Space-Filling Functions and Davenport Series

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Summary. In this paper, we study the pointwise Hölder regularity of some space-filling functions. In particular, we give some general results concerning the pointwise regularity of the Davenport series.

1 Introduction

In the seminal paper [1′], Cantor showed that there exists a one-to-one mapping between the unit interval and the unit square. A few years later, Peano discovered a continuous map from the unit interval onto the unit square (see [13]). Such "Peano curves" have been used in connection with different branches of mathematical analysis and are still used in data transmission and mathematical programming, where one looks for functions from $\mathbb{R}$ to $\mathbb{R}^2$ which are onto and preserve some neighborhood relationship. In other words, one searches for curves that go through all the elements of an array in a "regular way." Such considerations naturally lead to the use of Peano functions, satisfying some "regularity" conditions (see, for instance, [12, 14]).

We will study the pointwise Hölder regularity of some space-filling functions; since several of the historical ones were given by Davenport series, we will also give some general results concerning the pointwise regularity of such series. This work improves some previous results of [6, 7].

2 Definitions

In this section, we recall the definitions related to the space-filling functions and the pointwise Hölder regularity. Let $\mathcal{L}^d$ denote the d-dimensional Lebesgue measure.

Definition 1. *A function* $f : [0,1] \to \mathbb{R}^d$ $(d \geq 2)$ *is space-filling if* $\mathcal{L}^d(f([0,1])) > 0$*; a Peano function is a continuous space-filling function.*

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Morayne proved that there is no everywhere differentiable space-filling function [11]. We will be interested in obtaining the precise Hölder regularity of such functions.

2.1 Hölder Spaces and Hölder Exponents

The Hölder exponent refines the notion of a continuous, non-differentiable function.

Definition 2. *Let* $\alpha > 0$ *and* $x \in \mathbb{R}$*; a locally bounded function* f *belongs to the Hölder space* $C^\alpha(x)$ *if there exist* $C, R > 0$ *and a polynomial* P *such that*

$$|r| < R \Rightarrow |f(x+r) - P(r)| \leq C|r|^\alpha. \tag{1}$$

A regularity index of f at each point x is given by the following definition.

Definition 3. *The Hölder exponent of* f *at* x *is*

$$h(x) = h(x; f) = \sup\{\alpha : f \in C^\alpha(x)\}. \tag{2}$$

Obviously, $h(x) < 1$ implies that f is not differentiable at x. This exponent is sometimes called the *lower Hölder exponent* in order to emphasize the difference from the *upper Hölder exponent*, which is a counterpart of the Hölder exponent, and is a way to measure the irregularity of a function at a point.

Definition 4. *Let* $0 < \alpha \leq 1$*; a function* f *belongs to* $I^\alpha(x)$ *if there exist* $C, R > 0$ *such that*

$$r < R \Rightarrow \sup_{|r'|<r} |f(x+r') - f(x)| \geq C r^\alpha. \tag{3}$$

The upper Hölder exponent of f *at* x *is*

$$\overline{h}(x) = \overline{h}(x; f) = \inf\{\alpha : f \in I^\alpha(x)\}. \tag{4}$$

The spaces $I^\alpha(x)$ can be generalized for $\alpha > 1$ (see, e.g., [2]).

Definition 5. *The* r*-oscillation of a function* f *at* x *is*

$$\mathrm{osc}_r(x) = \mathrm{osc}_r(x; f) = \mathrm{diam} f(B(x, r)).$$

An equivalent definition of the spaces $C^\alpha(x)$ and $I^\alpha(x)$ can be given in terms of r-oscillation, which sheds light on the duality between these two notions; indeed one immediately checks that:

- A function f belongs to $C^\alpha(x)$ if and only if there exist $C, R > 0$ such that

$$r < R \Rightarrow \mathrm{osc}_r(x) \leq C r^\alpha. \tag{5}$$

- A function f belongs to $I^\alpha(x)$ if and only if there exist $C, R > 0$ such that

$$r < R \Rightarrow \mathrm{osc}_r(x) \geq C r^\alpha. \tag{6}$$

2.2 Uniform Hölder Spaces and Strongly Monohölder Functions

We give here the uniform versions of the pointwise Hölder spaces and introduce the important notion of *strongly monohölder functions* which formalizes the idea of a function which has everywhere the same regularity, in a way as uniform as possible.

The previous definitions concerning the pointwise regularity have a uniform counterpart.

Definition 6. *Let* $0 < \alpha < 1$*; a function* f *belongs to* C^α *if there exist* $C, R > 0$ *such that, for any* x*,*

$$|r| < R \Rightarrow |f(x+r) - f(x)| \leq C|r|^\alpha,$$

or equivalently,

$$r < R \Rightarrow \mathrm{osc}_r(x) \leq Cr^\alpha.$$

In the same way, $f \in I^\alpha$ *if there exist* $C, R > 0$ *such that, for any* x*,*

$$r < R \Rightarrow \sup_{|r'| \leq r} |f(x+r') - f(x)| \geq Cr^\alpha,$$

which can be rewritten as

$$r < R \Rightarrow \mathrm{osc}_r(x) \geq Cr^\alpha.$$

The regularity of most of the "historical Peano functions" is the same at every point; we will make an intensive use of the following notation.

