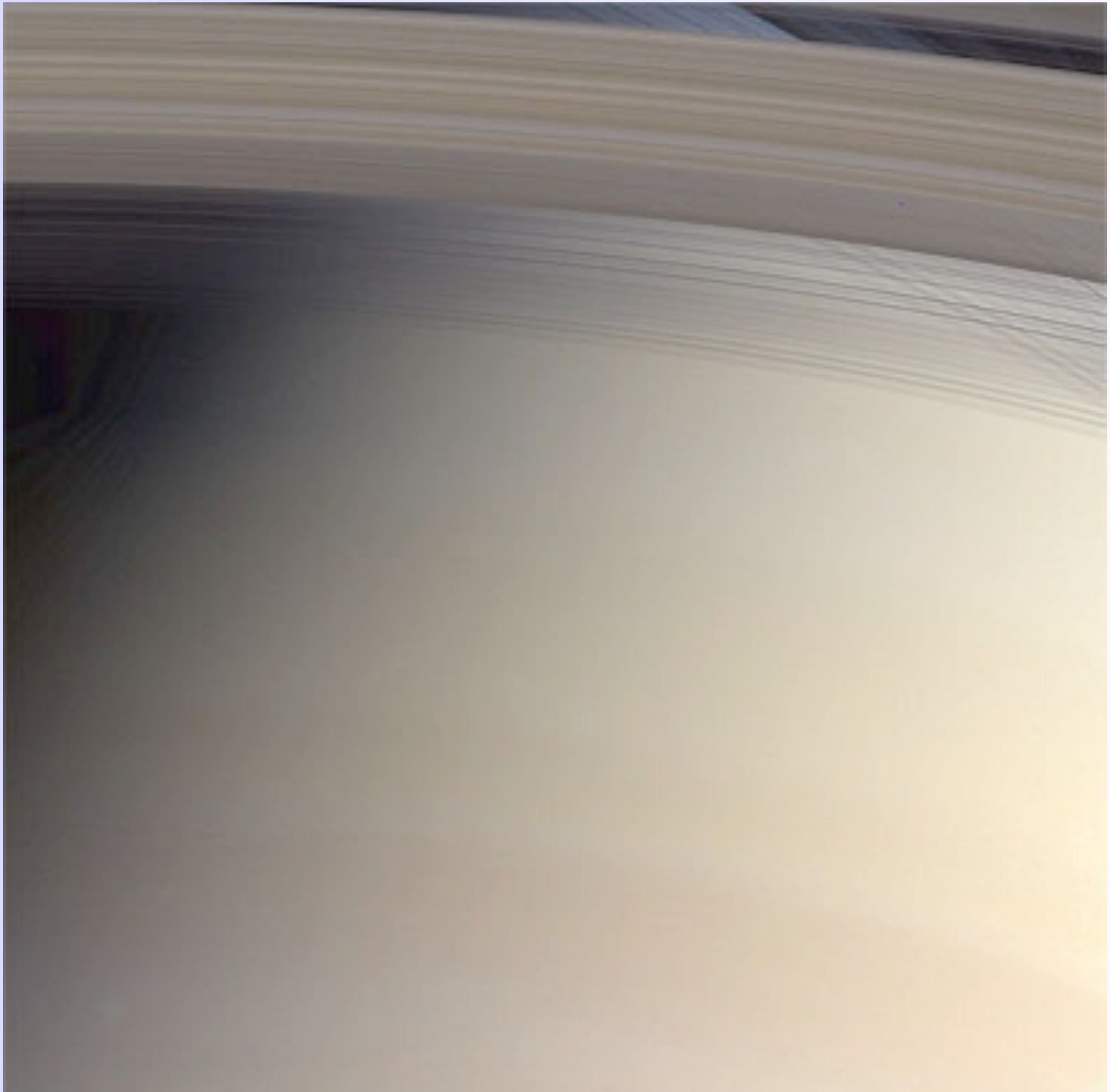
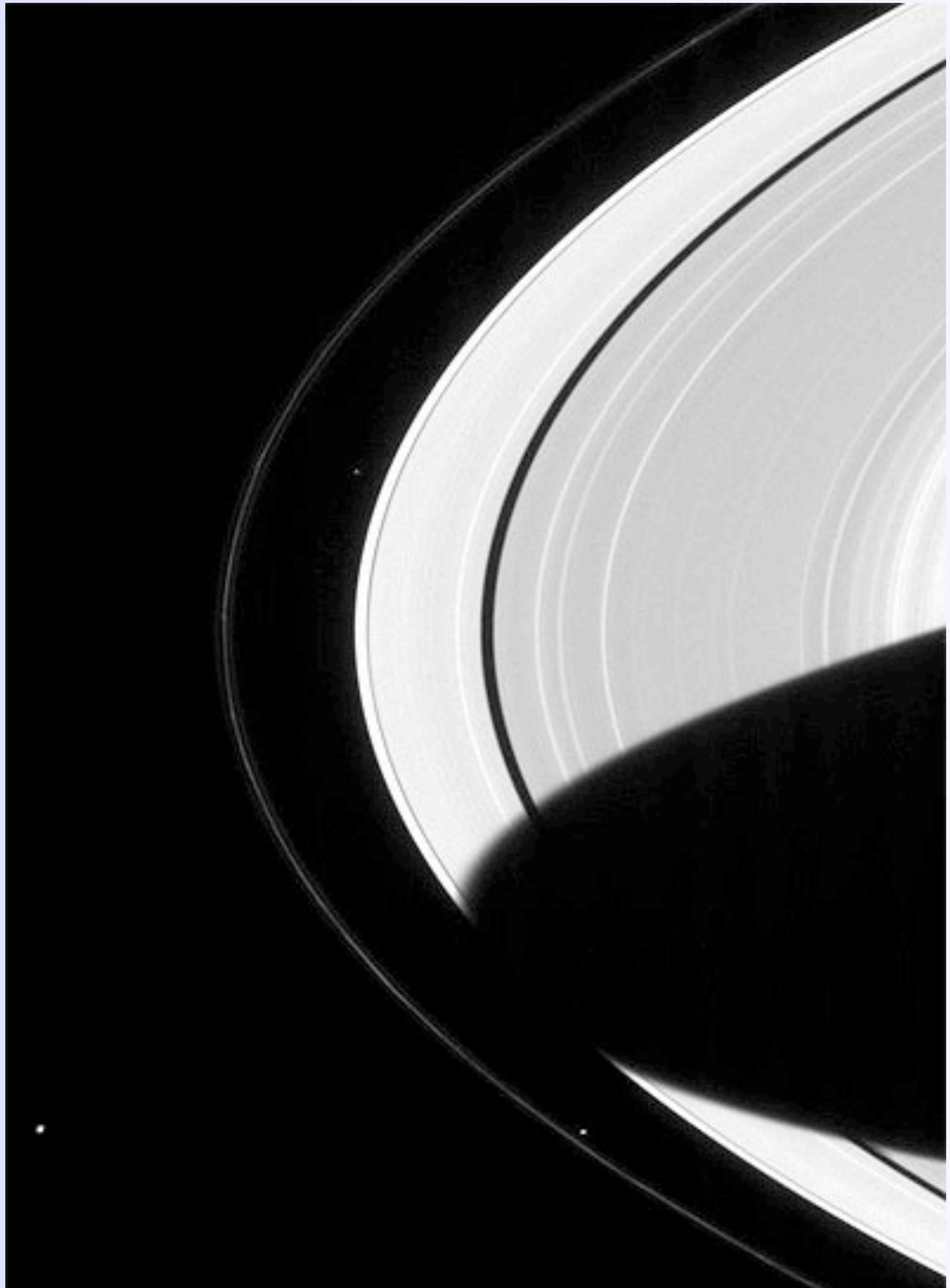
