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THE EQUILIBRIUM PRICING MODELS: THEIR RELATIONSHIPS AND THEIR APPLICATION TO PORTFOLIO THEORY

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The structure of this paper is the following. The first section deals with the Time State Preference model. Its assumptions will be presented and the model will be applied to the case of the valuation of a firm and to an asset pricing equilibrium model. Afterwards, the other two equilibrium models, the CAPM and the APT, are presented in the second section. Then, starting form the TSP equilibrium model, the third section will examine the relationships between these three models. The fourth section is devoted to the discussion of the disadvantages of the CAPM and the APT and will approach the problems related to their testability.

2. The Time State Preference Model

Any investment decision deals with the choice of resources allocation between consumption and investment over future time intervals. The difficulty in this choice arises from the uncertainty about the benefits in the future. In this context, the TSP model defines for each time interval or period a set of possible end-of-period payoffs, associated with mutually exclusive states of nature. Then the probability of an end-of-period payoff is the probability of occurrence of the state associated with this payoff. The uncertainty disappears once, at the end of the period, the state is revealed. So in the TSP framework, an asset is represented by successive sets of possible end-of-period payoffs over future time intervals, the payoffs of one set being associated with mutually exclusive states of nature.

If a pure security is defined as *a security which pays one dollar at the end of the period if a given state occurs and nothing if any other state occurs* [1], then a market asset can be considered as a combination of pure securities, and its set of payoffs is distributed over the states of nature. And, insofar as the capital market is complete, that is *the number of unique linearly independent securities is equal to the total number of alternative future states of nature* [2], and no arbitrage profit is possible, then

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[1]Copeland T.E. and Weston J.F., Financial Theory and Corporate Policy, Addison-Wesley, 1983, p.110.

[2]Copeland T.E. and Weston J.F., op. cit., p.111.

So L is function of the c_i and λ.Looking for a maximum, L is differentiated in respect to each variable, and each derivative is set equal to zero.

$$\frac{\delta L}{\delta c_i} = q_i\ U'(c_i) + \lambda\ P_i = 0 \quad \text{for each } i \tag{2.5}$$

$$\frac{\delta L}{\delta \lambda} = -W_0 + \sum_{i=1}^{N} c_i P_i = 0 \tag{2.6}$$

Summing the derivatives in respect to the c_i

$$\sum_{i=1}^{N} \frac{\delta L}{\delta c_i} = \sum_{i=1}^{N} q_i U'(c_i) + \lambda \sum_{i=1}^{N} P_i = 0$$

as $\sum_{i=1}^{N} q_i U'(c_i)$ is the expected marginal utility of U'(c) and given (2.3) then

$$E[U'(c)] + \lambda \frac{1}{r} = 0$$

and

$$\lambda = -r\ E[U'(c)] \tag{2.7}$$

Substituting (2.7) in (2.5)

$$\frac{\delta L}{\delta c_i} = q_i\ U'(c_i) - rE[U'(c)]\ P_i = 0$$

hence

$$P_i = \frac{q_i\ U'(c_i)}{r\ E[U'(c)]} \tag{2.8}$$

At this stage we can observe that the price the investor should pay for the pure security i depends on three elements. The first is the probability of occurrence q_i of the state associated to its payoff. The greater the probability, the more he should pay for the security. This implies the assumption of homogeneous expectations, that is, the probabilities of the states are known to all investors. The second element is the relative marginal utility of the end-of-period payoff. The higher this is, the higher the

The equation (2.10) can easily be extended to a multiperiod context. To do so, one must consider that this relationship is repeated for each period. So adding the time subscript t to the X_j and the c, and then summing the discounted values of the different periods yields the general relationship [4]:

$$P_j = \sum_t \frac{1}{r^t}\left(E[X_{jt}] + \frac{cov(X_{jt},U'(c_t))}{E[U'(c_t)]}\right) \qquad (2.11)$$

The problem with such multiperiod formulation is that the c_t, the whole consumptions of each period, are not determined. While in the single period economy the end-of-period wealth of the investors, their cash flows and their consumption are necessarily equal, in the multiperiod economy these variables may be different because of the time dimension involved in the consumption-investment decision. In this regard, Stapleton and Subrahmanyam (1980) made a survey of the different approaches solving this problem. These generally consist in defining a utility of wealth function or in considering the consumption as exogenous or non stochastic function of wealth.

