u^T y - d_1(y)$. Moreover, the zero equilibrium of the (closed) system obtained by setting $u = 0$ is asymptotically stable modulo a detectability condition. The consequences of this basic theorem are far-reaching and have made dissipativity theory the central tool in nonlinear control theory for the stability analysis (and design) of equilibria; see [24, 32, 16] for illustrations.

In ongoing work [27, 29], we aim to show that dissipativity theory also provides an interconnection theory for oscillators. The main idea is very straightforward: we characterize an oscillator by a dissipation inequality with a supply rate

$$w(u, y) = u^T y + a_k(y) - d(y), \quad a_k(y) \geq 0, \quad d(y) \geq 0. \tag{19}$$

The supply rate (19) differs from the (strictly) passive supply rate (18) by the activation term $a_k(y)$. The passive oscillator is viewed as a system that passively exchanges its energy with the environment but that contains an active internal element. The competition between the internal elements that dissipate the storage and the active element that restores the storage is viewed as the basic oscillation mechanism. A necessary condition for sustained oscillations is that the system restores energy at low energy, that is, $a_k(y) - d(y) > 0$ when $|y|$ is small, and that the system dissipates energy at high energy, that is, $a_k(y) - d(y) < 0$ when $|y|$ is large.

A convenient way to obtain dissipativity with the supply rate (19) is to consider the feedback interconnection of a passive system with the static nonlinearity illustrated in Figure 4: $\phi_k(y) = -ky + \phi(y)$ where $\phi(\cdot)$ is a smooth sector nonlinearity in the sector $(0, \infty)$, which satisfies $\phi'(0) = \phi''(0) = 0$, $\phi'''(0) := \kappa > 0$ and $\lim_{|s| \to \infty} \frac{\phi(s)}{s} = +\infty$ ("stiffening" nonlinearity). The storage S of the passive system Σ satisfies

$$\dot{S} \leq ky^2 - y\phi(y) + uy, \tag{20}$$

which corresponds to $a_k(y) = ky^2$ and $d(y) = y\phi(y)$ in (19).

The parameter k regulates the level of activation near the equilibrium $x = 0$. When $k \leq 0$, the feedback system is strictly passive, and under a suitable detectability condition, the equilibrium is globally asymptotically stable. Stability of the equilibrium is lost at a critical value $k^* \geq 0$.

Stability

Two distinct bifurcation scenarios provide a stable oscillation mechanism for the feedback system in Figure 4. The first one corresponds to a supercritical Hopf bifurcation: two complex conjugate eigenvalues cross the imaginary axis

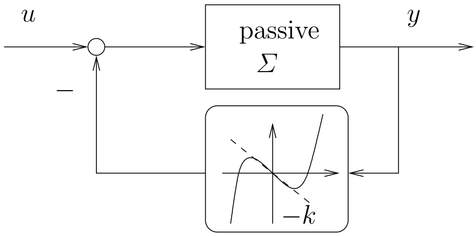

Fig. 4. Block diagram of a family of systems satisfying the dissipation inequality (20).

at $k = k^*$, giving rise to a stable limit cycle surrounding the unstable equilibrium $x = 0$ for $k > k^*$. The normal form of this bifurcation is obtained when the passive system Σ in the loop is a harmonic oscillator, characterized by the transfer function $H(s) = \frac{s}{s^2+\omega^2}$. The dynamics of the feedback system are then

$$\frac{d^2y}{dt^2} + \omega^2 y + \frac{d}{dt}(ky - \phi(y)) = \dot{u}. \tag{21}$$

Equation (21) is the model of a van der Pol oscillator when $\phi(y) = y^3$: the circuit interpretation is that the sustained exchange of energy between a capacitor and an inductor is regulated by a static element (a tunnel-diode circuit) that dissipates energy when the current is high and restores energy when the current is low.