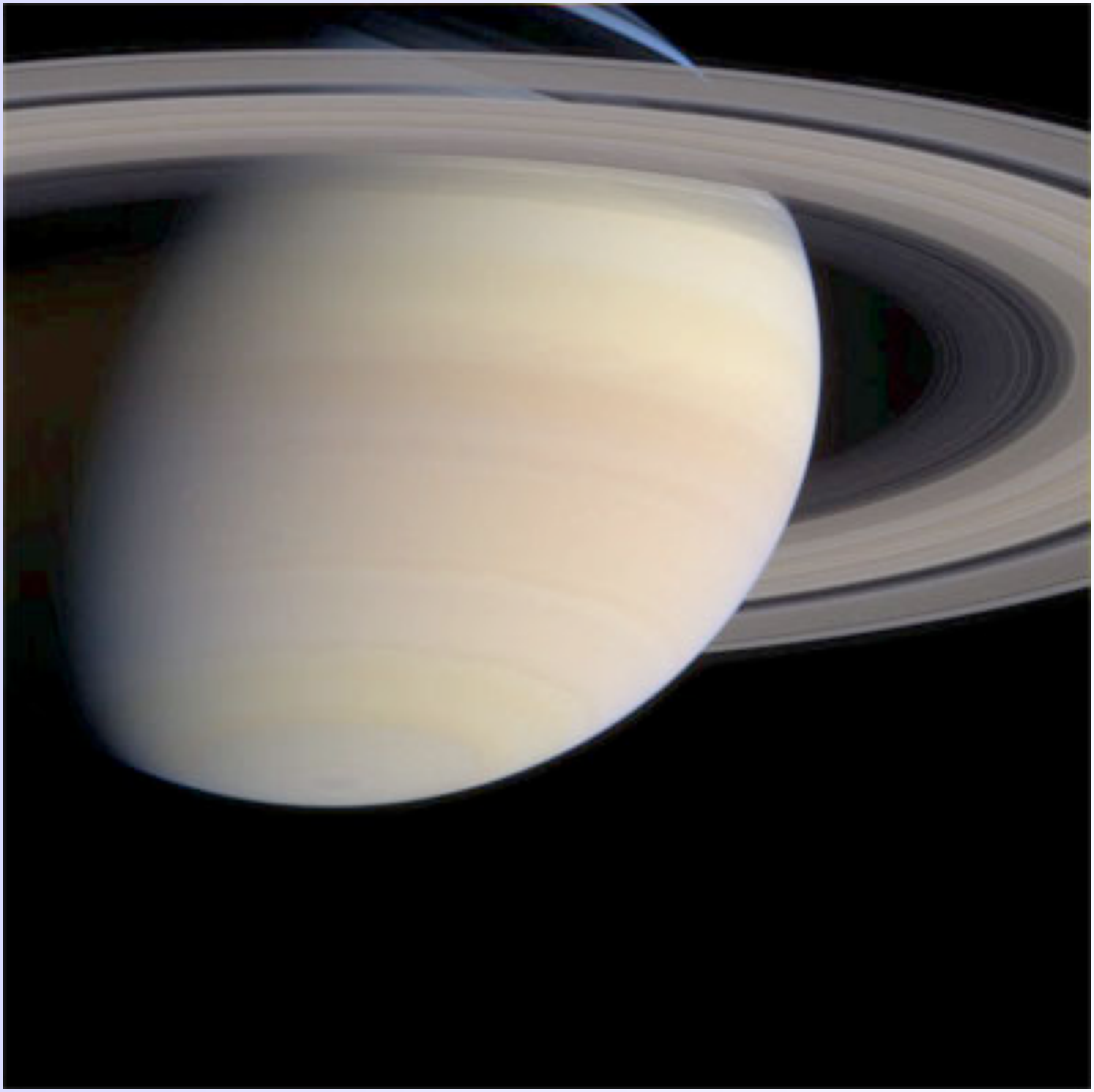
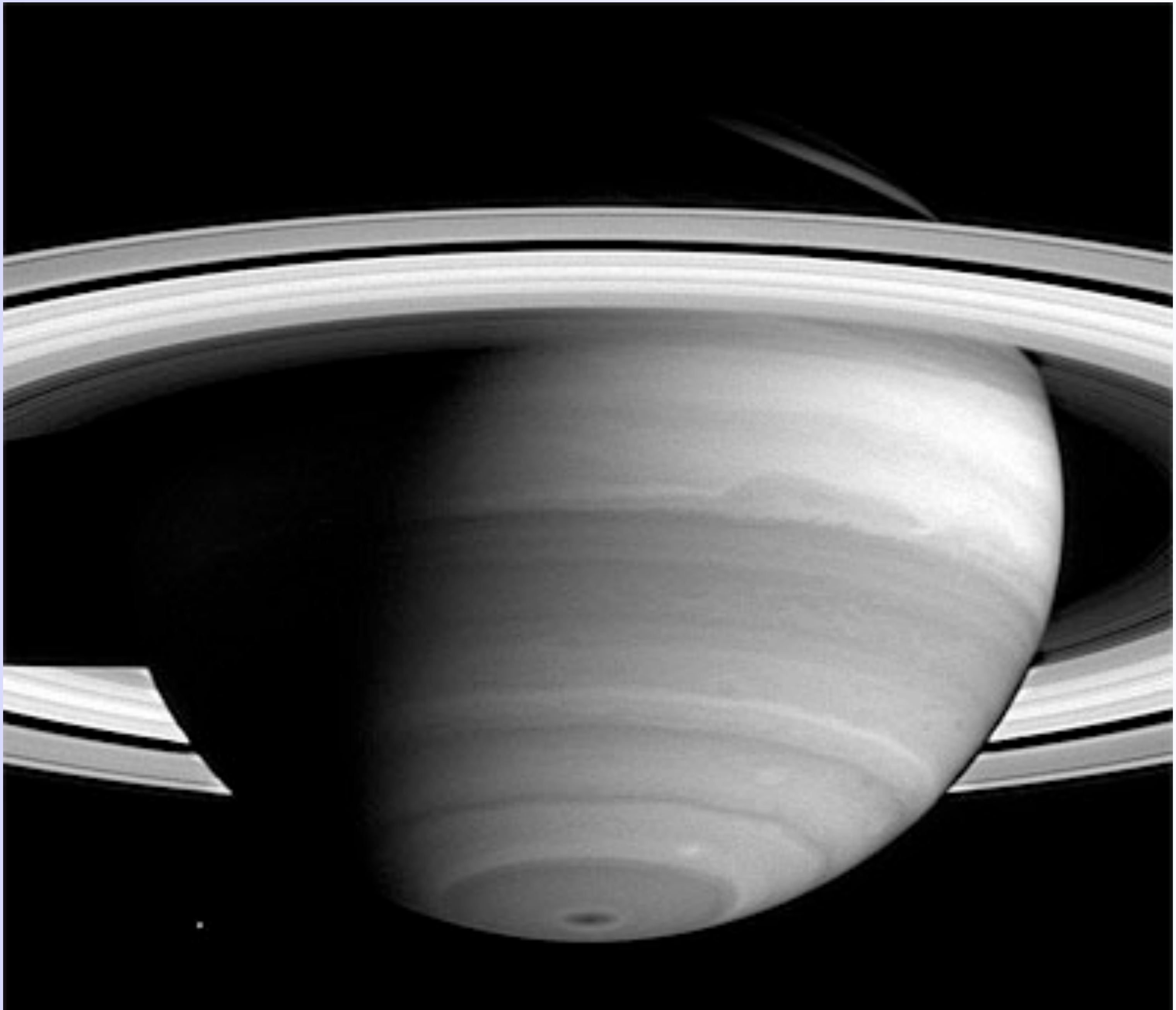
2ème cours de Mécanique Analytique (21 Septembre 2011)

