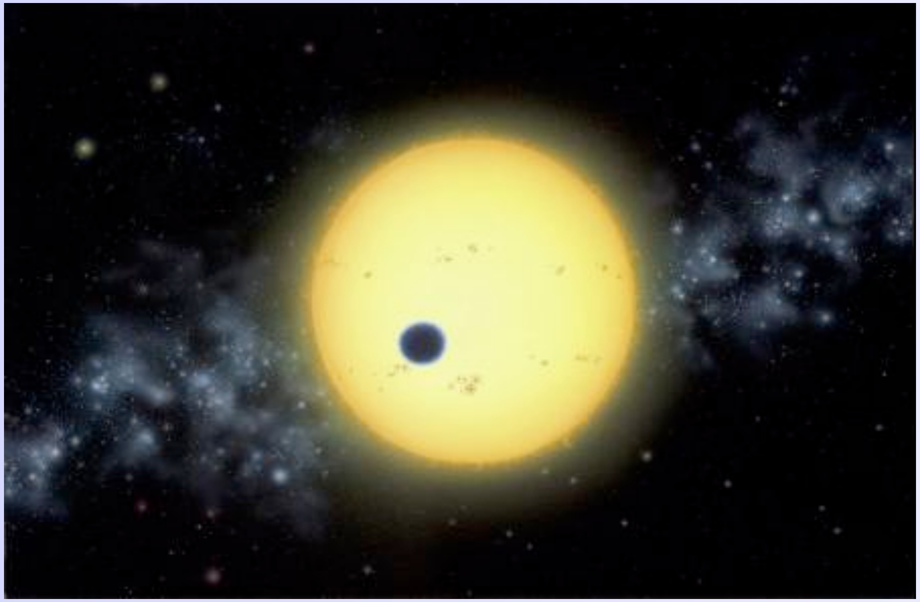
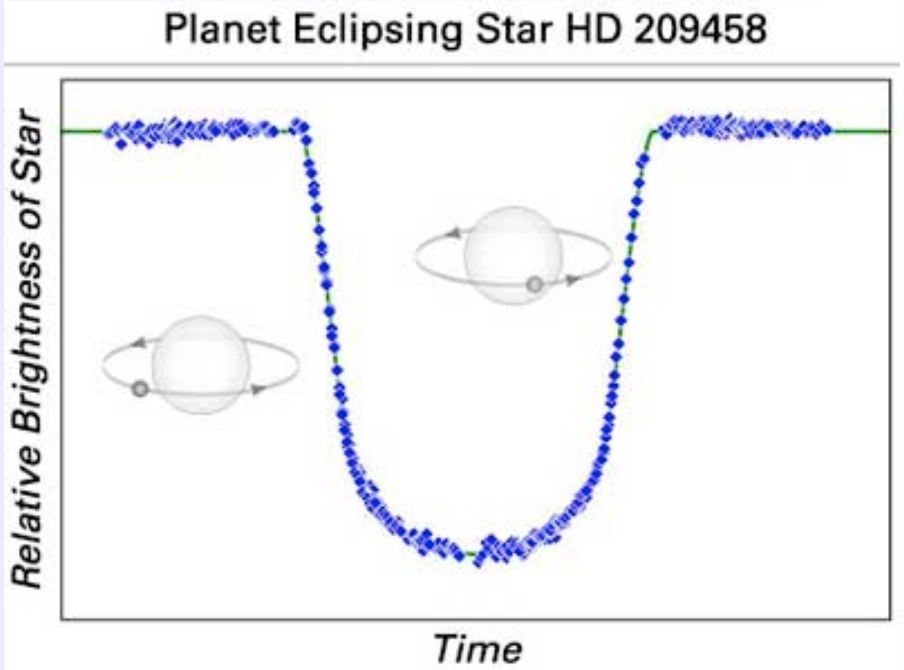
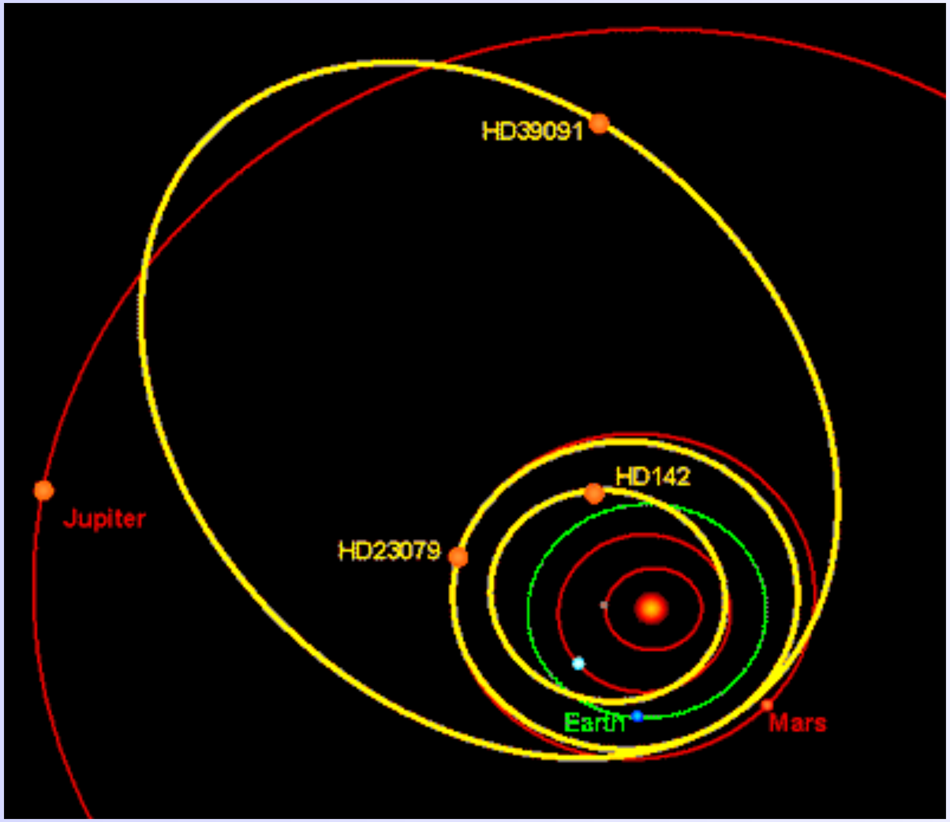
