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Rapid Communication

Asymptotic expansion of slightly coupled modal dynamic transfer functions

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ABSTRACT

In case of non-diagonal modal damping, normal modes of vibration do not decouple modal equations. The usual way to handle such a non-diagonal modal damping matrix is to neglect its off-diagonal elements. In this paper, we propose an approximate method based on an asymptotic expansion of the transfer function. It is intermediate between the classical decoupling approximation and the formal solution requiring a full matrix inversion. Indeed, on the one hand, it allows to partially account for modal coupling and, on the other hand, still allows the modal equations to be solved independently from each other. We first provide the mathematical background necessary to canvass the proposed method, then consider a benchmark against which the benefits of the method are measured.

1. Introduction

Various engineering applications are modeled by a set of coupled second-order ordinary differential equations such as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}. \tag{1}$$

In the particular case of mechanical or structural vibrations, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ represent the mass, damping and stiffness matrices, while $\mathbf{f}(t)$ and $\mathbf{x}(t)$ represent externally applied loads and structural displacements. The normal modes of vibrations, gathered here in a matrix $\mathbf{\Phi}$, are computed from $\mathbf{M}$ and $\mathbf{K}$ and normalized through the mass matrix, so that

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi} = \mathbf{I}; \quad \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi} = \mathbf{\Omega}, \tag{2}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{\Omega}$ is the diagonal matrix of squared circular frequencies [1]. In practical applications, a limited number of modes is considered. This is practically performed by keeping some columns of $\mathbf{\Phi}$ only, resulting therefore in a slender rectangular $n \times m$ matrix ($m \ll n$). The change of variables $\mathbf{x}(t) = \mathbf{\Phi}\mathbf{q}(t)$ together with the projection into the modal space (i.e. the pre-multiplication of (1) by $\mathbf{\Phi}^{\mathrm{T}}$) yields a set of m ordinary differential equations

$$\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{\Omega}\mathbf{q} = \mathbf{g}, \tag{3}$$

where $\mathbf{D} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi}$ is the modal damping matrix and $\mathbf{g} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{f}$ is the vector of modal forces. The frequency domain formulation of (3), obtained by Fourier transformation of both sides of (3), writes

$$\mathbf{Q}(\omega) = \mathbf{H}(\omega)\,\mathbf{G}(\omega), \tag{4}$$

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where $\mathbf{Q}(\omega)$ and $\mathbf{G}(\omega)$ are, respectively, the Fourier transforms of $\mathbf{q}$ and $\mathbf{g}$ and

$$\mathbf{H}(\omega) = (-\omega^2\mathbf{I} + \mathrm{i}\omega\mathbf{D} + \mathbf{\Omega})^{-1} \tag{5}$$

is the modal transfer matrix of the system.

The modal approach is classically used in order (i) to reduce the number of unknowns (as $m \ll n$) and (ii) to uncouple the set of equations (1). The normal modes of vibration are computed in such a way that $\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}$ and $\mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi}$ are precisely diagonal, see (2), but nothing can in principle be said about $\mathbf{D}$. Nevertheless, owing to the lack of knowledge about damping in structures, the damping matrix $\mathbf{C}$ is commonly assumed to take a particular formulation, as the most famous Rayleigh damping, allowing goal (ii) to be reached. However, in some circumstances as the presence of dash-pots or aerodynamic loading, a specific mathematical modeling of damping forces is available. Unfortunately it usually yields a non-diagonal modal damping matrix $\mathbf{D}$, which does not allow for the proper decoupling of (3). For simplicity in the following argumentation, it is convenient to decompose the modal damping matrix $\mathbf{D}$ as a sum of two matrices

$$\mathbf{D} = \mathbf{D}_d + \mathbf{D}_o \tag{6}$$

collecting exclusively diagonal ($\mathbf{D}_d$) and off-diagonal ($\mathbf{D}_o$) elements of $\mathbf{D}$.

A thorough review of the literature related to non-diagonal modal damping matrices has been recently presented in [2,3]. We therefore restrict the portrayal of the panorama to a brief description of three major philosophies.

In the first approach, being actually the formal one, the full modal damping matrix $\mathbf{D}$ is considered. The set of coupled equations (3), but of reduced size m however, has to be handled. The only remaining advantage of the modal approach is therefore a reduction of the number of unknowns, which may be substantial in some applications.

A second strategy is based on complex mode shapes [4,5], allowing a diagonalization of the system in a state space. Because it doubles the size of eigenvectors ($2n$) and due to the difficult interpretation of non-real mode shapes, this solution is often overlooked.