Definition 7. *Let* $0 < \alpha < 1$*; a function* f *is strongly monohölder of exponent* α *(*$f \in SM^\alpha$*) if* $f \in C^\alpha \cap I^\alpha$*, i.e., if there exist* $C, R > 0$ *such that, for any* x*,*

$$r < R \Rightarrow \frac{1}{C} r^\alpha \leq \sup_{y \in B(x,r)} |f(y) - f(x)| \leq Cr^\alpha, \tag{7}$$

or equivalently,

$$r < R \Rightarrow \frac{1}{C} r^\alpha \leq \mathrm{osc}_r(x) \leq Cr^\alpha.$$

Strongly monohölder functions share the following property (see [3]): Let $\dim_B$ denote the box-counting dimension; if $f : [0,1] \to \mathbb{R}$ is continuous, then

$$f \in SM^{2-h} \Rightarrow \dim_B(\mathrm{graph}(f)) = h.$$

Since the associated box-counting dimension is larger than one, the graph of such a function is usually qualified as a "fractal set."

3 The Peano Function

The Peano function [13] is the first continuous space-filling function ever exhibited.

Let K be the function defined by $K(j) = 2 - j$ $(0 \le j \le 2)$; we denote by K^j the jth iterate of K, and set by convention $K^0(j) = j$. The Peano function is defined in [13] as follows (Figs. 1 and 2):

$$P : [0,1] \to [0,1]^2 \quad x \mapsto (p_1(x), p_2(x)),$$

where, if

$$x = \sum_{k=1}^{\infty} \frac{x_k}{3^k}, \tag{8}$$

with $x_k \in \{0,1,2\}$ $(\forall k)$,

$$p_1(x) = \sum_{k=1}^{\infty} \frac{K^{\sum_{l=1}^{k-1} x_{2l}}(x_{2k-1})}{3^k}$$

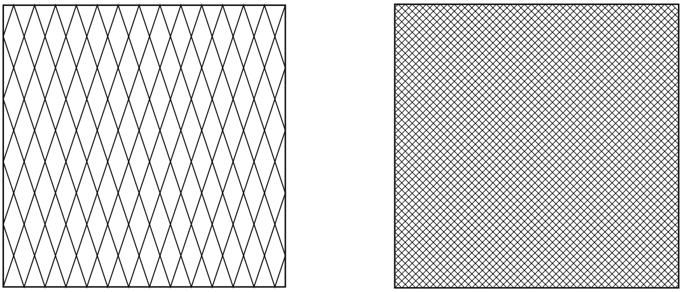

Fig. 1. Approximations of the Peano curve by polygonal curves: one sets $t_0^{(j)} = 0$ and $t_k^{(j)} = t_0 + k/j$; the polygonal curve parameterized by γ_j approximating the curve parameterized by γ is the piecewise linear curve made of the $j+1$ segments of extremities $\gamma(t_i)$, $\gamma(t_{i+1})$ $(0 \le i < j)$. Here are represented γ_{3^5} and γ_{3^8}

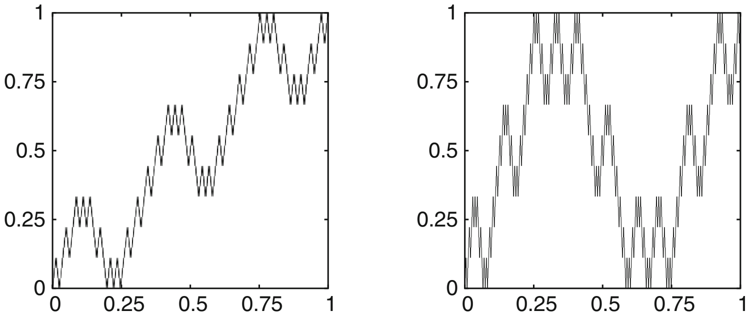

Fig. 2. The Peano functions p_1 and p_2

and

$$p_2(x) = \sum_{k=1}^{\infty} \frac{K^{\sum_{l=0}^{k-1} x_{2l+1}}(x_{2k})}{3^k}.$$

It is easy to check that this function is well defined, i.e.,

$$P\left(\sum_{k=1}^{k_0} \frac{x_k}{3^k}\right) = P\left(\sum_{k=1}^{k_0-1} \frac{x_k}{3^k} + \frac{x_{k_0}-1}{3^{k_0}} + \sum_{k=k_0+1}^{\infty} \frac{2}{3^k}\right)$$

for any $k_0 > 1$ and any sequence $(x_k)_{k\in\mathbb{N}}$ defined on $\{0,1,2\}$ such that $x_{k_0} \neq 0$, thanks to the operator K (in other words, $P(x)$ does not depend on the sequence chosen to represent x).

Peano proved the following result in [13].

