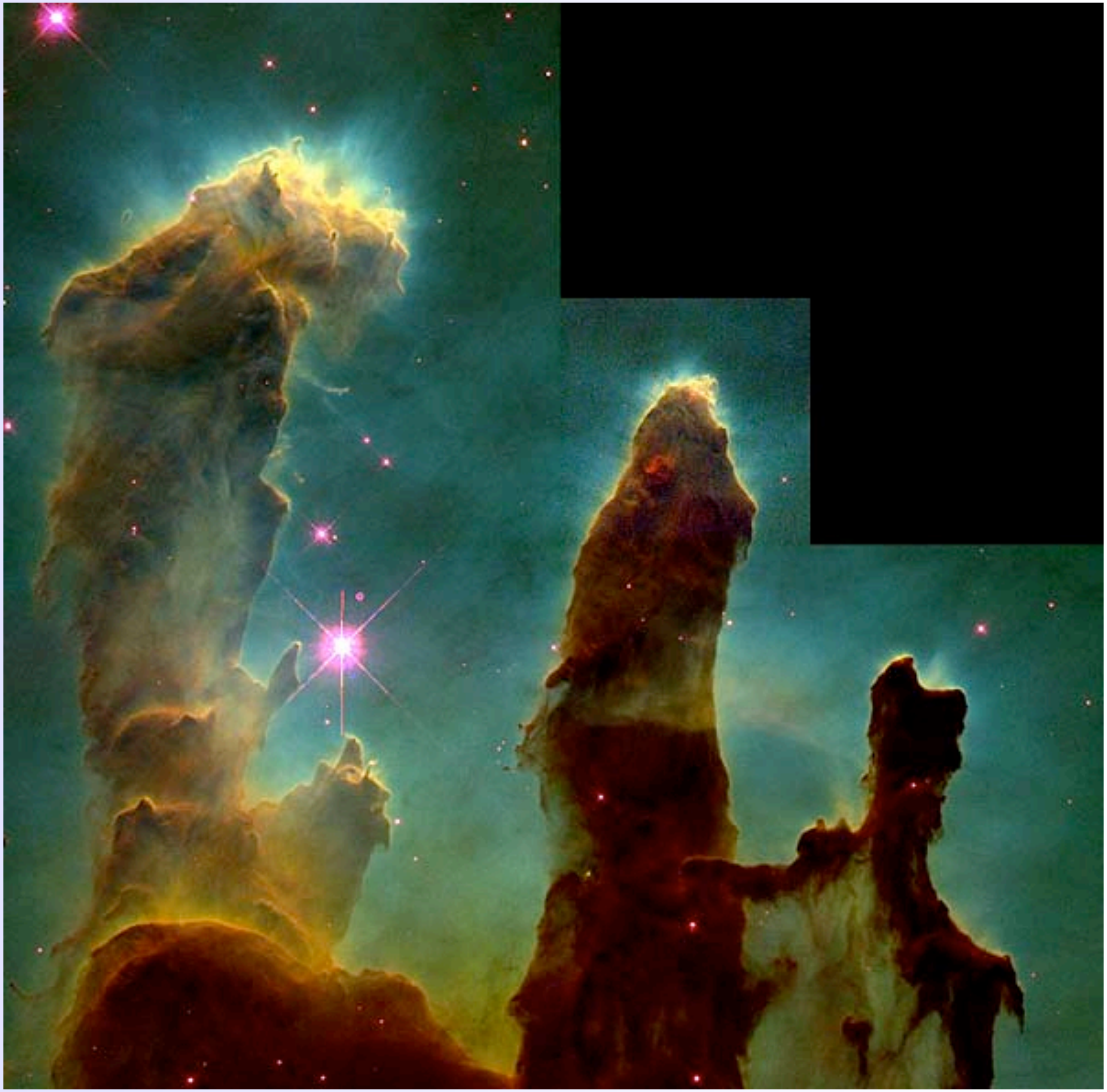
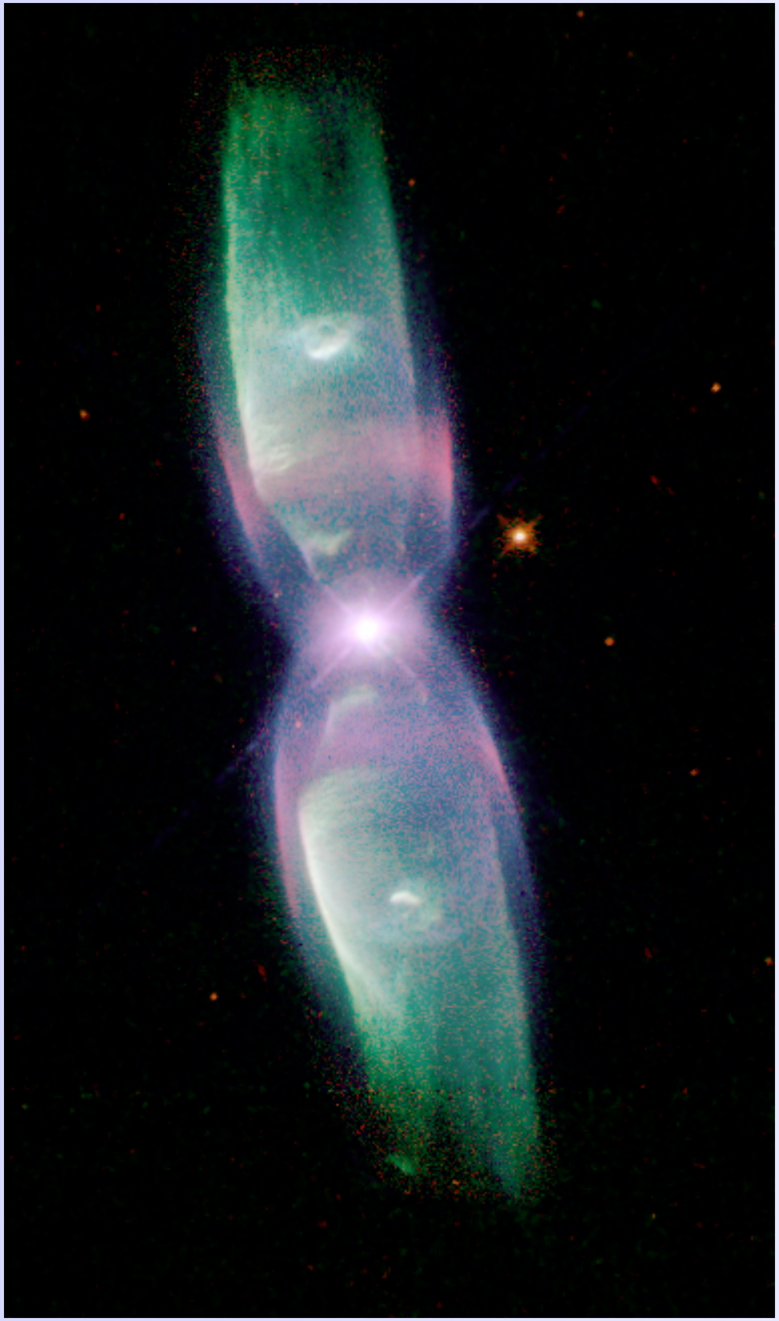