At last but not least, the third solution, apparently due to Lord Rayleigh [6], is to neglect the off-diagonal terms of the modal damping matrix and therefore to simply set $\mathbf{D}_o = 0$ in (6). The origin of the method is based on the observation that elements of $\mathbf{D}_o$ are usually much smaller than those of $\mathbf{D}_d$. There exists several ways to formally express the smallness of $\mathbf{D}_o$ compared to $\mathbf{D}_d$. However they seem to be strongly interconnected and therefore equivalent [7]. Among them, we will thus only exploit the diagonality index $\rho(\mathbf{D})$ defined as the largest eigen value of $\mathbf{D}_d^{-1}\,\mathbf{D}_o$ in absolute value

$$\rho(\mathbf{D}) = \max(|\mathrm{eig}\mathbf{D}_d^{-1}\,\mathbf{D}_o|). \tag{7}$$

It is demonstrated that this index is connected to the mathematical concept of diagonal dominance [7].

This third method, referred to as *the decoupling approximation next*, has been prominently applied despite its evident lack of formalism but condoned by virtue of its basement on an engineering thinking. In particular the modal superposition method—a deterministic time domain approach—requires the modal equations (3) to be uncoupled and hinges therefore on this approximation in case of non-diagonal modal damping matrix. Similarly a stochastic dynamic analysis in frequency domain [8], for which the analysis requires pre- and post-multiplication of the power spectral density matrix of modal forces by the transfer matrix $\mathbf{H}$, is evidently faster when $\mathbf{H}$ is diagonal, i.e. when $\mathbf{D}_o$ is neglected.

One could believe that the smaller the off-diagonal terms, the smaller the error, concerning the estimation of modal responses $\mathbf{q}$. This is not necessarily true. Indeed a recent research came to the puzzling conclusion that the error committed when neglecting off-diagonal elements might grow as the diagonality index decreases [3]. Clearly these findings trigger the questioning of decades of use of the decoupling approximation initially formulated by Lord Rayleigh.

In this paper we propose a novel approximate method to partially account for modal coupling. The method builds up on the smallness of off-diagonal terms to construct an approximate expression of the modal dynamic transfer function $\mathbf{H}$. Actually we show that the classical decoupling approximation is the leading order solution of the asymptotic expansion of (5) for small $\rho(\mathbf{D})$, whereas the proposed method simply consists in enriching it with the first-order correction.

The proposed model is presented in Section 2. Then, in Section 3, the major benefits brought by this first correction are enlightened by comparison with results obtained with the decoupling approximation.

2. Mathematical formulation

Let us write the modal transfer matrix

$$\mathbf{H} = (\mathbf{J}_d + \mathbf{J}_o)^{-1}, \tag{8}$$

where the introduction of $\mathbf{J}_d = -\mathbf{I}\omega^2 + \mathrm{i}\omega\mathbf{D}_d + \mathbf{\Omega}$ and $\mathbf{J}_o = \mathrm{i}\omega\mathbf{D}_o$ allows a formal separation of diagonal and off-diagonal elements. Because $\mathbf{J}_d$ is a non-singular matrix for every ω, we may also write

$$\mathbf{H} = (\mathbf{I} + \mathbf{J}_d^{-1}\,\mathbf{J}_o)^{-1}\mathbf{H}_d, \tag{9}$$

where $\mathbf{H}_d = \mathbf{J}_d^{-1}$ is the modal transfer matrix obtained with the decoupling approximation. The factor $(\mathbf{I} + \mathbf{J}_d^{-1}\,\mathbf{J}_o)^{-1}$ appears therefore as a formal correction that needs to be applied to the decoupling approximation when non-diagonal modal damping takes place ($\mathbf{J}_o \neq 0$). It is interesting to notice that this factor regularly tends towards the identity matrix as $\mathbf{J}_o$ tends towards zero. In the following developments, we perform an asymptotic expansion of this factor for small diagonality index, with the main objective of avoiding any full matrix inversion.

The eigenvalue decomposition of $\mathbf{D}_d^{-1}\ \mathbf{D}_o$ is

$$\mathbf{D}_d^{-1}\ \mathbf{D}_o = \mathbf{\Psi}\boldsymbol{\lambda}\mathbf{\Psi}^{-1}, \tag{10}$$

where $\mathbf{\Psi}$ is a matrix of L_∞-normalized eigenvectors and $\boldsymbol{\lambda}$ is a diagonal matrix with eigenvalues. The definition of the diagonality index ρ, see (7), is such that

$$\rho(\mathbf{D}) = \|\boldsymbol{\lambda}\|_\infty. \tag{11}$$

Consideration of a small diagonality index suggests to write

$$\boldsymbol{\lambda} = \varepsilon\mathbf{\Lambda}, \tag{12}$$

where $\mathbf{\Lambda} = o(1)$ and $\varepsilon \ll 1$ is the small parameter. The off-diagonal component of modal damping $\mathbf{D}_o$ is expressed by substitution of (12) into (10)

$$\mathbf{D}_o = \varepsilon\mathbf{D}_d\mathbf{\Psi}\mathbf{\Lambda}\mathbf{\Psi}^{-1}. \tag{13}$$