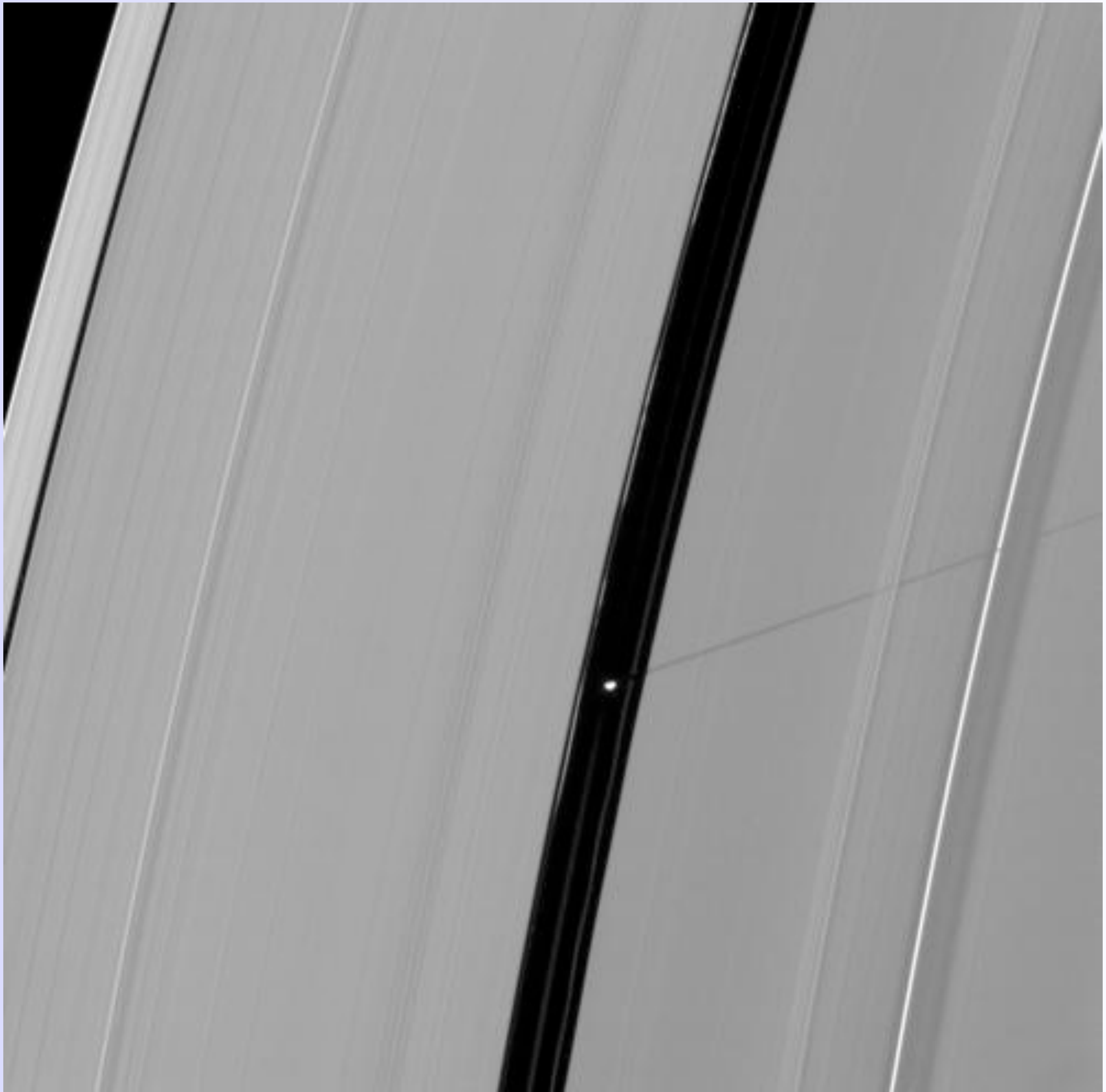
A ce jour: 62 lunes!