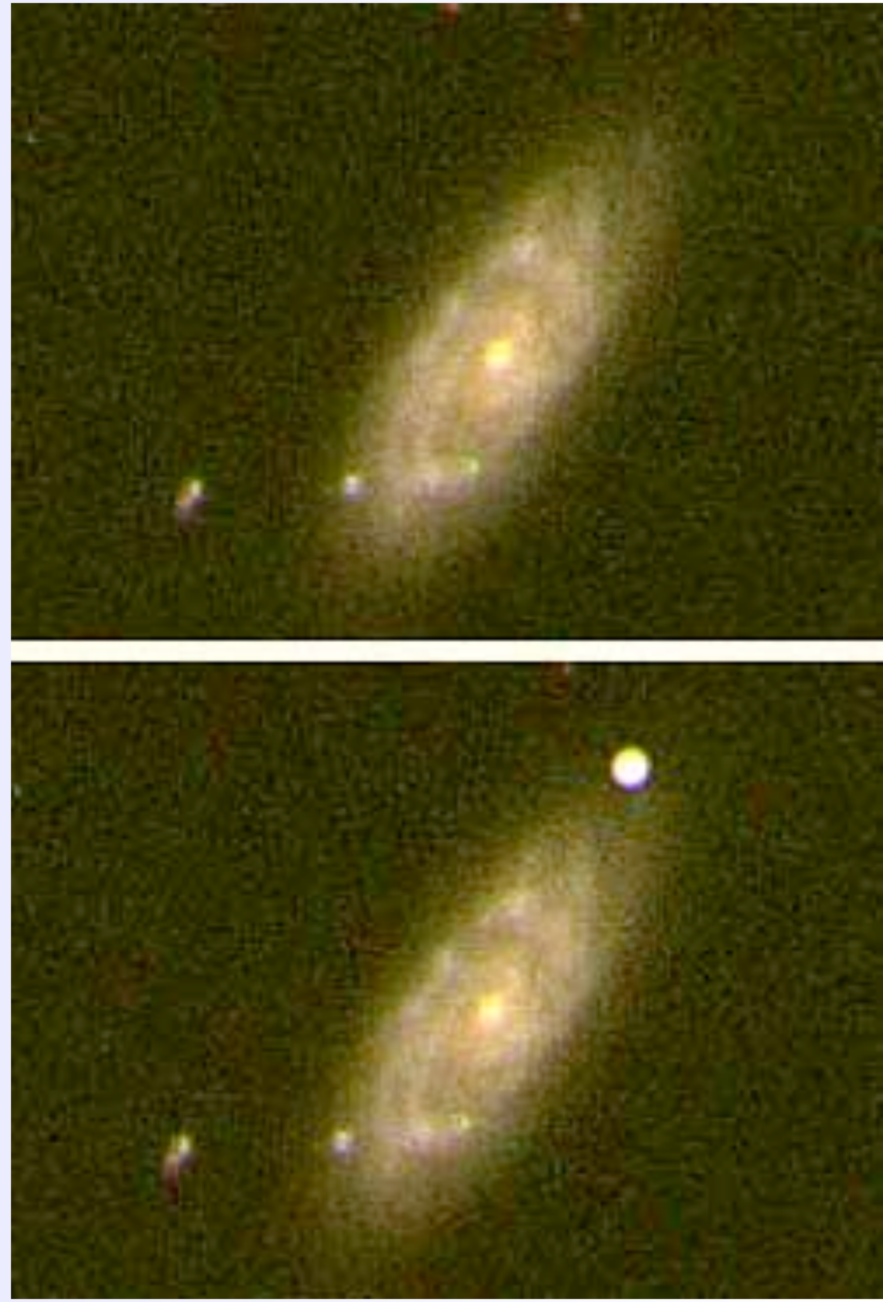
4ème cours de Mécanique Analytique (28 Septembre 2011)