Similarly the eigenvalue decomposition of $\mathbf{J}_d^{-1}\ \mathbf{J}_o$ is expressed as

$$\mathbf{J}_d^{-1}\ \mathbf{J}_o = \mathbf{\Upsilon}\boldsymbol{\mu}\mathbf{\Upsilon}^{-1}, \tag{14}$$

where $\mathbf{\Upsilon}(\omega)$ collects the L_∞-normalized eigenvectors and $\boldsymbol{\mu}(\omega)$ is a diagonal matrix with eigenvalues. A major difference between (10) and (14) is the dependency upon frequency: (10) is an intrinsic property of modal damping and hence independent from frequency whereas (14) is related to the frequency response of the system resulting therefore in frequency dependent eigenvalues $\boldsymbol{\mu}(\omega)$.

Two interesting limit behaviors of (14) are

$$\lim_{\omega\to 0}\mathbf{J}_d^{-1}\ \mathbf{J}_o = \lim_{\omega\to 0}\mathrm{i}\omega\mathbf{\Omega}^{-1}\mathbf{D}_o = 0$$

and

$$\lim_{\omega\to +\infty}\mathbf{J}_d^{-1}\ \mathbf{J}_o = \lim_{\omega\to +\infty}\frac{-\mathrm{i}}{\omega}\mathbf{D}_o = 0. \tag{15}$$

They indicate that $\mu(0) = 0$ and $\mu(+\infty) = 0$, and that $\mathbf{\Upsilon}$ tends towards the eigenmatrix of $\mathbf{\Omega}^{-1}\mathbf{D}_o$ for small ω and towards that of $\mathbf{D}_o$ for large ω.

Further developments are based on the fact that eigenvalues of $\mathbf{J}_d^{-1}\ \mathbf{J}_o$ are small when the diagonality index is small. To justify this postulate, the expression of $\mathbf{J}_d^{-1}\ \mathbf{J}_o$ is expanded as follows:

$$\begin{aligned}\mathbf{J}_d^{-1}\ \mathbf{J}_o &= \mathrm{i}\omega(\mathbf{\Omega} - \mathbf{I}\omega^2 + \mathrm{i}\omega\mathbf{D}_d)^{-1}\mathbf{D}_o = \mathrm{i}\omega\left[\mathrm{i}\omega\mathbf{D}_d\left(\frac{1}{\mathrm{i}\omega}\mathbf{D}_d^{-1}\ (\mathbf{\Omega} - \mathbf{I}\omega^2) + \mathbf{I}\right)\right]^{-1}\mathbf{D}_o \\ &= \left(\frac{1}{\mathrm{i}\omega}\mathbf{D}_d^{-1}\ (\mathbf{\Omega} - \mathbf{I}\omega^2) + \mathbf{I}\right)^{-1}\mathbf{D}_d^{-1}\ \mathbf{D}_o = \varepsilon\left(\frac{1}{\mathrm{i}\omega}\mathbf{D}_d^{-1}\ (\mathbf{\Omega} - \mathbf{I}\omega^2) + \mathbf{I}\right)^{-1}\mathbf{\Psi}\mathbf{\Lambda}\mathbf{\Psi}^{-1}.\end{aligned} \tag{16}$$

As the limit behaviors $\mathbf{J}_d^{-1}\ \mathbf{J}_o$ for $\omega\to 0$ and $+\infty$ are established, see (15), and because only three different regimes (quasi-static, resonant, inertial) are expected in a linear dynamical system, all there is left to find out is the behavior in the vicinity of natural frequencies, where resonance phenomena can eventually take place. In frequency bands centered around natural frequencies, the following condition holds:

$$\mathbf{\Omega} - \mathbf{I}\omega^2 \simeq 0, \tag{17}$$

which indicates, after substitution into (16), that $\mathbf{J}_d^{-1}\ \mathbf{J}_o \simeq \varepsilon\mathbf{\Psi}\mathbf{\Lambda}\mathbf{\Psi}^{-1}$ in the vicinity of natural frequencies. As $\mathbf{\Psi}$ and $\mathbf{\Lambda}$ are of order 1, we may therefore conclude that the eigenvalues of $\mathbf{J}_d^{-1}\ \mathbf{J}_o$ are of order ε at most, from where we write

$$\boldsymbol{\mu}(\omega) = \varepsilon\mathscr{M}(\omega), \tag{18}$$

with $\mathscr{M}(\omega) = o(1)$. The eigenvalue decomposition of $\mathbf{J}_d^{-1}\ \mathbf{J}_o$ is therefore written

$$\mathbf{J}_d^{-1}\ \mathbf{J}_o = \varepsilon\mathbf{\Upsilon}\mathscr{M}\mathbf{\Upsilon}^{-1}. \tag{19}$$