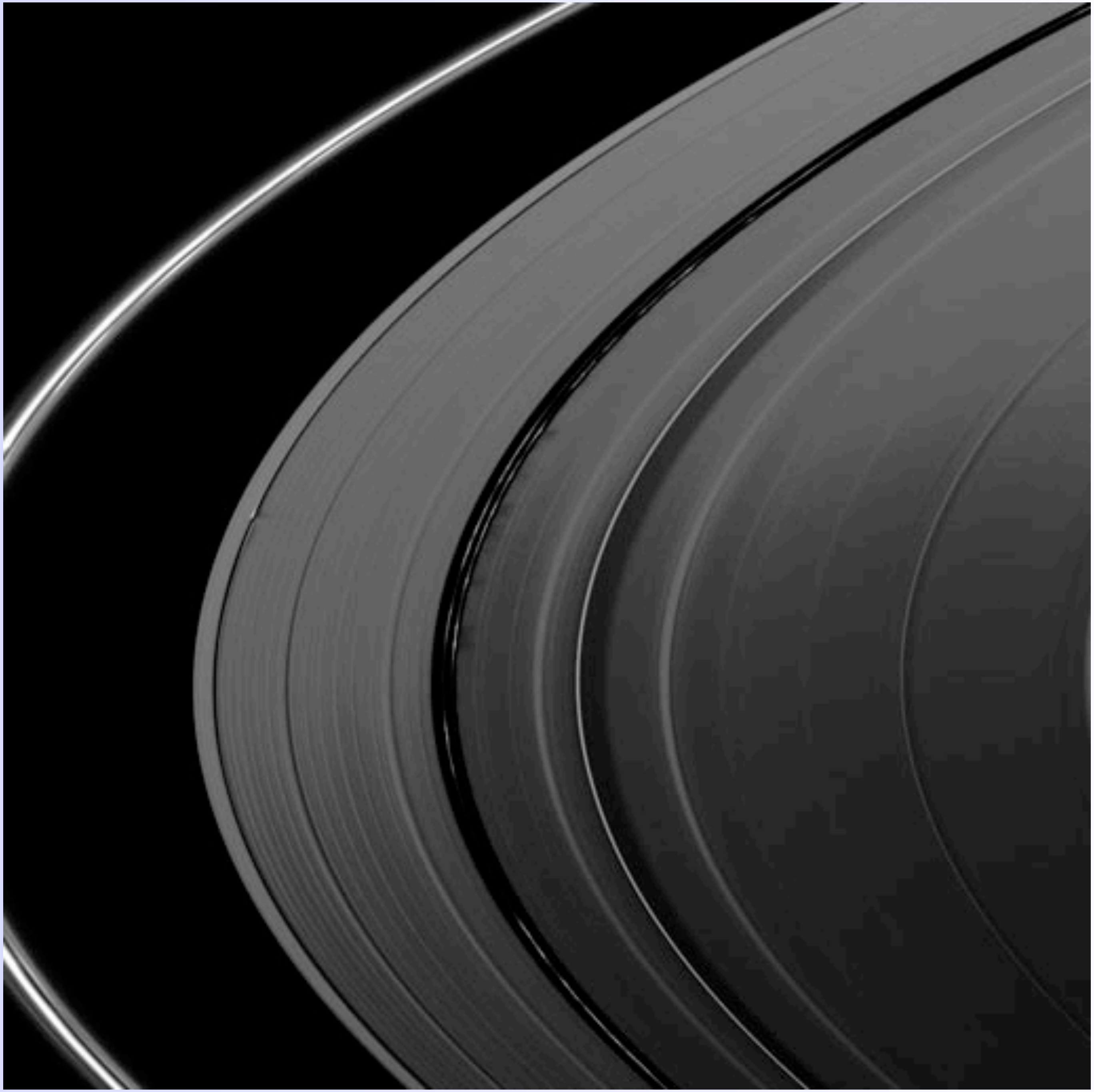


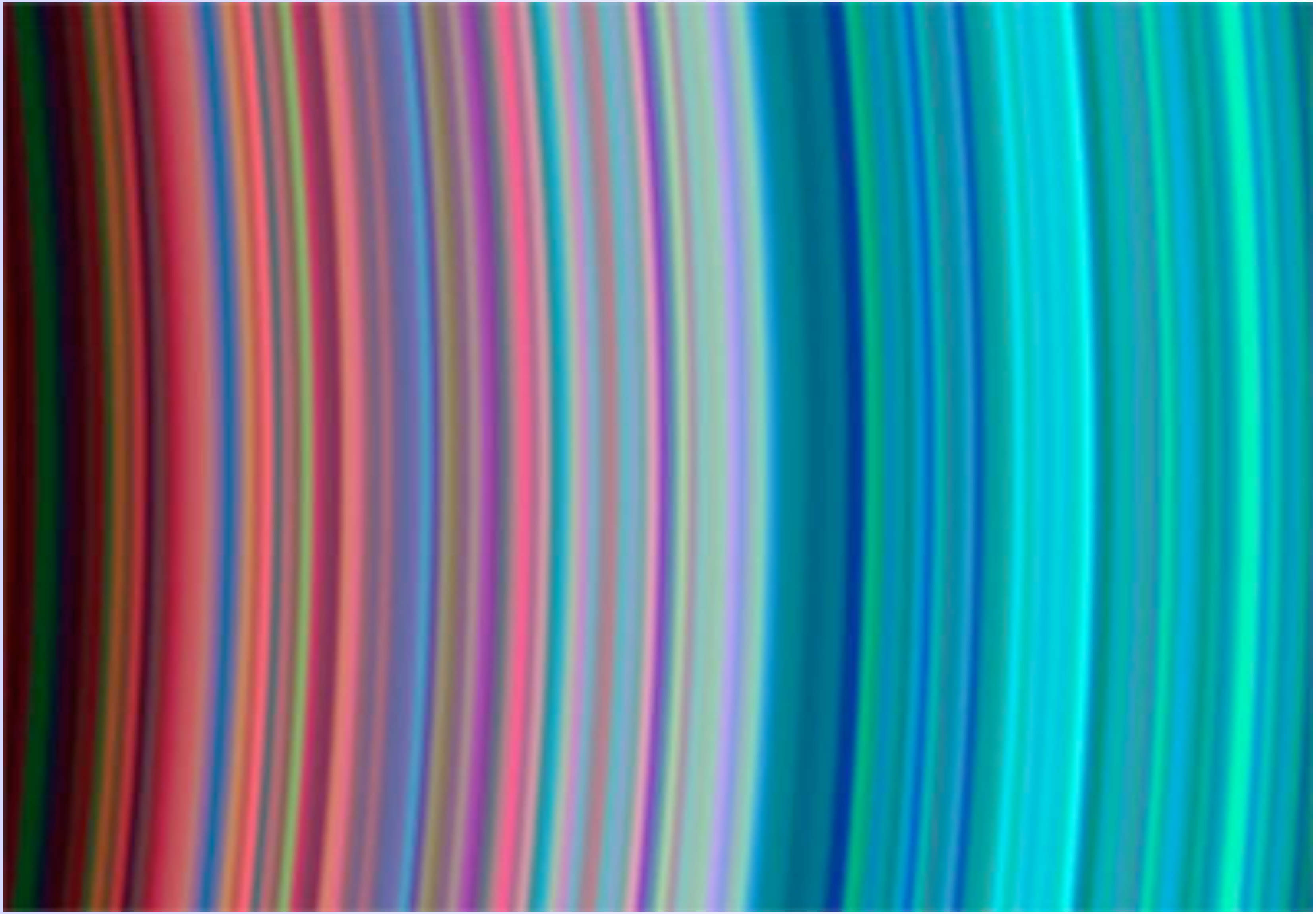












Leçon précédente ...

Introduction

Chapitre 1 : Les équations de Lagrange

- 1.1 Rappel de quelques notions fondamentales
- 1.2 Statique et principe des travaux virtuels
- 1.3 Liaisons holonomes et coordonnées généralisées
- 1.4 Généralisation du principe des travaux virtuels et le principe de d'Alembert

Aujourd'hui ...

Chapitre 1 : Les équations de Lagrange

- 1.4 Généralisation du principe des travaux virtuels et le principe de d'Alembert
- 1.5 Les équations de Lagrange
- 1.6 Exemples d'utilisation des équations de Lagrange

• 1.4 Généralisation du principe des travaux virtuels et le principe de d'Alembert

$$\sum_{\alpha=1}^{N} \vec{F}_\alpha^{\ell} \cdot \delta\vec{r}_\alpha = 0$$