• 1.6 Exemples d'utilisation des équations de Lagrange

- Choisir un ensemble de f coordonnées généralisées (q_1, q_2, …, q_f)

- Exprimer T, Q_i et V en fonction des q_i, $\dot{q}_i$ et t

- Calculer L = T - V

- Ecrire les équations de Lagrange

• 1.5 Les équations de Lagrange

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i} \quad (1.19)$$

$$\boxed{Q_i = \sum_{\alpha=1}^{N} \vec{F}^{a}_{\alpha} \cdot \frac{\partial \vec{r}_{\alpha}}{\partial q_i}} \quad (1.20)$$

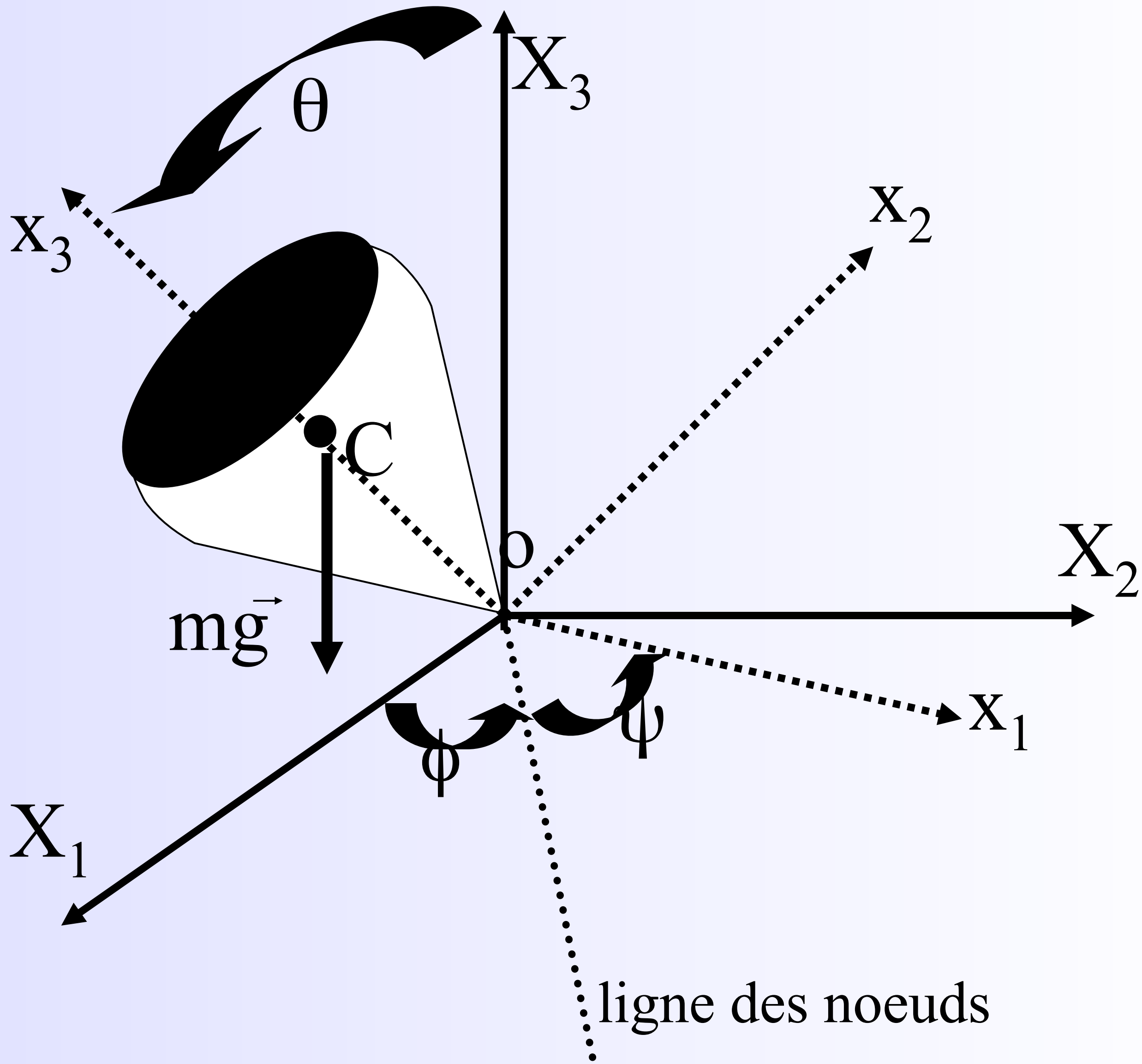
$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0} \quad (1.28)$$

$$\boxed{L = T - V} \quad (1.29)$$

- 1.6 Exemple 1 : le pendule plan
- 1.6 Exemple 2 : la cuvette sphérique
- 1.6 Exemple 3 : la fronde en contraction
- 1.6 Exemple 4 : problème de Lagrange-Poisson
- 1.6 Exemple 5 : la particule chargée dans un champ EM

- 1.6 Exemple 4 : problème de Lagrange-Poisson

(ψ, θ, ϕ)



- 1.6 Exemple 4 : problème de Lagrange-Poisson

$$L = \frac{1}{2}\left[I_1\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2\right] - mgl\left(\cos\theta + 1\right)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I_1 \dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1 \dot{\phi}\sin^2\theta + I_3 \cos\theta\left(\dot{\psi} + \dot{\phi}\cos\theta\right)$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)$$

• 1.6 Exemple 4 : problème de Lagrange-Poisson

$$\frac{dp_\theta}{dt} = \frac{\partial L}{\partial \theta}$$

$$\frac{dp_\phi}{dt} = \frac{\partial L}{\partial \phi} = 0$$

$$\frac{dp_\psi}{dt} = \frac{\partial L}{\partial \psi} = 0$$

$$I_1 \ddot{\theta} = I_1 \dot{\phi}^2 \sin\theta \cos\theta - I_3 \dot{\phi} \sin\theta \left(\dot{\psi} + \dot{\phi}\cos\theta\right) + mgl \sin\theta$$

$$p_\phi = I_1 \dot{\phi} \sin^2\theta + I_3 \cos\theta \left(\dot{\psi} + \dot{\phi}\cos\theta\right) = Cte_2 = \vec{L} \cdot \vec{E}_3$$

$$p_\psi = I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta\right) = Cte_3 = \vec{L} \cdot \vec{e}_3$$