A second bifurcation scenario in the feedback system in Figure 4 is a supercritical pitchfork bifurcation: the stable equilibrium $x = 0$ becomes a saddle beyond the bifurcation value $k = k^*$ and two new stable equilibria appear for $k > k^*$. The normal form of this bifurcation is obtained when the passive system Σ in the loop is an integrator, characterized by the transfer function $H(s) = \frac{1}{s}$. The dynamics of the feedback system are then

$$\dot{y} = ky - \phi(y) + u. \tag{22}$$

When the input $u = 0$, the equilibrium $y = 0$ is stable for $k \leq 0$ and unstable for $k > 0$. Two other equilibria exist for $k > 0$ and create a bistable behavior: the positive (respectively, negative) equilibrium attracts all solutions with positive (respectively, negative) initial condition. For every $k > 0$, the bistable behavior persists over a range of (constant) inputs R, causing hysteresis in the static response of the system, as shown in Figure 5.

This hysteresis is turned into a relaxation oscillation when the input slowly *adapts* to follow the hysteresis loop, resulting in the closed-loop dynamics

$$\begin{aligned} \dot{y} &= ky - \phi(y) + R \\ \tau\dot{R} &= -R - y. \end{aligned} \tag{23}$$

Equation (23) is the Fitzhugh–Nagumo model recalled in the previous section. The same adaptation in the model of Figure 4 is illustrated in Figure 6.

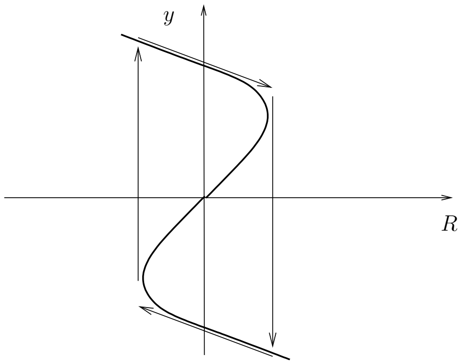

Fig. 5. The hysteresis associated to the bistable system (22) with constant input $u = R$.

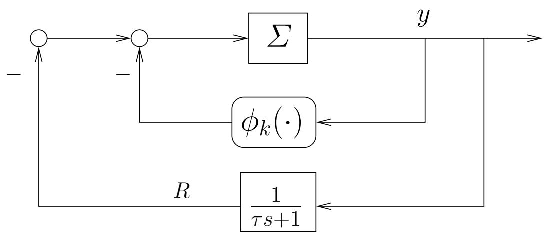

Fig. 6. Converting the pitchfork scenario into a relaxation oscillator with a slow adaptation mechanism ($\tau \gg 0$). The case $\Sigma = \frac{1}{s}$ corresponds to the Fitzhugh–Nagumo oscillator.

Oscillations resulting from slow adaptation in a bistable system seem to constitute a prevalent mechanism in models of biological oscillators (see, for instance, [4] for several illustrations).

The main result in [27] provides higher-dimensional generalization of the two above examples by showing that the dissipativity characterization of the feedback system forces one of the two bifurcation scenarios. The results are local in the parameter space (they hold for values of the parameter in the vicinity of the bifurcation), but they are global in the state space, that is, convergence to the stable limit cycle is proven for all initial conditions that do not belong to the stable manifold of the (unstable) equilibrium at the origin.

Interconnections

Dissipativity theory not only provides a stability theory for oscillators that admit the representation in Figure 4 but also an interconnection theory. As a direct consequence of passivity theory, any passive interconnection of dissipative systems with the supply rate (19) provides a new dissipative system with a supply rate of the same form.

For the sake of illustration, we only consider the interconnection of two identical passive oscillators, each characterized by a dissipation inequality

$$\dot{S}_i \leq k y_i^2 - \phi(y_i) y_i + u_i y_i, \quad i = 1, 2.$$

We consider the interconnections in Figure 7 and assume linear coupling

$$u = -\Gamma y + v,$$

using the vector notation $u = (u_1, u_2)^T$, $y = (y_1, y_2)^T$, and $v = (v_1, v_2)^T$.