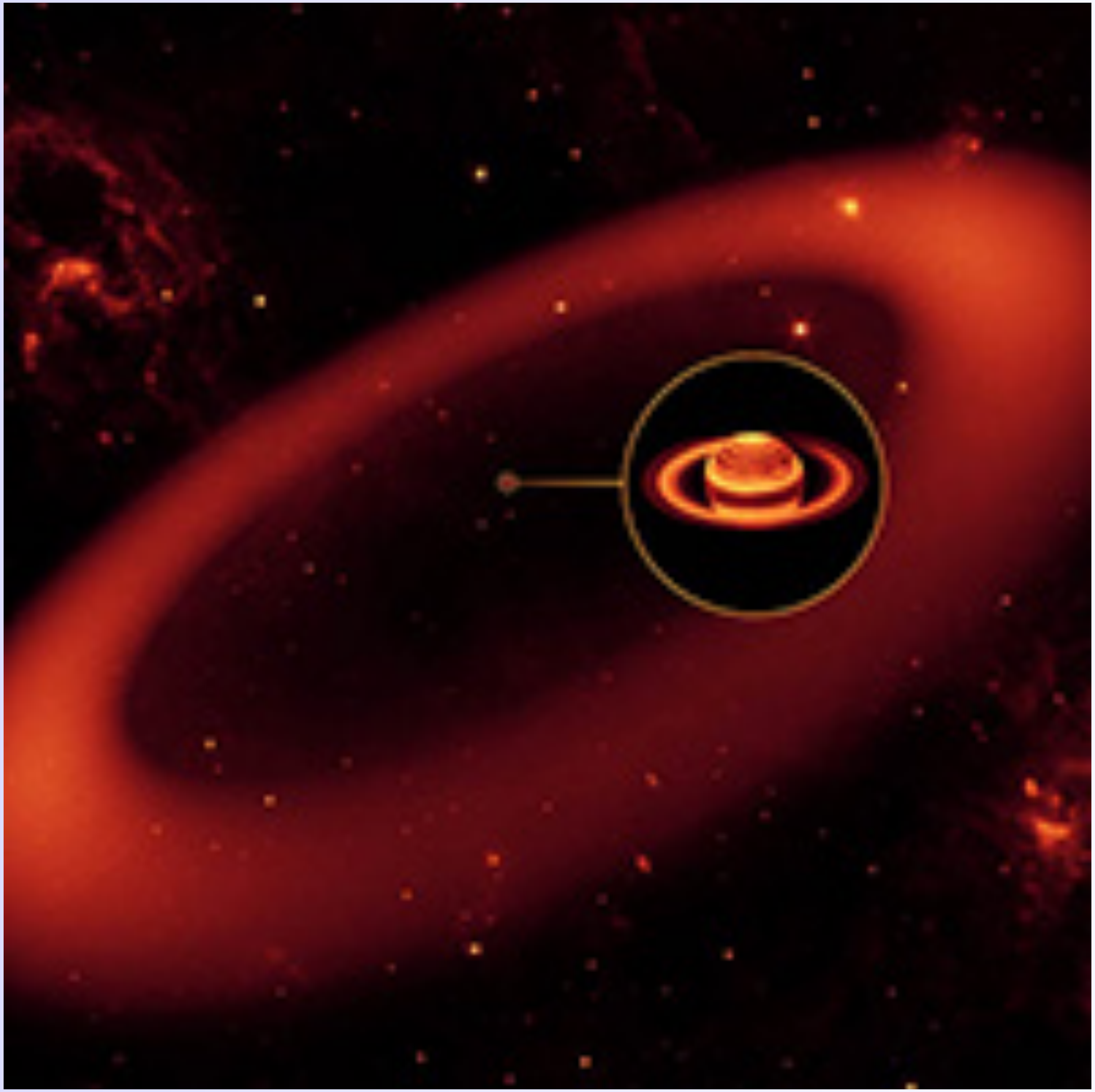
5ème cours de Mécanique Analytique (29 Septembre 2011)