• 1.6 Exemple 4 : problème de Lagrange-Poisson

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0$$

$$H = T + V = \frac{1}{2}\left[I_1\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + I_3\left(\dot{\psi} + \dot{\phi}\cos(\theta)\right)^2\right]$$

$$+ mgl\left(\cos\theta + 1\right) = E = C_1^{te}$$

• 1.6 Exemple 4 : problème de Lagrange-Poisson

$$\frac{1}{2} I_1 \dot{\theta}^2 + U(\theta) = E'$$

$$\dot{\phi} = \frac{C_2^{te} - C_3^{te} \cos\theta}{I_1 \sin^2\theta}$$

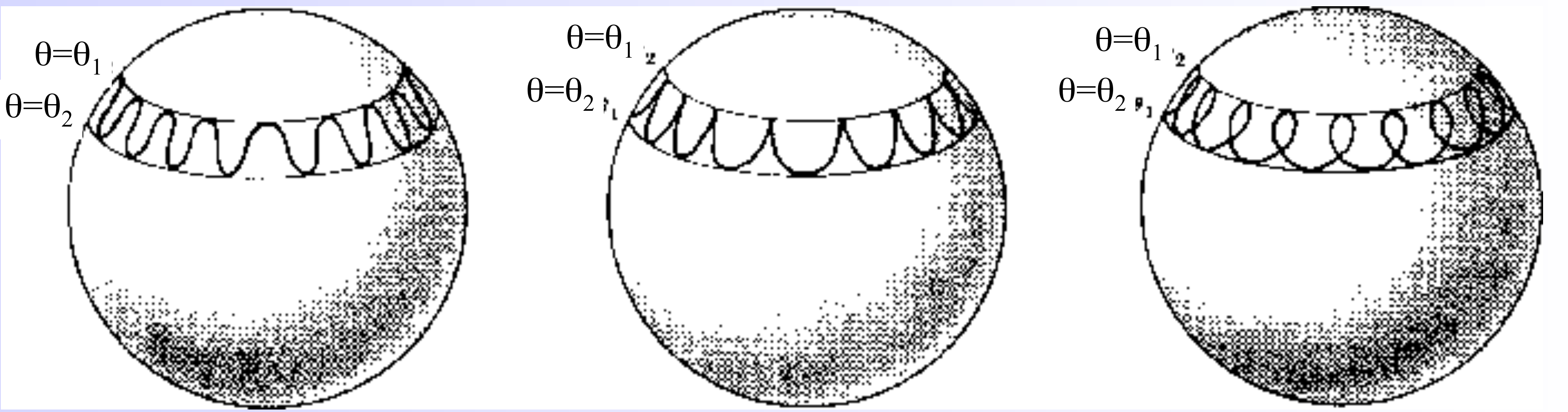
$$\dot{\psi} = \frac{C_3^{te}}{I_3} - \dot{\phi} \cos\theta$$

$$U(\theta) = mgl(1 + \cos\theta) + \frac{1}{2} \frac{\left(C_2^{te} - C_3^{te} \cos\theta\right)^2}{I_1 \sin^2\theta}$$

$$E' = E - \frac{1}{2} \frac{C_3^{te\,2}}{I_3}$$

• 1.6 Exemple 4 : problème de Lagrange-Poisson

(b) : $\cos\theta_1 = C_2^{te}/C_3^{te}$



(a) : $\cos\theta_1 < C_2^{te}/C_3^{te} < \cos\theta_2$

(c) $\cos\theta_2 < C_2^{te}/C_3^{te} < \cos\theta_1$

- 1.6 Application : la précession des équinoxes

$$M_L = \frac{GM_L}{d_L^3}(I_3 - I_1)$$

$$\Omega = \frac{M_L}{L} = \frac{GM_L}{d_L^3}\frac{I_3 - I_1}{I_3\omega} = \frac{R_T}{d_L}^3 \frac{M_L}{M_T}\omega$$