Finally we turn back to the correction factor introduced in (9), for which an approximate expression is sought. As the spectral radius of $\mathbf{J}_d^{-1}\ \mathbf{J}_o$ is of order ε, henceforth smaller than 1, we can define the following series

$$\mathbf{Y} = \mathbf{I} + \sum_{k=1}^{+\infty}(-1)^k(\mathbf{J}_d^{-1}\ \mathbf{J}_o)^k \tag{20}$$

and foresee that it presents a high convergence rate for small diagonality index. As the product $\mathbf{Y}(\mathbf{I}+\mathbf{J}_d^{-1}\ \mathbf{J}_o)$

$$\begin{aligned}\mathbf{Y}(\mathbf{I}+\mathbf{J}_d^{-1}\ \mathbf{J}_o) &= \left(\mathbf{I}+\sum_{k=1}^{+\infty}(-1)^k(\mathbf{J}_d^{-1}\ \mathbf{J}_o)^k\right)(\mathbf{I}+\mathbf{J}_d^{-1}\ \mathbf{J}_o)\\ &= \mathbf{I}+\mathbf{J}_d^{-1}\ \mathbf{J}_o+\sum_{k=1}^{+\infty}(-1)^k(\mathbf{J}_d^{-1}\ \mathbf{J}_o)^k+\sum_{k=1}^{+\infty}(-1)^k(\mathbf{J}_d^{-1}\ \mathbf{J}_o)^{k+1}\\ &= \mathbf{I}+\mathbf{J}_d^{-1}\ \mathbf{J}_o+\sum_{k=1}^{+\infty}(-1)^k(\mathbf{J}_d^{-1}\ \mathbf{J}_o)^k-\sum_{k=0}^{+\infty}(-1)^k(\mathbf{J}_d^{-1}\ \mathbf{J}_o)^k=\mathbf{I}\end{aligned} \tag{21}$$

is strictly equal to the identity matrix, it turns out that $\mathbf{Y}$ is precisely the correction factor we want to approximate, see (9).

Substitution of (19) into (20) yields the series expansion

$$\mathbf{Y}=(\mathbf{I}+\mathbf{J}_d^{-1}\ \mathbf{J}_o)^{-1}=\mathbf{I}+\sum_{k=1}^{+\infty}\varepsilon^k(-1)^k\boldsymbol{\Upsilon}\mathscr{M}^k\boldsymbol{\Upsilon}^{-1} \tag{22}$$

from which, on basis of the smallness of ε, we propose to keep only the first term of the summation. After consideration of (19), we finally obtain the approximate expression of the transfer function including the first correction term

$$\mathbf{H}_c=(\mathbf{I}-\mathbf{J}_d^{-1}\ \mathbf{J}_o)=(\mathbf{I}-\mathrm{i}\omega\mathbf{H}_d\mathbf{D}_o)\mathbf{H}_d. \tag{23}$$

As announced, this expression is the sum of the leading order term $\mathbf{H}_d$ (the only one considered in the decoupling approximation) and the first correction.

Substitution of (23) into (4) yields the approximate expression of modal responses

$$\mathbf{Q}_c=(\mathbf{I}-\mathrm{i}\omega\mathbf{H}_d\mathbf{D}_o)\mathbf{H}_d\mathbf{G}. \tag{24}$$