HD39091
HD142
Jupiter
HD23079
Earth
Mars



- 1.6 Exemple 1 : le pendule plan
- 1.6 Exemple 2 : la cuvette sphérique
- 1.6 Exemple 3 : la fronde en contraction
- 1.6 Exemple 4 : problème de Lagrange-Poisson
- 1.6 Exemple 5 : la particule chargée dans un champ EM

• 1.7 Le principe variationnel d'Hamilton

Le mouvement réel d'un système mécanique à f degrés de liberté $q = \{q_1, ..., q_f\}$ entre les temps t_1 et t_2 est tel que l'intégrale (dite *intégrale d'action)*

$$I[q] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)\, dt \qquad (1.46)$$

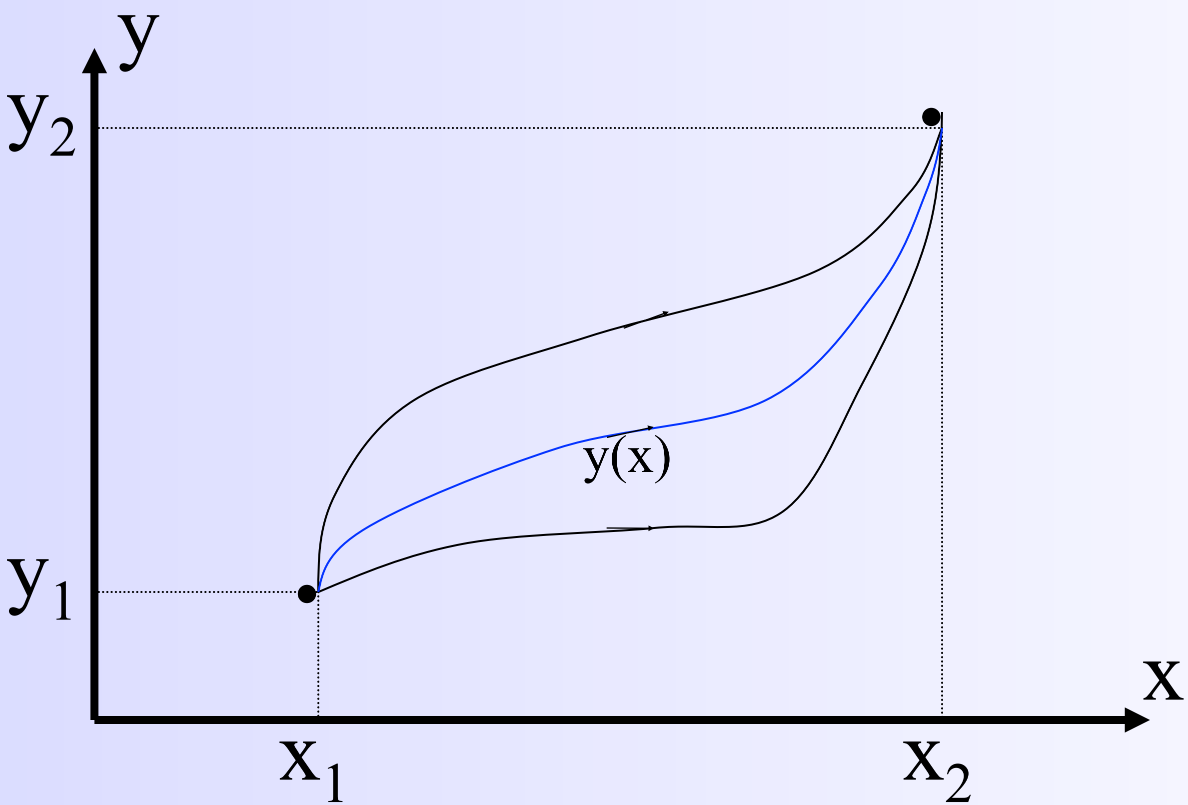
où $L(q(t), \dot{q}(t), t)$ est le lagrangien de système, est extrémale.

$$I[y] = \int_{x_1}^{x_2} f[y(x), y'(x), x] dx \qquad (1.47)$$

• 1.7 Le principe variationnel d'Hamilton

$y = y(\alpha, x)$ $\alpha = 0,$ $y(0, x) = y(x)$.

$y(\alpha, x) = y(0, x) + \alpha\eta(x)$ (1.48) $\eta(x_1) = \eta(x_2) = 0$



• 1.7 Le principe variationnel d'Hamilton

$$I(\alpha) = \int_{x_1}^{x_2} f[y(\alpha, x), y'(\alpha, x), x]\, dx \quad (1.49)$$

$$\left.\frac{\partial I}{\partial \alpha}\right|_{\alpha=0} = 0 \quad (1.50)$$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} \right) dx \quad (1.51)$$

$$\frac{\partial y}{\partial \alpha} = \eta(x), \qquad \frac{\partial y'}{\partial \alpha} = \frac{d\eta}{dx} \quad (1.52)$$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta}{dx} \right) dx \quad (1.53)$$