Proposition 1. *The Peano function is onto* $[0,1]^2$*.*

Proof. If x has (8) as an expansion, let us denote $x_k^{(1)} = K^{\sum_{l=1}^{k-1} x_{2k}}(x_{2k-1})$ and $x_k^{(2)} = K^{\sum_{l=1}^{k-1} x_{2l-1}}(x_{2k})$. We have $K^j(l) = l$ if j is odd and $K^j(l) = 2-l$ otherwise. Since the operator K leaves the parity unchanged, $x_k^{(1)} = K^{\sum_{l=1}^{k-1} x_k^{(2)}}(x_{2k-1})$ and $x_k^{(2)} = K^{\sum_{l=1}^{k-1} x_k^{(1)}}(x_{2k})$. Therefore,

$$x_{2k-1} = K^{\sum_{l=1}^{k-1} x_k^{(2)}}(x_k^{(1)}), \qquad x_{2k} = K^{\sum_{l=1}^{k-1} x_k^{(1)}}(x_k^{(2)}) \tag{9}$$

and any element $(x^{(1)}, x^{(2)}) = (\sum_{k=1}^{\infty} x_k^{(1)} 3^{-k}, \sum_{k=1}^{\infty} x_k^{(2)} 3^{-k})$ of $[0,1]^2$ gives rise to an element $x = \sum_{k=1}^{\infty} x_k 3^{-k}$ of $[0,1]$, using relation (9). □

Proposition 2. *The Peano function belongs to* $SM^{1/2}$*.*

Proof. Let us work with p_1. The case of p_2 can be treated in the same way. Let $x \in [0,1]$ and $r > 0$. Define $k_0 \in \mathbb{N}$ such that

$$\frac{1}{3^{k_0}} \leq r < \frac{1}{3^{k_0-1}}.$$

If $y = \sum_{k=1}^{\infty} y_k 3^{-k}$ belongs to $B(x,r)$, we have

$$|x-y| = \sum_{k=k_0}^{\infty} \frac{\delta_k}{3^k},$$

for a sequence $(\delta_k)_{k\in\mathbb{N}}$ defined on $\{0,1,2\}$. One immediately gets

$$p_1(x) - p_1(y) = \sum_{\lceil k_0/2\rceil+1}^{\infty} \frac{K^{\sum_{l=1}^{k-1} x_{2l}}(x_{2k-1})}{3^k} - \sum_{\lceil k_0/2\rceil+1}^{\infty} \frac{K^{\sum_{l=1}^{k-1} y_{2l}}(y_{2k-1})}{3^k}.$$

Therefore, $|p_1(x) - p_1(y)| \leq C\sqrt{|x-y|} \leq C' 3^{-k_0/2}$ and $\operatorname{osc}_r(x) \leq C''\sqrt{r}$. Moreover, if $\delta_{k_0} \neq 0$, setting $\beta = \sum_{l=1}^{\lceil k_0/2\rceil} x_{2l}$, one has $K^\beta(x_{2\lceil k_0/2\rceil+1}) \neq K^\beta(y_{2\lceil k_0/2\rceil+1})$. This implies $\operatorname{osc}_r(x) \geq C' 3^{-k_0/2} \geq C''\sqrt{r}$. □

4 A Strong Monohölderianity Criterion

We now prove a general strong monohölderianity criterion. This criterion extends a less general one proved in [7]. It immediately implies that most of the "historical Peano functions" (e.g., the functions of Peano, Wunderlich, Hilbert, Moore, and Sierpinski (see [15]) and also the time-changed Polya function introduced in [10]) are strongly monohölder with Hölder exponent 1/2.

A tree T of subintervals of $[0,1)$ will be called regular if it satisfies the following requirements. There exists $C>0$ such that:

1. The root of T is $[0,1)$.
2. Each element of T has at most C children, which form a subdivision of their parent.
3. For each generation G_j, $\forall e, f \in G_j$, $|e| \leq C|f|$.

Proposition 3. *Let $f : [0,1) \to \mathbb{R}^d$ be a continuous function satisfying the following conditions: There exists a constant $C' > 0$ and $\alpha \in (0,1)$ such that*

$$\forall e \in T; \qquad \frac{1}{C'}(\mathrm{diam}(e))^\alpha \leq \mathrm{diam}(f(e)) \leq C'(\mathrm{diam}(e))^\alpha, \tag{10}$$

where T is a regular tree. Then, the function f belongs to SM^α.

Proof. Let $x, y \in [0,1)$; there exists $e \in T$ such that

$$x \in e \qquad \text{and} \qquad |e| \leq |x-y| \leq C|e|.$$

The points x and y are separated by at most C intervals $e_1 = e, e_2, \ldots, e_k$ of the same generation, with endpoints $x_1, \ldots, x_k$. It follows that

$$|f(x)-f(y)| \leq |f(x)-f(x_1)| + |f(x_1)-f(x_2)| + \cdots + |f(x_k)-f(y)|$$
$$\leq (C+1)C'|e|^\alpha \leq (C+1)C'|f(x)-f(y)|^\alpha.$$

Let us now show the uniform irregularity. Because of the lower bound in (10) there exists two points u and v in the interval e_1 such that

$$|f(u)-f(v)| \geq C'|e_1|^\alpha.$$

Since the interval e_1 can be chosen including any point x, and at any scale, the uniform irregularity follows. □

The Peano functions introduced by Peano, Wunderlich, Hilbert, Moore, Sierpinski, and the time-changed Polya function are defined in a recursive way, so that the images of p-adic intervals are exactly triangles or squares; therefore one immediately checks in these examples that the assumptions of Proposition 3 are satisfied with $\alpha = 1/2$.