It is essential to notice that the proposed method allows to partially account for modal coupling and is numerically very efficient. Indeed its application allows to compute the response in each mode independently from the others (according to (24)), contrarily to a rigorous approach that would require a simultaneous determination of modal responses in all modes, see (3). From a practical viewpoint, the decomposition as the sum of a leading order term and a first correction is also written for the modal response as

$$\mathbf{Q}_c=\mathbf{Q}_d+\Delta\mathbf{Q}, \tag{25}$$

with $\mathbf{Q}_d(\omega)=\mathbf{H}_d(\omega)\mathbf{G}(\boldsymbol{\omega})$ and $\Delta\mathbf{Q}(\omega)=\mathbf{H}_d(\omega)(-\mathrm{i}\omega\mathbf{D}_o(\omega)\mathbf{Q}_d(\omega))$. This decomposition suggests therefore that the analysis be performed in two steps: (i) first, to consider modal forces $\mathbf{G}$ and to compute the modal response $\mathbf{Q}_d$ as would be obtained with the decoupling approximation, (ii) then to compute the correction $\Delta\mathbf{Q}$ as the response of the *same uncoupled modal system* subjected to modal forces $-\mathrm{i}\omega\mathbf{D}_o\mathbf{Q}_d$. This way of organizing the computation of the modal responses indicates that two consecutive analyses of the decoupled system are used, contrary to the formal approach which requires inversion of a full matrix.

This two-step procedure shows that the mind of proposed method fits with usual asymptotic expansion methods for which successive corrections are traditionally expressed recursively from the knowledge of previously established corrections and the leading order solution [9].

3. Performances of the proposed model

3.1. Benchmark problem

The performance of the proposed model is numerically assessed with a benchmark problem, previously considered in [3,10], consisting of a couple of 4-DOF systems. The two considered systems have the same modal stiffness matrix $\boldsymbol{\Omega}$

$$\boldsymbol{\Omega}=\begin{bmatrix}3.95^2 & & & \\ & 3.98^2 & & \\ & & 4.00^2 & \\ & & & 4.10^2\end{bmatrix} \tag{26}$$

and their modal damping matrices are

$$\mathbf{D}_1=\begin{bmatrix}1.61 & -0.1865 & -0.1742 & 0.3838\\ -0.1865 & 1.7 & 0.3792 & -0.1773\\ -0.1742 & 0.3792 & 1.8 & -0.1742\\ 0.3838 & -0.1773 & -0.1742 & 1.75\end{bmatrix};\quad \mathbf{D}_2=\begin{bmatrix}1.61 & 0.0009 & 0.04 & 0.041\\ 0.0009 & 1.7 & 0.0227 & 0.0376\\ 0.04 & 0.0227 & 1.8 & 0.04\\ 0.041 & 0.0376 & 0.04 & 1.75\end{bmatrix}. \tag{27}$$

The diagonal elements of the modal damping matrices are strictly identical for both systems; modal damping matrices differ from each other only by their off-diagonal elements. In this benchmark problem the modal forces are considered to be harmonic with unit amplitude and forcing frequency equal to $\omega_f = 4.16\,\mathrm{rad/s}$, that is

$$\mathbf{G}(\omega) = [1;\ \ 1;\ \ 1;\ \ 1]^{\mathrm{T}}\delta(\omega - \omega_f). \tag{28}$$

The diagonality indices of the modal damping matrices, as defined in (11), are

$$\rho(\mathbf{D}_1) = 0.431; \quad \rho(\mathbf{D}_2) = 0.055.$$

3.2. Illustration of eigenvalues λ and $\boldsymbol{\mu}(\omega)$

Fig. 1 depicts the ordered eigenvalues λ_k of $\mathbf{D}_d^{-1}\ \mathbf{D}_o$ as horizontal lines and the frequency dependent eigenvalues $\mu_k(\omega)$ of $\mathbf{J}_d^{-1}\ \mathbf{J}_o$. The development of the proposed method hinges on the fact that the smallness of λ_k's necessarily involves the smallness of μ_k's. This was formally proved in Section 2 and is illustrated in Fig. 1a for $\mathbf{D}_1$ and Fig. 1b for $\mathbf{D}_2$, where the largest absolute value of μ_1, obtained in the vicinity of natural frequencies—in conformity with developments of Section 2, is well smaller than λ_1. The largest eigenvalue of $\mathbf{D}_d^{-1}\ \mathbf{D}_o$ denoted by λ_1 corresponds to the diagonality index ρ, namely $\rho(\mathbf{D}_1) = 0.431$ and $\rho(\mathbf{D}_2) = 0.055$.

As a prelude to the forthcoming analysis of accuracy, we may focus on the smallness of the diagonality indices. It is evident that a large diagonality index, more exactly a large eigenvalue μ_1, cuts down the convergence rate of series (22) and therefore decreases the accuracy of the proposed method. As $\mu_1(\omega_f) = 0.298$ for $\mathbf{D}_1$ and $\mu_1(\omega_f) = 0.051$ for $\mathbf{D}_2$ have different orders of magnitude, we should expect a larger discrepancy for $\mathbf{D}_1$ than for $\mathbf{D}_2$. This is illustrated in the following section.