• 1.7 Le principe variationnel d'Hamilton

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \frac{d\eta}{dx} dx = \frac{\partial f}{\partial y'} \eta(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \eta(x)\, dx$$

(1.54)

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \frac{d\eta}{dx} \right) dx \quad (1.53)$$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right) \eta(x) dx \quad (1.55)$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \quad (1.56)$$

• 1.7 Le principe variationnel d'Hamilton

$y_i \; (i = 1, ..., f)$

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \qquad (i = 1, ..., f) \qquad (1.57)$$

$$\begin{cases} x \to t \\ y_i \to q_i \\ y_i' \to \dot{q}_i \\ f(y_i(x), y_i'(x), x] \to L(q_i(t), \dot{q}_i(t), t) \end{cases} \qquad (1.58)$$

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad (i = 1, ..., f)} \qquad (1.59)$$

• 1.8 Lois de conservation et symétries

$$L = T - V$$

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}} \qquad (1.60)$$

$$\boxed{H = p_i \dot{q}_i - L} \qquad (1.61)$$

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

$$\vec{p} = (p_x,\ p_y,\ p_z) = \left(\frac{\partial L}{\partial \dot{x}}, \frac{\partial L}{\partial \dot{y}}, \frac{\partial L}{\partial \dot{z}}\right) = (m\dot{x},\ m\dot{y},\ m\dot{z})$$