$$\vec{F}_\alpha = \vec{F}_\alpha^{\ell} + \vec{F}_\alpha^{a}$$

- 3N équations de Newton
- ℓ équations holonomes
- f (= 3N - ℓ) équations pour les forces de liaisons

Soient 6N équations pour déterminer 6N inconnues (les $x_{\alpha i}$ et les $F\ell_{\alpha i}$)

Très compliqué !!!

$$\boxed{\sum_{\alpha=1}^{N} \left(m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2} - \vec{F}_\alpha^{a}\right) \cdot \delta\vec{r}_\alpha = 0} \qquad (1.9)$$

$$\delta\vec{r}_\alpha = \vec{v}_\alpha dt$$

- 1.4 Généralisation du principe des travaux virtuels et le principe de d'Alembert

$$\sum_{\alpha=1}^{N}(m_\alpha \frac{d\vec{v}_\alpha}{dt} - \vec{F}^a_\alpha)\cdot \vec{v}_\alpha dt = 0$$

$$\frac{d}{dt}\left(\sum_{\alpha=1}^{N}\frac{1}{2}m_\alpha \vec{v}^2_\alpha\right)dt - \sum_{\alpha=1}^{N}\vec{F}^a_\alpha \cdot d\vec{r}_\alpha = 0$$