3.3. Accuracy of the proposed model

Fig. 2 provides the Fourier transform of the response in mode 1, obtained for $\mathbf{D}_1$ and $\mathbf{D}_2$, regardless of the narrow-band nature of the applied force considered in the benchmark problem. It is actually the response of systems 1 and 2 to unit modal forces with uniform frequency content (white noise). This alternative case is studied here for the sake of a more comprehensive discussion on the influence of modal coupling.

The results obtained with the proposed method (represented by bullets) compares very well with the exact solution (represented by dashed lines). The thick solid line is the result obtained with the decoupling approximation; it is obviously identical for systems 1 and 2 as they just differ from each other by their off-diagonal elements, which are disregarded in the decoupling approximation. Referring to the typical pattern of Fig. 1, it is evident to observe a very good matching of all methods outside resonance frequency bands. In other words, both the decoupling approximation and the proposed method provide accurate results in the quasi-static and inertial frequency ranges.

A closer look at the resonance frequency band, see insert in Fig. 2, reveals that the proposed method provides better estimates for $\mathbf{D}_2$ than for $\mathbf{D}_1$ (bullets virtually lie on the dashed line for $\mathbf{D}_2$). This is naturally expected as our proposition is based on the smallness of the diagonality index, which is much smaller for $\mathbf{D}_2$ than for $\mathbf{D}_1$. On the contrary, the inaccuracy of the decoupling approximation is more serious for $\mathbf{D}_2$ than for $\mathbf{D}_1$, which goes beyond common sense as pointed out in [3].

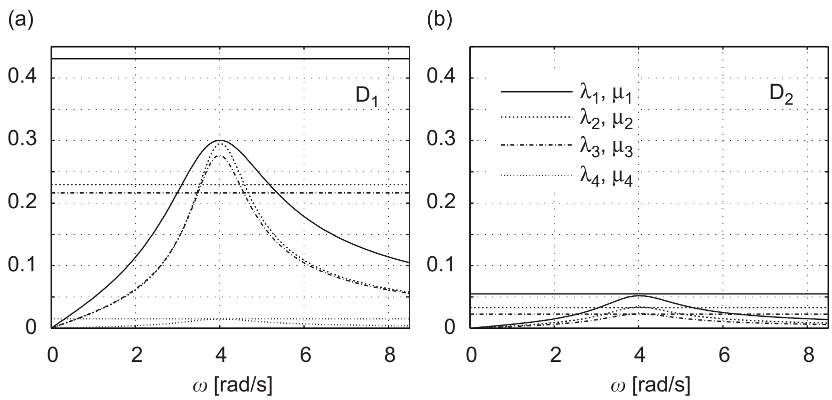

Fig. 1. Constant eigenvalues λ_k of $\mathbf{D}_d^{-1}\ \mathbf{D}_o$ (horizontal lines) and frequency dependent eigenvalues $\mu_k(\omega)$ of $\mathbf{J}_d^{-1}\ \mathbf{J}_o$. They are represented for two different modal damping matrices, $\mathbf{D}_1$(a) and $\mathbf{D}_2$(b).

Next we consider a harmonic force and study precisely the benchmark problem. The relevance of the proposed model is assessed by the relative steady-state error due to the decoupling approximation, expressed as

$$\mathcal{E}_c = \frac{|\mathbf{Q}(\omega_f) - \mathbf{Q}_c(\omega_f)|}{|\mathbf{Q}(\omega_f)|}, \tag{29}$$

where $\mathbf{Q}(\omega_f)$ is the exact modal response, see (4), and $\mathbf{Q}_c(\omega_f)$ is the approximate response obtained with the proposed model, see (24). For comparison with the usual application of the decoupling approximation, we also define

$$\mathcal{E}_d = \frac{|\mathbf{Q}(\omega_f) - \mathbf{Q}_d(\omega_f)|}{|\mathbf{Q}(\omega_f)|}, \tag{30}$$

with $\mathbf{Q}_d(\omega) = \mathbf{H}_d(\omega)\mathbf{G}(\omega)$, as defined in (25).