5 The Lebesgue Function

The Lebesgue function [9] is a classical example of an almost everywhere differentiable continuous space-filling function.

Let us first recall the definition of the triadic Cantor set.

Definition 8. *The triadic Cantor set K is the subset of $[0,1]$ such that*

$$x \in K \Leftrightarrow x = \sum_{k=1}^{\infty} \frac{2x_k}{3^k},$$

for a binary sequence $(x_k)_{k\in\mathbb{N}}$.

The Lebesgue function is defined on K as follows:

$$L_{|K} : K \to [0,1]^2 \quad x \mapsto (l_1(x), l_2(x)),$$

where, if

$$x = \sum_{k=1}^{\infty} \frac{2x_k}{3^k},$$

for a binary sequence $(x_k)_{k\in\mathbb{N}}$,

$$l_1(x) = \sum_{k=1}^{\infty} \frac{x_{2k-1}}{2^k}, \qquad \text{and} \qquad l_2(x) = \sum_{k=1}^{\infty} \frac{x_{2k}}{2^k}.$$

The Lebesgue function can be continuously extended to $[0,1]$ as follows. If $x \notin K$, let I_x denote the largest (open) interval of K^c containing x. The Lebesgue function L is defined as the continuous function satisfying (Figs. 3 and 4)

$$L : [0,1] \to [0,1]^2 \quad x \mapsto \begin{cases} L_{|K}(x) & \text{if } x \in K \\ L \text{ is linear on } \bar{I}_x & \text{if } x \in K^c \end{cases}.$$

Lebesgue showed in [9] that L is onto the unit square. We show here a slightly stronger result [7].

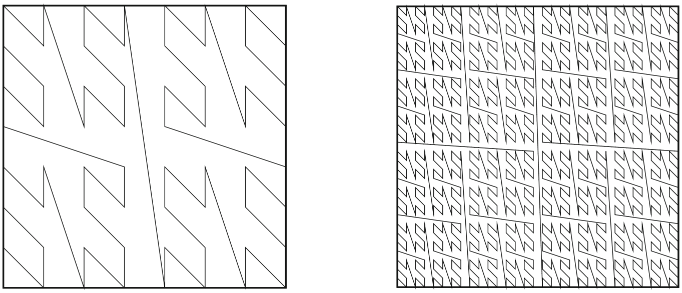

Fig. 3. Approximations of the Lebesgue curve by polygonal curves (see Fig. 1). Here are represented γ_{3^6} and $\gamma_{3^{10}}$

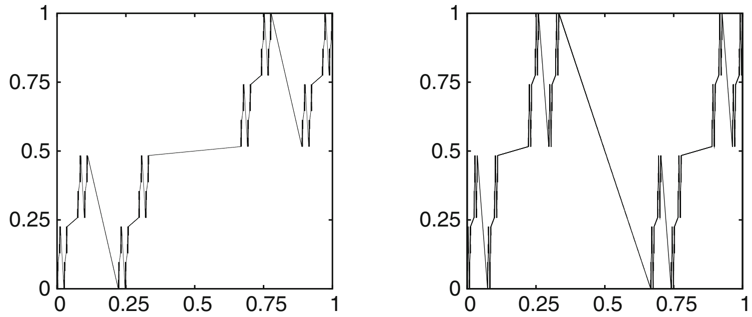

Fig. 4. The Lebesgue functions l_1 and l_2

Proposition 4. *Let K^* be the subset of $[0,1]$ such that*

$$x \in K^* \Leftrightarrow x = \sum_{k=1}^{\infty} \frac{2x_k}{3^k},$$

where $(x_k)_{k\in\mathbb{N}}$ is a binary sequence for which there is no k_0 such that $k > k_0 \Rightarrow x_k = 1$. The restriction of the Lebesgue function to K^* is onto $[0,1]^2 - \{(1,1)\}$, but is not a one-to-one function.

Proof. Let $(y,z) \in [0,1]^2 - \{(1,1)\}$. If $y = \sum_{k=1}^{\infty} y_k 2^{-k}$ and $z = \sum_{k=1}^{\infty} z_k 2^{-k}$, for two binary sequences $(y_k)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$, (y,z) defines a number $x \in K$,

$$x = \sum_{k=1}^{\infty} \frac{2y_k}{3^{2k-1}} + \sum_{k=1}^{\infty} \frac{2z_k}{3^{2k}}.$$

Moreover, since $(y,z) \neq (1,1)$, one of the associated binary sequences ($(y_k)_k$, say) can be chosen to be not ultimately equal to 1 (i.e., such that there is no k_0 such that $k > k_0 \Rightarrow y_k = 1$). As a consequence, $x \in K^*$ and, by construction, $L(x) = (y,z)$. If y or z is a dyadic number, it will have two different binary representations, which give rise to two different pre-images in K^*. □

The regularity of the Lebesgue function is given by the following result, which was obtained in a different way in [7].