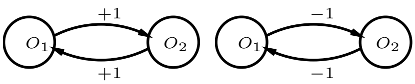

Fig. 7. Positive and negative feedback interconnection of two oscillators.

The interconnections in Figure 7 correspond to a symmetric matrix $\Gamma = \Gamma^T$ given by

$$\Gamma = \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix},$$

with $-y^T \Gamma y \leq y^T y$ for the two considered situations. The interconnection therefore satisfies the following dissipation inequality with storage $S = S_1 + S_2$:

$$\dot{S} \leq (k+1) y^T y - \Phi(y)^T y + v^T y,$$

which is dissipativity with the supply rate (19) for the system with input $v = (v_1, v_2)^T$ and output $y = (y_1, y_2)^T$. (We also use the notation $\Phi(y) = (\phi(y_1), \phi(y_2))^T$.) The bifurcation for the feedback interconnection of the two oscillators is the same as for each individual oscillator; only the bifurcation value $k = k^*$ is shifted by one for the interconnection. The general case of symmetric interconnections is treated in detail in [29]. The main observation is that any symmetric coupling $u = -\Gamma y + v$ can be written as $u = -(\Gamma + \lambda I) y + \lambda y + v$ with $\Gamma' = \Gamma + \lambda I$ a nonnegative symmetric matrix. The sum of the storages therefore satisfies

$$\dot{S} \leq (k+\lambda) y^T y - \Phi(y)^T y + v^T y,$$

which is dissipativity with the supply rate (19) for the system with input v and output y.

Synchronization

Dissipativity theory also provides a synchrony analysis for networks of oscillators. Synchrony is a convergence property for the difference between the solutions of different systems. Convergence properties for the *difference* between solutions of a closed system are characterized by notions of *incremental* stability [1, 14, 18]. For open systems, the corresponding notion is *incremental* dissipativity. Consider two different solutions $x_1(t)$ and $x_2(t)$ of the system (16) with inputs and outputs $(u_1(t), y_1(t))$ and $(u_2(t), y_2(t))$, respectively. Denote the incremental variables by $\delta x = x_1 - x_2$, $\delta u = u_1 - u_2$, and $\delta y = y_1 - y_2$. The system is *incrementally* dissipative if it satisfies a dissipation inequality

$$\dot{\delta S} \leq w(\delta u, \delta y) \tag{24}$$

for the *incremental* scalar storage function $\delta S(\delta x) \geq 0$.

Consider two copies of the same system (16) with the difference coupling

$$u_i = -K(y_i - y_j),\ i, j = 1, 2,$$

which corresponds to the interconnection matrix

$$\Gamma = K \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Incremental dissipativity of the system (16) with the supply rate

$$w(\delta u, \delta y) = k\delta y^2 - \delta y \phi(\delta y) + \delta u \delta y \tag{25}$$

implies output synchronization for the interconnected system: substituting $\delta u = -2K\delta y$ in the inequality (24) yields

$$\dot{\delta S} \leq (k - 2K)\delta y^2 - \delta y \phi(\delta y),$$

which implies asymptotic convergence of δy to zero, that is, output synchronization, when $2K > k$. State synchronization follows from output synchronization modulo a detectability condition.

In [29], we show that the implications of incremental dissipativity for synchronization extend to the interconnection of N identical systems with network topologies that include S_N symmetry (all-to-all topology), D_N symmetry (bidirectional ring topology), and Z_N symmetry (unidirectional ring topology). These results are closely related to other recent synchronization results in the literature [28, 19, 1], all based on incremental stability notions.

Synchronization of passive oscillators

We have shown that the Lure-type system in Figure 4 satisfies a dissipation inequality with the supply rate (19). We conclude this section by showing that the same system is also incrementally dissipative with a supply rate of the form (25) when the passive system Σ is linear and when the static nonlinearity ϕ is monotone.