Let us now come back to a single period economy, assuming that there is only one investor in the market who possesses all, or that all investors act in the same way, in that case c, the whole consumption, is equal to the cash flow of the market portfolio, that is, $c = X_m$. Therefore the relationship (2.11) becomes :

$$P_j = \frac{1}{r}\left(E[X_j] + \frac{cov(X_j,U'(X_m))}{E[U'(X_m)]}\right) \qquad (2.12)$$

This is the one-period TSP equilibrium whose characteristics are :

- there is no assumption concerning the distributions
- the utility can be state dependent

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[4] r is assumed to be constant across the periods

the rate of return R_j of an asset j and the rate of return R_m of the market portfolio by the relationships :

$$R_j = \frac{X_j - P_j}{P_j} = \frac{X_j}{P_j} - 1$$

$$R_m = \frac{X_m - P_m}{P_m} = \frac{X_m}{P_m} - 1$$

We also know that the risk free rate Rf is equal to r minus one. Then dividing the two members of the equation (3.1) by P_j and dividing and multiplying the second element to the right of the equal sign by P_m^2, and substituting the three above relationships yields :[5]

$$E(R_j) = Rf + \frac{cov(R_j,R_m)}{var(R_m)} [E(R_m) - Rf] \qquad (3.4)$$

So, the expected rate of return of an asset is equal to the risk free rate, or price of time, plus a reward that is proportional to the risk of this asset measured by its beta (β).

The assumptions of the standard CAPM are :

- investors are risk averse,
- investors make decisions in terms of expected values and standard deviations of the cash flows which have a normal distribution,

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5 $$P_j = \frac{1}{r}\left(E[X_j] - \frac{(E[X_m] - r\,P_m)\,cov(X_j,X_m)}{var(X_m)}\right)$$

$$1 = \frac{1}{r}\left(\frac{E[X_j]}{P_j} - \frac{P_m^2\,(E[X_m] - r\,P_m)\,cov(X_j,X_m)}{P_m^2\,P_j\,var(X_m)}\right)$$

$$1 = \frac{1}{1+R_f}\left(E(R_j+1) - \frac{[E(R_m+1)-(1+Rf)]\,cov(R_j+1,R_m+1)}{var(R_m+1)}\right)$$

as $cov(R_j,1)=0$, $cov(R_m,1)=0$, $var(1)=0$

$$1 + Rf = E(R_j) + 1 - \frac{(E(R_m) - Rf)\,cov(R_j,R_m)}{var(R_m)}$$

hence $$E(R_j) = Rf + \frac{cov(R_j,R_m)[E(R_m) - Rf]}{var(R_m)}$$

Similarly to the CAPM, the APT model can be defined in terms of prices as expressed by the relationship

$$P_j = \frac{1}{r}\left(E(X_j) + \sum b_{ji} F_i\right) \tag{3.6}$$

The price of an asset is defined as the discounted value of a certainty equivalent of a cash payment. This cash payment is equal to the expected cash flow corrected by a compensation for risk taking, compensation which is linearly related to k factors.

Again, dividing (3.6) by P_j and substituting the value of R_j and R_f, we derive the APT in terms of rates of return.[6]

$$E(R_j) = Rf + \sum c_{ji} F_i \tag{3.7}$$

where $c_{ji} = -\frac{b_{ji}}{P_j}$

So, the expected rate of return of an asset is equal to the risk free rate plus a reward that is a linear combination of the risk factors F_i.

4. Derivation of the APT and the CAPM from the TSP

4.1. Derivation of the APT

Starting from the one period TSP equilibrium equation (2.12),

$$P_j = \frac{1}{r}\left(E[X_j] + \frac{\operatorname{cov}(X_j, U'(X_m))}{E[U'(X_m)]}\right)$$

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6 $P_j = \frac{1}{r}\left(E(X_j) + \sum_{i=1}^{N} b_{ji} F_i\right)$

$$\frac{P_j}{P_j} = \frac{1}{r}\left(\frac{E(X_j)}{P_j} + \frac{\sum b_{ji} F_i}{P_j}\right)$$

$$1+R_f = E(R_j) + 1 + \frac{\sum b_{ji} F_i}{P_j}$$

assuming that $c_{ji} = -\frac{b_{ji}}{P_j}$ then $E(R_j) = R_f + \sum c_{ji} F_i$

4.2. *Derivation of the CAPM*

Starting again from the one period equilibrium equation of the TSP (2.12),

$$P_j = \frac{1}{r}\left(E[X_j] + \frac{cov(X_j,U'(X_m))}{E[U'(X_m)}\right) \qquad (2.12)$$

Let us now assume that the utility function $U(X_m)$ is quadratic or that X_m and X_j are joint normal, then[9]

$$cov[X_j,U'(X_m)] = E[U''(X_m)].cov[X_j,X_m]$$

Substituting this result in (2.12) we obtain the CAPM

$$P_j = \frac{1}{r}\left(E[X_j] + \frac{E[U''(X_m)]\ cov(X_j,X_m)}{E[U'(X_m)]}\right) \qquad (4.3)$$

or

$$P_j = \frac{1}{r}\left(E[X_j] - \lambda\ cov(X_j,X_m)\right) \qquad (4.4)$$

where

$$\lambda = - \frac{E[U''(X_m)]}{E[U'(Xm)]} \qquad (4.5)$$

is the market price per unit of risk.