Proposition 5. *The Lebesgue function L belongs to C^h, with $h = \log 2/2\log 3$. If $x \notin K$, $L \in C^\infty(x)$; if $x \in K$, the function belongs to $C^h(x) \cap I^h(x)$.*

Proof. The uniform regularity was obtained in [7], using a generic result (Proposition 17). Let us show the pointwise regularity. We will work with l_1 (defined on $[0,1]$); the case of l_2 is similar. We can suppose that $x \in K$; let $\alpha = \log 2/2\log 3$. If $y \in K$ is such that

$$|x - y| = \sum_{k=k_0}^{\infty} \frac{2\delta_k}{3^k},$$

for a binary sequence $(\delta_k)_{k\in\mathbb{N}}$, one has

$$|l_1(x) - l_1(y)| = \sum_{k=\lceil k_0/2\rceil+1}^{\infty} \frac{\delta_k}{2^k} \le C|x-y|^{\alpha}. \tag{11}$$

Now, if $y \notin K$, let $I_y = (a, b)$ and set $c = a$ if $y > x$, $c = b$ otherwise. One has

$$|l_1(x)-l_1(y)| \le |l_1(x)-l_1(c)|+|l_1(c)-l_1(y)| \le C(|x-c|^{\alpha}+|c-y|^{\alpha}) \le C|x-y|^{\alpha},$$

where we have used either the relation (11) or the linearity of l_1. Let us now show the pointwise irregularity of l_1. Let $x \in K$, $r > 0$ and let $k_0 \in \mathbb{N}$ defined by the relations

$$\frac{1}{3^{k_0-1}} \le r < \frac{1}{3^{k_0-2}}.$$

Let also $y \in B(x, r) \cap K$ such that

$$|x-y| = \sum_{k=k_0}^{\infty} \frac{2\delta_k}{3^k},$$

for a binary sequence $(\delta_k)_{k\in\mathbb{N}}$, with $\delta_{k_0} \neq 0$ (this can be done by choosing a point y in a different triadic interval of generation k_0 coming up in the construction of K). For such a number,

$$|l_1(x) - l_1(y)| = \sum_{k=\lceil k_0/2\rceil+1}^{\infty} \frac{\delta_k}{2^k} \ge C|x-y|^{\alpha} \ge C' 3^{-k_0\alpha} \ge C'' r^{\alpha}.$$

6 The Schoenberg Function

Let Λ be the 2-periodic even function such that

$$\Lambda(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1/3 \\ 3x-1 & \text{if } 1/3 \le x \le 2/3 \\ 1 & \text{if } 2/3 \le x \le 1. \end{cases}$$

The Schoenberg function [16] is defined by (Figs. 5 and 6)

$$S : [0,1] \to [0,1]^2 \quad x \mapsto (s_1(x), s_2(x)),$$

where

$$s_1(x) = \sum_{k=1}^{\infty} \frac{\Lambda(3^{2(k-1)}x)}{2^k}$$

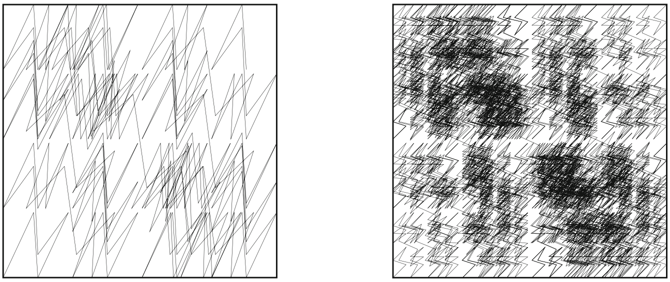

Fig. 5. Approximations of the Schoenberg curve by polygonal curves (see Fig. 1). Here are represented γ_{3^8} and $\gamma_{3^{10}}$

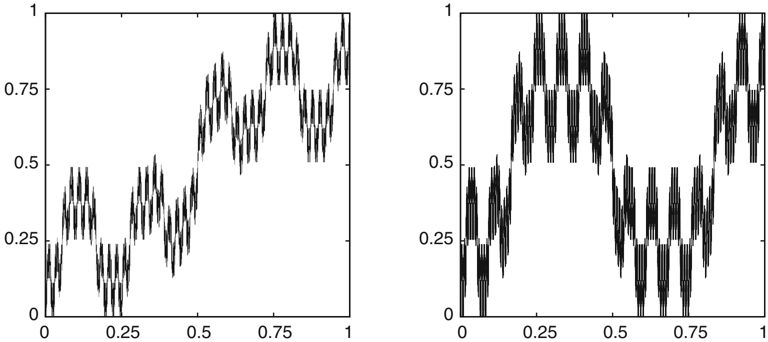

Fig. 6. The Schoenberg functions s_1 and s_2

and

$$s_2(x) = \sum_{k=1}^{\infty} \frac{\Lambda(3^{2k-1}x)}{2^k}.$$

The fact that S is onto the unit square can be deduced from Proposition 4 and the following result, which was obtained by Schoenberg in [16].