For linear systems, dissipativity is equivalent to incremental dissipativity, that is, $S(x)$ satisfies the dissipation inequality $\dot{S} \leq w(u,y)$ if and only if the incremental storage $\delta S = S(\delta x)$ satisfies the incremental dissipation inequality $\dot{\delta S} \leq w(\delta u, \delta y)$.

The static nonlinearity $y = \phi(u)$ is passive if and only if $\phi(s)s \geq 0$ for all s. It is also *incrementally passive* if it satisfies the additional monotonicity property $(\phi(s_1) - \phi(s_2))(s_1 - s_2) = \delta\phi(s)\delta s \geq 0$ for all $\delta s = s_1 - s_2$.

If the passive system in Figure 4 is linear, it has a quadratic storage $S(x) = x^T P x$ and the feedback system satisfies the dissipation inequality

$$\dot{S} \leq ky^2 - y\phi(y) + uy.$$

The incremental storage is $\delta S = \delta x^T P \delta x$, which satisfies the incremental dissipation inequality

$$\dot{\delta S} \leq k\delta y^2 - \delta y \delta\phi(y) + \delta u \delta y.$$

If $\phi(\cdot)$ is monotone, then $\delta y \delta\phi(y) \geq \delta y \psi(\delta y) \geq 0$ for some static nonlinearity $\psi(\cdot)$ and the feedback system satisfies the incremental dissipation inequality

$$\dot{\delta S} \leq k\delta y^2 - \delta y \psi(\delta y) + \delta u \delta y.$$

Combining the global convergence result to a stable limit cycle for one system with the synchronization results for a network of interconnected identical systems, one obtains global convergence results to a synchrone oscillation for passive oscillators that admit the feedback representation in Figure 4.

6 Conclusion

Oscillators are important building blocks of dynamical systems. When suitably interconnected, they robustly produce ensemble phenomena with synchrony properties not encountered in equilibrium systems. The chapter has described two stabilization problems that illustrate the role of synchrony as a design principle: a rhythmic control task and the design of group motions for moving particles in a plane.

A system theory for oscillators requires an interconnection theory. The external characterization of oscillators adopted in this chapter follows the fundamental characterization of open systems by a dissipation inequality and proposes a supply rate that enables limit cycle oscillations in the isolated system. The proposed dissipation inequality has implications for the stability properties of the oscillator, both in isolation and when interconnected to other oscillators. In its incremental form, the same dissipation inequality has implications for the synchrony properties of networks of identical oscillators.

Acknowledgements. The author acknowledges several collaborators involved in the research projects reported in this paper: Manuel Gerard, Naomi Leonard, Derek Paley, Renaud Ronsse, and Guy-Bart Stan. He also wishes to acknowledge Jan Willems for several discussions and a continuing source of inspiration.

References

1. Angeli D (2002) A Lyapunov approach to incremental stability properties. IEEE Trans. on Automatic Control 47:410–422
2. Brown E, Moehlis J, Holmes P (2004) On phase reduction and response dynamics of neural oscillator populations. Neural Computation 16(4):673–715
3. Gérard M, Sepulchre R (2004) Stabilization through weak and occasional interactions: a billiard benchmark. In: 6th IFAC Symposium on Nonlinear Control Systems, Stuggart, Germany
4. Goldbeter A (1996) Biochemical oscillations and cellular rhythms. Cambridge University Press, Cambridge, UK
5. Hodgkin AL, Huxley AF (1952) A quantitative description of membrane current and its application to conduction and excitation in nerve. J. Physiology 117:500–544
6. Holmes PJ (1982) The dynamics of repeated impacts with a sinusoidally vibrating table. Journal of Sound and Vibration 84(2):173–189
7. Holmes PJ, Full RJ, Koditschek D, Guckenheimer J (2006) Dynamics of legged locomotion: Models, analyses and challenges. SIAM Review, to appear
8. Hopfield JJ (1982) Neural networks and physical systems with emergent collective computational abilities. Proceedings of the National Academy of Sciences USA 79:2554–2558
9. Justh EW, Krishnaprasad PS (2002) A simple control law for UAV formation flying Technical report Institute for Systems Research, University of Maryland
10. Justh EW, Krishnaprasad PS (2003) Steering laws and continuum models for planar formations. In: IEEE 42nd Conf. on Decision and Control, Maui, HI
11. Kumar V, Leonard N, Morse A (eds) (2004) Cooperative control, vol. 309 of Lecture Notes in Control and Information Sciences. Springer-Verlag, London
12. Kuramoto Y (1984) Chemical oscillations, waves, and turbulence. Springer-Verlag, London
13. Lehtihet BN, Miller HE (1986) Numerical study of a billiard in a gravitational field. Physica 21D:93–104
14. Lohmiller W, Slotine JJE (1998) On contraction analysis for nonlinear systems. Automatica 34(6):683–696
15. Ogren P, Fiorelli E, Leonard NE (2004) Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment. IEEE Trans. on Automatic Control 49(8):1292–1302
16. Ortega R, Loria A, Nicklasson PJ, Sira-Ramirez H (1998) Passivity-based Control of Euler-Lagrange Systems. Springer-Verlag, London
17. Paley D, Leonard N, Sepulchre R (2004) Collective motion: bistability and trajectory tracking. In: IEEE 43rd Conf. on Decision and Control, Atlantis, Bahamas