To find again the CAPM relationship (3.1), one must consider that (4.4) also holds for the market portfolio,

$$P_m = \frac{1}{r}\left(E[X_m] - \lambda cov(X_m,X_m)\right)$$

$$P_m = \frac{1}{r}\left(E[X_m] - \lambda var(X_m)\right)$$

hence

$$\lambda = \frac{E(X_m) - r\ P_m}{var(X_m)}$$

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[9]If the utility function is quadratic: $U(X_m) = a + bX_m + c\ X_m^2$
then its first derivative is linear in X_m and we have $cov(X_j,U'(X_m)) = 2c\ cov(X_m,X_m)$
since the second derivative of $U(X_m)$ is equal to 2c, then $cov(X_j,U'(X_m)) = U''(X_m)\ cov(X_j,X_m)$
As for the joint normality case, see Rubinstein M., 1976.

It is worth noting that the risk premia associated with one APT factor is function of the sensitivity b_{ji} of the cash flow to the value of the factor, and of the covariance between this factor and the market portfolio. It is clear then that although it is often claimed that the APT does not require the market portfolio, the market portfolio plays a role in both equilibrium models, especially for their empirical testing.

As we have made an additional assumption to derive the CAPM from the APT, one could be tempted to define the CAPM as a special case of the APT, but as it was pointed out in the introduction, it is more relevant to deal with the testability of these two equilibrium models than with the superiority of one to the other. We could also have followed another approach in this section, starting first with the assumption of normality in order to derive the CAPM from the TSP, and afterwards making the additional factor model assumption so as to derive the APT. In its turn, this second approach could let suppose that the CAPM is the most general. In fact, the CAPM and the APT have been derived under different assumptions. Therefore they are different equilibrium models and they can hold either simultaneously or separately. It was proved in this present section that they are identical only if one assumes that the cash flows are generated by a factor model, and that they are normally distributed or the utility function is quadratic.

What is more, in order to define a logical sequence of the two models, it has to be proved that the factor model assumption is less restricting than the normality assumption. In this regard, Stapleton and Subrahmanyam (1983) proved that a one factor model assumption is almost identical to the normality assumption if the factor is the market portfolio.

5. CAPM versus APT

The object of this section is to examine the main reasons that incline most of the researchers to believe that the APT model is more robust than the CAPM, and in return, to present the different criticisms on the APT. It will be shown that the normally distributed, the CAPM is identical to the APT, the k factors being only a disaggregation of risk premia. Therefore, in this context, allowing the equilibrium to be dependent on a factor model, has sense to the extent it could help to better circumscribe the unknown market portfolio.

The question is first to know which of the assumption of normality and linearity of the cash flows is likely to exist, and consequently if mean and variance are the relevant parameters to describe the distributions of the cash flows.

2) The APT makes no assumption concerning the utility function of individuals.

But the nature of the factors, as well as their associated measures of risk, are not defined and they cannot be interpreted. Since anything can be included in the factor structure, we are wondering about the use of the APT with regards to theory such as the cost of capital.

3) The APT allows the equilibrium to be dependent on k factors. This is attractive, but :

- The number of factors is not defined and it rises as the number of firms in the subset is increased.
- The limitation of factor analysis, due to processing capacity for computation, impedes the use of large subset of assets. As a consequence the factors are not necessarily identical from one subset to another, their number can vary and they are also influenced by the way of grouping the assets.
- Similarly to the CAPM, which requires the stationarity of the betas, the sensitivity coefficients b_{ji} should also be stationary and, furthermore, the factor structure should be replicable across various time periods.

4) The APT does not give the market portfolio any special role.

measured. Any portfolio, which is an average of the assets included in the subset, can play the role of the market portfolio insofar as it is mean variance efficient. In this regard, let us recall the criticism of Roll concerning the testability of the CAPM, that is, the CAPM will hold if the portfolio used as the market portfolio is mean variance efficient.

Finally, it is worth noting that most empirical tests of the CAPM concern data about the assets of the stock exchange of a determined country. And the index used is often a proxy of the market portfolio. Consequently, such tests are tests of the CAPM as a local equilibrium model.

6) The APT can be extended to a multiperiod framework.

Several studies (Stapleton and Subrahmanyam, 1978; Constantinides, 1980) have also extended the standard CAPM to a multiperiod framework. Furthermore, the extension of the one period APT to a multiperiod APT requires, like the CAPM, additional sets of assumptions (Ohlson and Garman, 1980; Connor, 1984).

6. Conclusion

As far as we are concerned, our ambition in this paper is not to take a position in the debate CAPM versus APT. The relationships between these two equilibrium models, as well as with a more general one, the TSP, were defined. It appeared that under both assumptions of factor linearity and of normality in the cash flows, APT and CAPM are identical and they only differ in their presentation, the APT using a decomposition of the risk premia into k factors. The assumptions and the main criticisms against both models were presented, and it appeared that the APT, in spite of its attractive simplicity, presents a number of difficulties when it is tested. More fundamentally again, it seems that the APT, as an equilibrium model, suffers from the same deficiency as the CAPM, that is, the entire universe of assets has to be measured.

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