- 1.6 Autres systèmes gyroscopiques +
- Théorie élémentaire de la Toupie

• 1.6 La particule chargée dans un champ EM

$$\vec{F} = e\,(\vec{E} + \frac{1}{c}\vec{v}\wedge\vec{B})$$

$$\text{div}\ \vec{B} = 0, \qquad \text{rot}\ \vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{B} = \ \text{rot}\ \vec{A}, \qquad \vec{E} = -\ \text{grad}\ \phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$$B_i = \varepsilon_{ijk}\frac{\partial A_k}{\partial x_j}, \qquad E_i = -\frac{\partial\phi}{\partial x_i} - \frac{1}{c}\frac{\partial A_i}{\partial t} \qquad (1.35)$$

- 1.6 La particule chargée dans un champ EM

$$\vec{F} = e\left(\vec{E} + \frac{1}{c}\vec{v} \wedge \vec{B}\right)$$

$$B_i = \varepsilon_{ijk} \frac{\partial A_k}{\partial x_j}, \qquad E_i = -\frac{\partial \phi}{\partial x_i} - \frac{1}{c}\frac{\partial A_i}{\partial t} \qquad (1.35)$$

$$\begin{aligned} F_i &= eE_i + \frac{e}{c}\varepsilon_{ijk} v_j B_k \\ &= e\left(-\frac{\partial \phi}{\partial x_i} - \frac{1}{c}\frac{\partial A_i}{\partial t}\right) + \frac{e}{c}\varepsilon_{ijk}\dot{x}_j \varepsilon_{rsk}\frac{\partial A_s}{\partial x_r} \end{aligned}$$

• 1.6 La particule chargée dans un champ EM

$$\begin{aligned} F_i &= eE_i + \frac{e}{c}\varepsilon_{ijk}v_jB_k \\ &= e\left(-\frac{\partial\phi}{\partial x_i} - \frac{1}{c}\frac{\partial A_i}{\partial t}\right) + \frac{e}{c}\varepsilon_{ijk}\dot{x}_j\varepsilon_{rsk}\frac{\partial A_s}{\partial x_r} \end{aligned}$$

$$\varepsilon_{ijk}\varepsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$$

$$F_i = -e\frac{\partial\phi}{\partial x_i} - \frac{e}{c}\left[\frac{\partial A_i}{\partial t} + \dot{x}_j\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right)\right] \qquad (1.36)$$

$$-\frac{\partial\phi}{\partial x_i} = \left(\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}_i}\right) - \frac{\partial}{\partial x_i}\right)\phi$$

$$\frac{\partial A_i}{\partial t} + \dot{x}_j\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right) = \left(\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}_i}\right) - \frac{\partial}{\partial x_i}\right)(\vec{A}\cdot\vec{v})$$

• 1.6 La particule chargée dans un champ EM

$$F_i = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}_i}\right) - \frac{\partial V}{\partial x_i} \qquad (1.37)$$

$$\boxed{V = e(\phi - \frac{1}{c}\vec{A}\cdot\vec{v})} \qquad (1.38)$$

$$m\ddot{x}_i = F_i$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} = F_i$$

• 1.6 La particule chargée dans un champ EM

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad (1.39)$$

$$L = T - V = \frac{1}{2}mv^2 - e(\phi - \frac{1}{c}\vec{A}\cdot\vec{v}) \qquad (1.40)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

• 1.6 La particule chargée dans un champ EM

$$\phi'(q,t) = \phi(q,t) - \frac{1}{c}\frac{\partial}{\partial t}\chi(q,t) \quad (1.41)$$

$$\vec{A}' = \vec{A}(q,t) + \mathrm{grad}\,\chi(q,t) \quad (1.42)$$

$$\begin{aligned} L' &= \frac{1}{2}mv^2 - e\phi' + \frac{e}{c}\vec{A}'\cdot\vec{v} \\ &= L + \frac{e}{c}\left[\frac{\partial\chi}{\partial t} + \vec{v}\cdot\mathrm{grad}\,\chi\right] \\ &= L + \frac{d}{dt}\left(\frac{e}{c}\chi\right) \end{aligned} \quad (1.43)$$

• 1.6 La particule chargée dans un champ EM

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dM}{dt} \qquad (1.44)$$

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial q_i}\dot{q}_i \qquad (1.45)$$