• 1.8 Lois de conservation et symétries

$$\boxed{\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}} \qquad (1.62)$$

Si $L = L(q_1, q_3, q_4, \ldots q_f, \dot{\mathbf{q}}, t)$

$$\frac{dp_2}{dt} = 0 \quad \Longrightarrow \quad p_2(\dot{\mathbf{q}}, \mathbf{q}, t) = C^{te}$$

Première règle d'or

• 1.8 Lois de conservation et symétries

$$\frac{dH}{dt} = \dot{p}_i \dot{q}_i + p_i \ddot{q}_i - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial q_i}\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}} \qquad (1.63)$$

Si $L = L(\mathbf{q}, \dot{\mathbf{q}})$

$\dfrac{dH}{dt} = 0$ ====> $H(\mathbf{q}, \dot{\mathbf{q}}, t) = C^{te'}$

Seconde règle d'or

• 1.8 Lois de conservation et symétries

Si $\vec{r}_\alpha = \vec{r}_\alpha(q)$

$$T = \frac{1}{2} a_{ij}(q)\dot{q}_i\dot{q}_j \qquad (1.64)$$

Si $V = V(q,t)$

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \qquad (1.65)$$

$$p_i\dot{q}_i = \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i = 2T \qquad (1.66)$$

$$\boxed{H = T + V} \qquad (1.68)$$

- 1.8 Lois de conservation et symétries

Si $V = V(q)$

$$H = T + V = \text{constante} \qquad (1.69)$$

Analogie !

$$\left\{ \begin{array}{ll} \boxed{\dfrac{dp_i}{dt} = \dfrac{\partial L}{\partial q_i}} & (1.62) \\ \boxed{\dfrac{dH}{dt} = -\dfrac{\partial L}{\partial t}} & (1.63) \end{array} \right.$$

$p_i \leftrightarrow q_i$

$H \leftrightarrow t$

• 1.9 Le théorème de Noether

$$q_i \to q_i' = q_i + \varepsilon X_i(q,t) \qquad (1.70)$$

$$L(q',\dot{q}',t) = L(q,\dot{q},t) \qquad (1.71)$$

$$L\left(q+\varepsilon X(q,t),\dot{q}+\varepsilon\frac{d}{dt}X(q,t),t\right) = L(q,\dot{q},t) \qquad (1.72)$$

$$\frac{\partial L}{\partial \dot{q}_i}\frac{dX_i}{dt} + \frac{\partial L}{\partial q_i}X_i = 0 \qquad (L = L(q,\dot{q},t) \text{ et } X_i = X_i(q,t)) \qquad (1.73)$$

• 1.9 Le théorème de Noether

$$\frac{\partial L}{\partial \dot{q}_i}\frac{dX_i}{dt} + \frac{\partial L}{\partial q_i}X_i = 0 \qquad (L = L(q,\dot{q},t) \text{ et } X_i = X_i(q,t))$$

(1.73)

$q(t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}X_i\right) = 0$$ (1.74)

==> $p_i\, X_i = C^{te}$

Théorème de Noether

• 1.9 Le théorème de Noether

$$L(q_2, \ldots, q_f, \dot{q}_1, \ldots, \dot{q}_f, t) \qquad (1.75)$$

$$q_1 \to q_1 + \varepsilon \,,\; q_2 \to q_2, \ldots, q_f \to q_f$$

$$\dot{q}_1 \to \dot{q}_1 ,\; \dot{q}_2 \to \dot{q}_2, \ldots, \dot{q}_f \to \dot{q}_f$$