$$\sum_{\alpha=1}^{N}\frac{1}{2}m_\alpha \vec{v}^2_\alpha$$

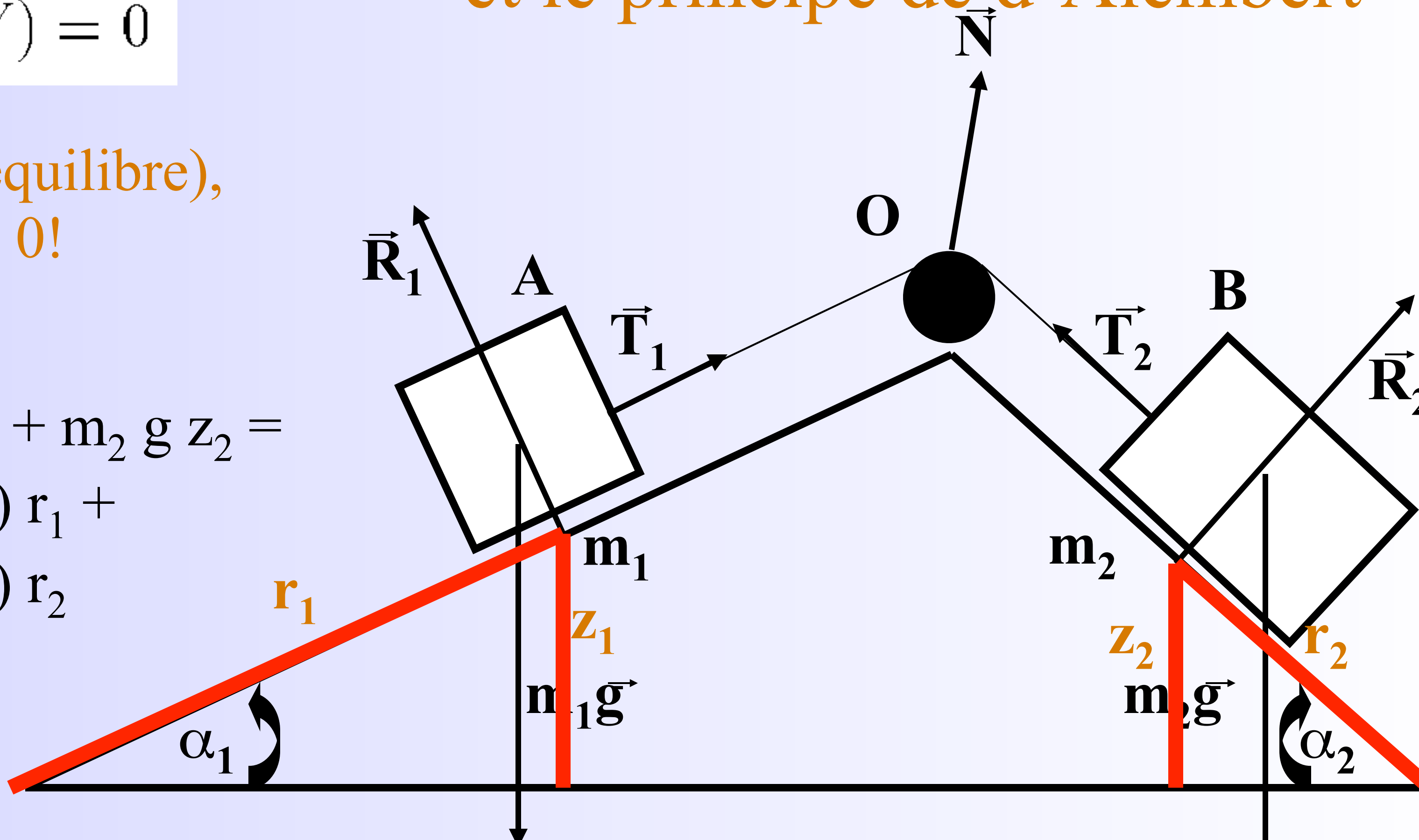
$$d\left(T+V\right) = 0$$

Si T = 0 (équilibre), alors dV = 0!

• 1.4 Généralisation du principe des travaux virtuels et le principe de d'Alembert

$$d\,(T+V)=0$$

Si T = 0 (équilibre), alors dV = 0!



$V = m_1\, g\, z_1 + m_2\, g\, z_2 =$
$m_1\, g \sin(\alpha_1)\, r_1 +$
$m_2\, g \sin(\alpha_2)\, r_2$

$dV = m_1\, g \sin(\alpha_1)\, dr_1 + m_2\, g \sin(\alpha_2)\, dr_2 =$
$m_1\, g \sin(\alpha_1)\, dr_1 - m_2\, g \sin(\alpha_2)\, dr_1 = 0$

$$m_1 \sin(\alpha_1) = m_2 \sin(\alpha_2)$$

• 1.5 Les équations de Lagrange

1.3) Liaisons holonomes et coordonnées généralisées

1.4) Généralisation du principe des travaux virtuels:
le principe de d'Alembert

1.5) Equations de Lagrange !

• 1.5 Les équations de Lagrange

$$\boxed{\sum_{\alpha=1}^{N}\left(m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2} - \vec{F}_\alpha^a\right)\cdot\delta\vec{r}_\alpha = 0} \quad (1.9)$$

$$\delta\vec{r}_\alpha = \frac{\partial\vec{r}_\alpha}{\partial q_i}\delta q_i \quad (1.10)$$

$$\sum_{\alpha=1}^{N} m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2}\cdot\frac{\partial\vec{r}_\alpha}{\partial q_i}\delta q_i = \sum_{\alpha=1}^{N}\vec{F}_\alpha^a\cdot\frac{\partial\vec{r}_\alpha}{\partial q_i}\delta q_i \quad (1.11)$$

$$Q_i = \sum_{\alpha=1}^{N}\vec{F}_\alpha^a\cdot\frac{\partial\vec{r}_\alpha}{\partial q_i} = \sum_{\alpha=1}^{N}\vec{F}_\alpha\cdot\frac{\partial\vec{r}_\alpha}{\partial q_i} \quad (1.12)$$

$$\sum_{\alpha=1}^{N} m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2}\cdot\frac{\partial\vec{r}_\alpha}{\partial q_i} = Q_i \quad (1.13)$$

• 1.5 Les équations de Lagrange

$$\sum_{\alpha=1}^{N} m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2}\cdot\frac{\partial\vec{r}_\alpha}{\partial q_i} = \sum_{\alpha=1}^{N} m_\alpha \left[\frac{d}{dt}\left(\frac{d\vec{r}_\alpha}{dt}\cdot\frac{\partial\vec{r}_\alpha}{\partial q_i}\right) - \frac{d\vec{r}_\alpha}{dt}\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}_\alpha}{\partial q_i}\right)\right] = \sum_{\alpha=1}^{N} m_\alpha \left[\frac{d}{dt}\left(\vec{v}_\alpha\cdot\frac{\partial\vec{r}_\alpha}{\partial q_i}\right) - \vec{v}_\alpha\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}_\alpha}{\partial q_i}\right)\right] \quad (1.14)$$

$$\vec{v}_\alpha = \frac{d\vec{r}_\alpha}{dt} = \frac{\partial\vec{r}_\alpha}{\partial q_j}\dot{q}_j + \frac{\partial\vec{r}_\alpha}{\partial t} \quad (1.15)$$

$$\frac{\partial\vec{r}_\alpha}{\partial q_i} = \frac{\partial\vec{v}_\alpha}{\partial\dot{q}_i} \quad , \quad \frac{d}{dt}\left(\frac{\partial\vec{r}_\alpha}{\partial q_i}\right) = \frac{\partial\vec{v}_\alpha}{\partial q_i} \quad (1.16)$$