18. Pavlov A, Pogromsky AY, van de Wouw N, Nijmeijer H (2004) Convergent dynamics, a tribute to Boris Pavlovich Demidovich. Systems & Control Letters 52:257–261
19. Pogromsky A (1998) Passivity based design of synchronizing systems. Int. J. Bifurcation and Chaos 8:295–319
20. Ronsse R, Lefevre P, Sepulchre R (2004a) Open-loop stabilization of 2D impact juggling. In: 6th IFAC Symposium on Nonlinear Control Systems, Stuggart, Germany
21. Ronsse R, Lefevre P, Sepulchre R (2004b) Sensorless stabilization of bounce juggling Technical report, Department of Electrical Engineering and Computer Science, University of Liège
22. Sastry S (1999) Nonlinear systems. Springer-Verlag, London
23. Sepulchre R, Gérard M (2003) Stabilization of periodic orbits in a wedge billiard. 1568–1573. In: IEEE 42nd Conf. on Decision and Control Maui, HI
24. Sepulchre R, Jankovic M, Kokotovic P (1997) Constructive nonlinear control. Springer-Verlag, London, UK
25. Sepulchre R, Paley D, Leonard N (2004a) Collective motion and oscillator synchronization. 189–205. In: Cooperative control, Kumar V, Leonard N, Morse A (eds), Springer-Verlag, London
26. Sepulchre R, Paley D, Leonard N (2004b) Collective stabilization of n steered particles in the plane, in preparation
27. Sepulchre R, Stan GB (2004) Feedback mechanisms for global oscillations in Lure systems. Systems & Control Letters 54(8):809–818
28. Slotine JJ, Wang W (2004) A study of synchronization and group cooperation using partial contraction theory. 207–228. In: Cooperative control, Kumar V, Leonard N, Morse A (eds) Springer-Verlag, London
29. Stan GB, Sepulchre R (2004) Dissipativity theory and global analysis of limit cycles, Technical report, Department of Electrical Engineering and Computer Science, University of Liège
30. Strogatz S (2003) Sync: the emerging science of spontaneous order. Hyperion
31. Strogatz SH (2000) From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators. Physica D 143:1–20
32. van der Schaft AJ (2000) L_2-gain and passivity techniques in nonlinear control. Springer-Verlag, London
33. Willems JC (1972) Dissipative dynamical systems. Arch. Rational Mechanics and Analysis 45:321–393
34. Wilson H (1999) Spikes, decisions, and actions. Oxford University Press, Oxford, UK
35. Winfree A (2000) The geometry of biological time, second edition. Springer-Verlag, London
36. Wojtkowski MP (1998) Hamiltonian systems with linear potential and elastic constraints. Communications in Mathematical Physics 194:47–60