In order to assess the relation between the decoupling error and the diagonality of the modal damping matrix, the error ($\mathcal{E}_c$ or $\mathcal{E}_d$) and the diagonality index are plotted against each other, see Fig. 3. To this purpose, it is interesting to construct

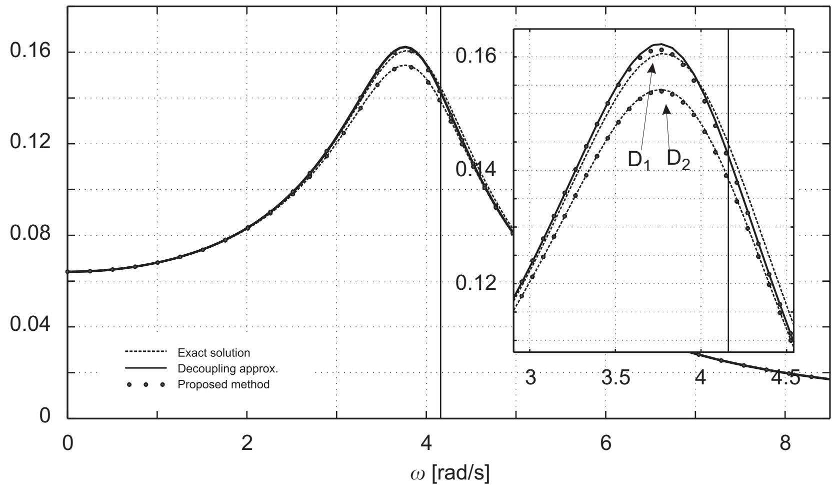

Fig. 2. Fourier transform of the response in the first mode $\mathbf{Q}_1(\omega)$ obtained by the decoupling approximation (solid line) and the proposed method (bullets). They are given for $\mathbf{D}_1$ and $\mathbf{D}_2$ and should be compared to the exact result obtained by inversion of the full matrix (dashed line). NB: the result of the decoupling approximation is identical for $\mathbf{D}_1$ and $\mathbf{D}_2$.

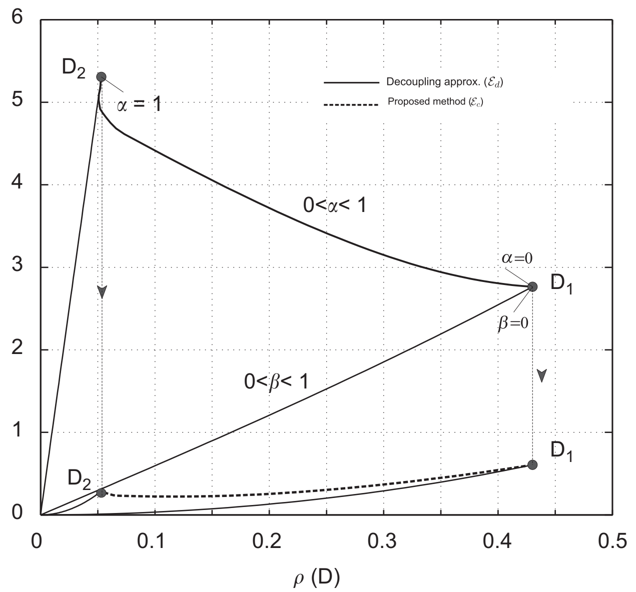

Fig. 3. (a) Relative steady-state error on the modal response computed with the usual decoupling approximation ($\mathcal{E}_d$) and the proposed method ($\mathcal{E}_c$). The four curves passing through the origin correspond to errors related to matrices $\mathbf{D}_{\beta,1}$ and $\mathbf{D}_{\beta,2}$ defined in (33).

virtual matrices $\mathbf{D}_\alpha$, obtained by linear interpolation between $\mathbf{D}_1$ and $\mathbf{D}_2$ as

$$\mathbf{D}_\alpha = (1-\alpha)\mathbf{D}_1 + \alpha\mathbf{D}_2, \quad 0 \leq \alpha \leq 1 \tag{31}$$

and to plot their representative points $(\rho(\mathbf{D}_\alpha), \mathscr{E}_d)$ and $(\rho(\mathbf{D}_\alpha), \mathscr{E}_c)$ on the error-diagonality graph. This yields the thick continuous, and respectively dashed, lines between representative points of $\mathbf{D}_1$ and $\mathbf{D}_2$. In particular both extremities of the upper solid curve correspond precisely to $\mathbf{D}_1$ and $\mathbf{D}_2$ and represent the error $\mathscr{E}_d$ related to the decoupling approximation. Similarly extremities of the lower dashed line represent the discrepancies obtained for $\mathbf{D}_1$ and $\mathbf{D}_2$, with the proposed method.

The error $\mathscr{E}_d$ made with the classical decoupling approximation deserves an extensive discussion. Indeed, consideration of systems intermediate between 1 and 2 roughs out a trend of the evolution of error with the diagonality index. This unfortunately leads to the conclusion that larger errors may be obtained for smaller diagonality indices [3]. This looks paradoxical as it is expected that an effectively diagonal matrix produce no error, and hence that the trend pass through the origin of axes in Fig. 3, which is manifestly not the case. This conclusion may be demystified by noting that the order of magnitude of the first term in series (22) is not only governed by the smallness of the diagonality index but also by the eigenmatrix $\boldsymbol{\Upsilon}$ which is affected by the distribution of off-diagonal elements in the modal damping matrix. Both effects have to be considered before drawing decisive conclusions.