Proposition 6. *For any $x \in K$, $S(x) = L(x)$ (where L denotes the Lebesgue function).*

Proof. Let $x \in K$; if $x = \sum_{k=1}^{\infty} 2x_k 3^{-k}$ for a binary sequence $(x_k)_{k \in \mathbb{N}}$, we have

$$\Lambda(3^{k_0}x) = \Lambda\left(\sum_{k=k_0+1}^{\infty} \frac{2x_k}{3^k}\right) = x_{k_0+1},$$

by definition of Λ. Therefore,

$$s_1(x) = \sum_{k=1}^{\infty} \frac{\Lambda(3^{2(k-1)}x)}{2^k} = \sum_{k=1}^{\infty} \frac{x_{2k-1}}{2^k} = l_1(x),$$

by definition of l_1. In the same way, $s_2(x) = l_2(x)$. $\square$

Concerning the regularity, S is strongly monohölder. The following result was stated in [7], but can be obtained in many different ways, see e.g., [1,5,8].

Proposition 7. *The Schoenberg function belongs to* $SM^{\log 2/2\log 3}$.

7 The Cantor Function

In a letter to Dedekind, Cantor proposed the construction of a new function as a candidate to be the first one-to-one correspondence between the unit interval and the unit square [4]. Dedekind pointed out that this function actually is not one to one (as showed by Proposition 8).

In this section, any number $x \in [0, 1)$ will be implicitly associated with the sequence $(x_k)_{k\in\mathbb{N}}$ of its proper expansion in the decimal base; i.e., it takes values in $\{0, \ldots, 9\}$, satisfies

$$x = \sum_{k=1}^{\infty} \frac{x_k}{10^k}, \tag{12}$$

and there is no k_0 such that $k > k_0 \Rightarrow x_k = 9$.

The Cantor function $\mathcal{C} : [0, 1) \to [0, 1]^2$ is defined by $\mathcal{C}(x) = (c_1(x), c_2(x))$, where

$$c_1(x) = \sum_{k=1}^{\infty} \frac{x_{2k-1}}{10^k} \qquad \text{and} \qquad c_2(x) = \sum_{k=1}^{\infty} \frac{x_{2k}}{10^k}.$$

One extends $\mathcal{C}$ on $[0, 1]$ by picking $\mathcal{C}(1) = (1, 1)$ (Fig. 7).

Proposition 8. *The Cantor function is onto* $[0, 1]^2$ *but is not a one-to-one function.*

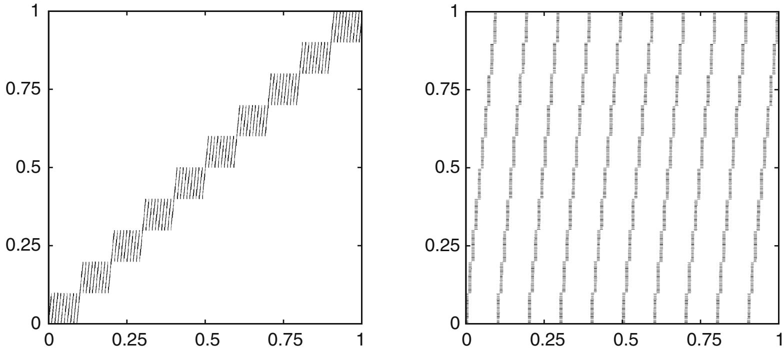

Fig. 7. The Cantor functions c_1 and c_2

Proof. Let $(y,z) \in [0,1]^2$. We can suppose that $(y,z) \neq (1,1)$. The expression

$$x = \sum_{k=1}^{\infty} \frac{y_k}{10^{2k-1}} + \sum_{k=1}^{\infty} \frac{z_k}{10^{2k}}$$

is the proper expansion of a number $x \in [0,1)$ such that $\mathcal{C}(x) = (y,z)$.

Let now $x \in [0,1)$ be a number defined by a sequence $(x_k)_{k\in\mathbb{N}}$ such that there exists an index $k_0 > 0$ for which $x_{k_0} < 9$ and $x_{k_0+2k} = 9$, $\forall k \in \mathbb{N}$. It is easy to check that the number y defined by the following sequence $(y_k)_{k\in\mathbb{N}}$,

$$y_k = \begin{cases} x_k & \text{if } k < k_0 \\ x_{k_0}+1 & \text{if } k = k_0 \\ x_k & \text{if } k = k_0 + 2l + 1, \text{ with } l \in \mathbb{N} \\ 0 & \text{if } k = k_0 + 2l, \text{ with } l \in \mathbb{N} \end{cases}$$

is such that $x \neq y$ and $\mathcal{C}(x) = \mathcal{C}(y)$. □

Proposition 8 provides an injective map from $[0,1]^2$ to $[0,1]$. Since it is trivial to find an injective map from $[0,1]$ to $[0,1]^2$, the Schröder–Bernstein theorem implies that there exists a one-to-one mapping from $[0,1]$ to $[0,1]^2$.

The regularity of the Cantor function at a given point x depends on the order of the approximation of the number x by numbers of the form $k/10^l$ $(k,l \in \mathbb{N})$.