$$X_i = (1, 0, \ldots, 0)$$

$$p_i X_i = p_1 \qquad (1.76)$$

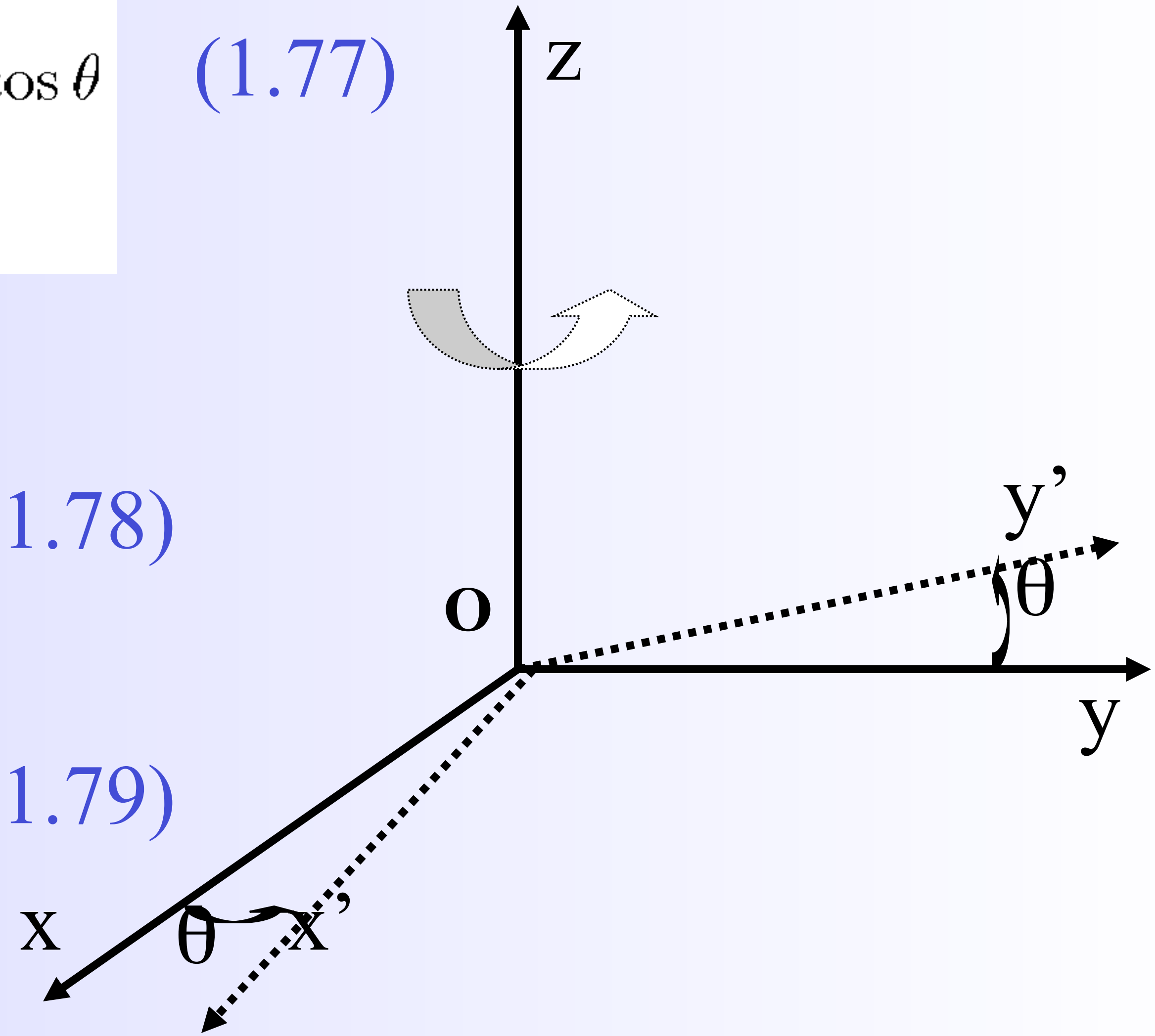
• 1.9 Le théorème de Noether

$$\begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \\ z' = z \end{cases} \quad (1.77)$$

$$\begin{cases} x' = x + \theta y \\ y' = -\theta x + y \\ z' = z \end{cases} \quad (1.78)$$

$$X_i = (y, -x, 0) \quad (1.79)$$

$$p_x y - p_y x$$



- 1.9 Le théorème de Noether

$$t \to t + \varepsilon$$

$$L(q, \dot{q}, t + \varepsilon) = L(q, \dot{q}, t) \quad)$$

$$\varepsilon \frac{\partial L}{\partial t} = 0$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0$$

c.q.f.d.

- Exercice proposé :

Ecrire l'équation de Lagrange permettant de déterminer le mouvement d'un pendule plan oscillant animé d'un mouvement de translation rectiligne uniforme