To throw light on this, we introduce another set of intermediate matrices, and consider, to this purpose, the diagonal/off-diagonal decomposition of $\mathbf{D}_1$ and $\mathbf{D}_2$

$$\mathbf{D}_1 = \mathbf{D}_{1,d} + \mathbf{D}_{1,o},$$

$$\mathbf{D}_2 = \mathbf{D}_{2,d} + \mathbf{D}_{2,o} \tag{32}$$

and define

$$\mathbf{D}_{\beta,1} = \mathbf{D}_{1,d} + \beta\,\mathbf{D}_{1,o},$$

$$\mathbf{D}_{\beta,2} = \mathbf{D}_{2,d} + \beta\,\mathbf{D}_{2,o}, \quad 0 \leq \beta \leq 1 \tag{33}$$

as a linear interpolation between $\mathbf{D}_1$ (or $\mathbf{D}_2$) and its diagonal elements, with $\mathbf{D}_1$ and $\mathbf{D}_2$ considered now separately. A practical situation corresponding to this parametric study would be the influence of wind velocity—responsible for non-diagonal modal damping—in a structure presenting a basic proportional damping [11].

The loci of couples $(\rho(\mathbf{D}_{\beta,1}), \mathscr{E}_d)$ and $(\rho(\mathbf{D}_{\beta,2}), \mathscr{E}_d)$ are represented by solid lines dropping from representative points of $\mathbf{D}_1$ and $\mathbf{D}_2$ to the origin. This monotonic behavior is now in agreement with intuition.

The results obtained with the proposed method are analyzed in a similar fashion, and are therefore reported in Fig. 3 too. Aside from the fact that the error $\mathscr{E}_c$ is much smaller than $\mathscr{E}_d$ (dashed line vs. solid line), the error $\mathscr{E}_c$ of matrices $\mathbf{D}_\alpha$ exhibits now a barely monotonically increasing behavior from $\mathbf{D}_2$ to $\mathbf{D}_1$. This brings to the conclusion that, for this particular example, consideration of the proposed correction enables to recover the expected trend. This is however of minor importance as the definition of $\mathbf{D}_\beta$ matrices is much more meaningful.

In this connection, Fig. 3 reveals that errors related to matrices $\mathbf{D}_{\beta,1}$ and $\mathbf{D}_{\beta,2}$ obtained with the proposed method start at the origin, like for the decoupling approximation, but now with a horizontal tangent. This observation is in agreement with the mathematical formulation of Section 2. Indeed as the classical decoupling approximation is the leading order term of series (22), it is expected to obtain an error of order ε, i.e. $\rho(\mathbf{D})$. On the contrary, in the proposed approach, we suggest to include the first correction in the formulation, which pushes back the order of magnitude of the error to ε^2, i.e. $\rho^2(\mathbf{D})$, and explains therefore the quadratic shape of the discussed error curve.

4. Conclusions

When non-diagonal modal damping takes place, the formal estimation of the modal transfer function requires the inversion of a full matrix, but of reduced size however. For this reason, modal equations are seldom simultaneously considered and alternative means such as complex eigenmodes are sometimes put forward. Nevertheless, for simplicity and because it is believed that small off-diagonal elements do not affect significantly the modal response, only the diagonal part of the modal damping matrix is usually considered. This allows naturally to perform the analysis with a decoupled set of equations.

We have shown that this common approach is actually the leading order solution of a matrix inversion problem and we propose to enrich the common solution with the first correction. In the proposed approach, no matrix inversion other than a diagonal one is required, no more than for the usual procedure. This makes its numerical implementation as efficient as simple. Furthermore, it still allows the computation of modal responses independently from each other, as in the usual decoupling approximation. This obviously makes a substantial difference with the rigorous approach which requires a simultaneous handling of all modal equations.

Because we include the first correction into the formulation, it is expected that the error committed on the estimation of modal responses be expressed as a quadratic expression of the diagonality index, in contrast with a linear dependency for the common decoupling approximation. This was illustrated with a benchmark problem.

The proposed method is therefore an intermediate solution between the inaccurate decoupling approximation, and the time-consuming formal approach. As it allows to significantly diminish the errors on modal responses (up to a factor of 10, see Fig. 3), while still offering a low computational cost, it is suggested to systematically apply the proposed method in case of non-diagonal modal damping.

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