Proposition 9. *If x is not of the form $k/10^l$ $(k,l \in \mathbb{N}_0)$, let $\phi(x)$ be the supremum of the exponents ϕ such that the equation*

$$\left| x - \frac{k}{10^l} \right| \leq 10^{-l\phi} \quad (k < 10^l)$$

has infinitely many solutions. If $x = k/10^l$ for two $k,l \in \mathbb{N}_0$, one sets $\phi(x) = \infty$. The Hölder exponent of $\mathcal{C}$ at x is

$$h(x) = \frac{1}{2\phi(x)}.$$

The upper Hölder exponent of $\mathcal{C}$ at x is

$$\overline{h}(x) = \begin{cases} 1/2 & \textit{if } x \neq k/10^l \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. Let $D = \{x : x = k/10^l,\ k,l \in \mathbb{N}\}$. Let us first show that the Cantor function is not left-continuous at numbers of the form $k/10^l$ and is continuous elsewhere. If $x \notin D$, let $N_x(y) = \inf\{j : x_j \neq y_j\} - 1$. The continuity of the Cantor function at x follows from the fact that for any sequence $(z_j)_{j\in\mathbb{N}}$,

$$\lim_j z_j = x \Leftrightarrow \lim_j N_x(z_j) \to \infty,$$

and from the inequality $N_{c_1(x)}(c_1(y)) \geq [N_x(y)/2]$ (the same relation holds for c_2). Suppose now that $x \in D$; there exists k_0 such that $x_{k_0} > 0$ and $x_k = 0$, $\forall k > k_0$. Let $x^{(l)}$ be the number satisfying

$$x_k^{(l)} = \begin{cases} x_k & \text{if } k < k_0 - 1 \\ x_{k_0} - 1 & \text{if } k = k_0 \\ 9 & \text{if } k_0 < k \leq k_0 + l \\ 0 & \text{if } k > k_0 + l \end{cases} .$$

One has

$$c_1(x^{(l)}) - c_1(x) = \sum_{k=k_0/2+1}^{k_0/2+\lceil l/2 \rceil} \frac{9}{10^k} \to 10^{-k_0/2}$$

if k_0 is even and $c_2(x^{(l)}) - c_2(x) \to 10^{-(k_0-1)/2}$ if k_0 is odd.

From now on, we can suppose that $x \notin D$. Since the Cantor function is continuous on $[0,1]-D$ but not on $[0,1]$, it is sufficient to look at $|\mathcal{C}(x)-\mathcal{C}(y)|$ where $y \in D$. For any $l \in I\!N_0$, let $k(l) \in I\!N_0$ be such that

$$\left| x - \frac{k(l)}{10^l} \right| = \min_{k<10^l} \left| x - \frac{k}{10^l} \right| \leq 10^{-\psi(l)},$$

where $\psi(l)$ is the largest integer such that the inequality holds. Let $y^{(l)} = k(l)/10^l$ if $k(l)/10^l > x$ and $y^{(l)} = k(l)/10^l - 10^{-\psi(l)}$ otherwise. Suppose that l is even (if l is odd, one can consider c_2 instead of c_1); it is easy to check that, as l goes to infinity,

$$|c_1(x) - c_1(y^{(l)})| \leq C10^{-l/2} \leq C(10^{-l\phi(x)})^{1/2\phi(x)} \leq C|x - y^{(l)}|^{1/2\phi(x)},$$

and $h(x) \leq 1/2\phi(x)$. Now, if $y = k/10^l$, with $k < 10^l$ and $k \neq k(l)$, one has, by definition of $k(l)$, $|c_1(x) - c_1(y)| \leq C|x-y|^{1/2\phi(x)}$, for y sufficiently close to x. Therefore $h(x) = 1/2\phi(x)$. The upper Hölder exponent is easy to get, since, by definition of $\mathcal{C}$, for any $l \in 2I\!N_0$, it is always possible to find a number y such that $10^{-l-1} \leq |x-y| < 10^{-l}$ and $|c_1(x) - c_1(y)| \geq 10^{(-l-1)/2}$. For any $r > 0$,

$$\sup_{y \in B(x,r)} |c_1(x) - c_1(y)| \geq C\sqrt{r}.$$

The case of c_2 is similar. □

Let us now show that the Cantor function is an example of a *Davenport series*. Such series are odd 1-periodic functions defined as follows: if $\{x\}$ denotes the "sawtooth function" $\{x\} = x - [x] - \frac{1}{2}$, then Davenport series are of the form

$$\sum_{n=1}^{\infty} a_n \{nx\} \qquad \text{with} \qquad (a_n) \in l^1, \tag{13}$$

see [6]; *p-adic Davenport series* correspond to the case where $a_n = 0$ except if $n = p^k$ for a p larger than 2, i.e., are of the form

$$f(x) = \sum_{j=1}^{\infty} a_j \{p^j x\}, \tag{14}$$

Recent results on Davenport series can be found in [6].