- 1.5 Les équations de Lagrange

$$\sum_{\alpha=1}^{N} m_\alpha \left[\frac{d}{dt}\left(\vec{v}_\alpha \cdot \frac{\partial \vec{v}_\alpha}{\partial \dot{q}_i}\right) - \vec{v}_\alpha \cdot \frac{\partial \vec{v}_\alpha}{\partial q_i}\right] = Q_i \qquad (1.17)$$

$$\boxed{T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha v_\alpha^2} \qquad (1.18)$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i} \qquad (1.19)$$

$$\boxed{Q_i = \sum_{\alpha=1}^{N} \vec{F}_\alpha^a \cdot \frac{\partial \vec{r}_\alpha}{\partial q_i}} \qquad (1.20)$$

- 1.5 Les équations de Lagrange

$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha v_\alpha^2, \qquad Q_i = \sum_{\alpha=1}^{N} \vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_i},$$

$$\vec{r}_\alpha = \vec{r}_\alpha(q, t), \qquad \vec{v}_\alpha = \frac{\partial \vec{r}_\alpha}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_\alpha}{\partial t}$$

$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \left(\frac{\partial \vec{r}_\alpha}{\partial q_i} \dot{q}_i + \frac{\partial \vec{r}_\alpha}{\partial t} \right) \cdot \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_\alpha}{\partial t} \right)$$

$$T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j + a_i \dot{q}_i + \frac{1}{2} a \qquad (1.21)$$

• 1.5 Les équations de Lagrange

$$\begin{cases} a_{ij} & = & \displaystyle\sum_{\alpha=1}^{N} m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j} \qquad (a_{ij} = a_{ji}) \\ a_i & = & \displaystyle\sum_{\alpha=1}^{N} m_\alpha \frac{\partial \vec{r}_\alpha}{\partial t} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_i} \\ a & = & \displaystyle\sum_{\alpha=1}^{N} m_\alpha \frac{\partial \vec{r}_\alpha}{\partial t} \cdot \frac{\partial \vec{r}_\alpha}{\partial t} \end{cases} \qquad (1.22)$$

$$\begin{cases} \vec{r}_\alpha & = & \vec{r}_\alpha(q) \\ a_i & = & 0 \quad , \quad a = 0 \end{cases} \qquad (1.23)$$

$$T = \frac{1}{2} a_{ij}(q) \dot{q}_i \dot{q}_j \qquad (1.24)$$

• 1.5 Les équations de Lagrange

$$\delta\vec{r}_\alpha = \frac{\partial\vec{r}_\alpha}{\partial q_i}\delta q_i,$$

$$\mathcal{T} = \sum_{\alpha=1}^{N} \vec{F}_\alpha \cdot \delta\vec{r}_\alpha = Q_i \delta q_i \qquad (1.25)$$

$$\boxed{Q_i = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right) - \frac{\partial V}{\partial q_i}} \qquad (1.26)$$

$$V = V(q, \dot{q}, t) \qquad (1.27)$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0} \qquad (1.28)$$

$$\boxed{L = T - V} \qquad (1.29)$$

• 1.5 Les équations de Lagrange

$$\vec{F}^a_\alpha = -\mathrm{grad}_\alpha V \qquad (1.30)$$

$$\begin{aligned} Q_i &= \sum_{\alpha=1}^{N} \vec{F}^a_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_i} = \sum_{\alpha=1}^{N}\sum_{(j)=1}^{3} F^a_{\alpha(j)} \frac{\partial x_{\alpha(j)}}{\partial q_i} \\ &= -\sum_{\alpha=1}^{N}\sum_{(j)=1}^{3} \frac{\partial V}{\partial x_{\alpha(j)}} \frac{\partial x_{\alpha(j)}}{\partial q_i} = -\frac{\partial V}{\partial q_i} \end{aligned} \qquad (1.31)$$

• 1.5 Les équations de Lagrange

$$L = T - V = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha - V(\vec{r}_1, ..., \vec{r}_N, t) \qquad (1.32)$$

$$\frac{\partial L}{\partial q_i} = -\frac{\partial V}{\partial q_i} \quad \text{et} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = m_{\alpha(i)}\ddot{q}_i \qquad (1.33)$$

$$m_\alpha \ddot{\vec{r}}_\alpha = -\mathrm{grad}_\alpha V \qquad (1.34)$$

$$\{q_1, ..., q_{f=3N}\} = \{\vec{r}_1, ..., \vec{r}_N\}$$

• 1.6 Exemples d'utilisation des équations de Lagrange

- Choisir un ensemble de f coordonnées généralisées $(q_1, q_2, \ldots, q_f)$
- Exprimer T, Q_i et V en fonction des q_i, $\dot{q}_i$ et t
- Calculer L = T - V
- Ecrire les équations de Lagrange