Let $\omega(x)$ be the 1-periodic function such that $\omega(x) = j$ if $x \in [j/10, (j+1)/10)$ $(0 \le j \le 9)$. Clearly, $x_n = \omega(10^{n-1}x)$, so that

$$c_1(x) = \sum_{k=1}^{\infty} \frac{\omega(10^{2k-2}x)}{10^k} \qquad \text{and} \qquad c_2(x) = \sum_{k=1}^{\infty} \frac{\omega(10^{2k-1}x)}{10^k}.$$

One easily checks that

$$\omega(x) = 10\{x\} - \{10x\} + \frac{9}{2}.$$

Therefore,

$$c_1(x) = \frac{1}{2} + \sum_{k=1}^{\infty} -\frac{\{(10^{2k-1}x)\}}{10^k} + \frac{\{(10^{2k-2}x)\}}{10^{k-1}}$$

$$c_2(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{\{(10^{2k-1}x)\}}{10^{k-1}} - \frac{\{(10^{2k}x)\}}{10^k}.$$

Thus, the coordinates of C are examples of 10-adic Davenport series. Another remarkable space-filling function that also turned out to be a p-adic Davenport series was the Lebesgue–Davenport function studied in [7]. We will now prove a general result which yields the pointwise Hölder regularity of any p-adic Davenport series (and therefore applies to these two space-filling functions). This results extends a previous one of [7], in which a regularity condition on the sequence of jumps was imposed, which turns out to be unnecessary.

8 Hölder Exponent of *p*-adic Davenport Series

Since $(a_j) \in l^1$, the function f defined by (14) is the sum of a normally convergent series; it follows that it is continuous at every non p-adic real number, and has a right and a left limit at every p-adic rational $k \cdot p^{-l}$ (where $\gcd(k,p) = 1$, which will be denoted $k \wedge p = 1$), with a jump of amplitude

$$b_l = a_l + a_{l+1} + \cdots .$$

Let $x_0 \in [0,1)$. We denote by ω_n the sequence of p-adic approximants of x_0, i.e., for each n, ω_n is the point of the form $k \cdot p^{-n}$ with $k \wedge p = 1$ which is closest to x_0.

Theorem 1. *Let f be given by (14), with $a_j \in l^1$. Let $x_0 \in \mathbb{R}$; if x_0 is not a p-adic rational, then*

$$h_f(x_0) = \liminf_{j\to\infty} \left(\frac{\log(|b_j|)}{\log(|x_0 - \omega_j|)} \right). \tag{15}$$

Assume now that $x_0 = k \cdot p^{-l}$ with $k \wedge p = 1$. If $b_l \neq 0$ then $h_f(t_0) = 0$, else (15) holds (and, in this case, $|x_0 - \omega_j| = p^{-j}$).

Proof. Denote by α the right-hand side of (15). First, note that f has a jump of amplitude b_j at ω_j; therefore, it follows from a classical lemma (see [6] for instance) that $h_f(x_0) \leq \alpha$. Therefore, we only have to prove the regularity at x_0. Let $\varepsilon > 0$. For j large enough,

$$|b_j| \leq (p^{-j})^{\alpha-\varepsilon}.$$

Let x be given; let J be defined by

$$p^{-J-1} < |x - x_0| \leq p^{-J},$$

and let l be the first integer such that x and x_0 are not in the same p-adic interval of length p^{-l}. We have $l \leq J$ and

$$f(x) - f(x_0) = \left(\sum_{j\leq J} a_j p^j \right)(x - x_0) + \sum_{j=l}^{J} a_j + \sum_{j>J} a_j(\{p^j x\} - \{p^j x_0\}).$$

Since $a_j = b_j - b_{j+1}$, the last term is bounded by

$$4 \sum_{j>J} |b_j| \leq C(p^{-J})^{\alpha-\varepsilon} \leq C|x - x_0|^{\alpha-\varepsilon}.$$

As regards the second term, since $|x_0 - \omega_l| \leq |x - x_0|$,

$$\left| \sum_{j=l}^{J} a_j \right| = |b_l - b_{J+1}| \leq |x_0 - \omega_l|^{\alpha-\varepsilon} + (p^{-J-1})^{\alpha-\varepsilon} \leq C|x - x_0|^{\alpha-\varepsilon}.$$

As regards the first term, we separate two cases; if $\alpha \leq 1$, then

$$\left| \sum_{j\leq J} a_j p^j \right| = \left| \sum_{j\leq J} (b_j - b_{j+1}) p^j \right| \leq C \cdot p^{(1-\alpha+\varepsilon)J} \leq C|x - x_0|^{\alpha-\varepsilon-1},$$

which yields the required bound for $|f(x) - f(x_0)|$.

If $\alpha > 1$, then the series $\sum a_j p^j = \sum (b_j - b_{j+1}) p^j$ is convergent; therefore, we can write

$$\sum_{j \leq J} a_j p^j = \sum_{j \in \mathbb{N}} a_j p^j - \sum_{j > J} a_j p^j$$

and

$$\left| \sum_{j > J} a_j p^j \right| = \left| \sum_{j > J} (b_j - b_{j+1}) p^j \right| \leq C \cdot p^{(1-\alpha+\varepsilon)J} \leq C |x - x_0|^{\alpha - \varepsilon - 1},$$

which yields the required bound for $|f(x) - f(x_0) - (\sum a_j p^j)(x - x_0)|$. □

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