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TRANSPORT OF DISPERSED PHASE IN CIVIL ENGINEERING: UNIFICATION OF THE MATHEMATICAL DESCRIPTION

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Abstract: In designing hydraulic structures and examining natural flows, civil and environmental engineers frequently rely on a pure water mathematical model coupled with an advection-diffusion equation. To some extent, such a single-phase model fails to accurately describe the dispersed phase and to take into account its interaction with the water flow. Consequently, we develop in this paper a multiphase model that addresses the shortcomings of standard models, improves their fidelity and unifies the mathematical description of transport phenomena. The text shows that the drift-flux model is an efficient multiphase basis for deriving mid-scale free-surface model. In particular, an original one-dimensional drift-flux model for free-surface flows is derived to address the specific problems arising in civil and environmental engineering. This approach succeeds in enhancing the mathematical fidelity of models for sediment, air and pollutant transport.

Keywords: Hydraulics, Two-phase flow, Drift-flux model, Aerated flow

INTRODUCTION

Hydraulic engineering treats of the conveyance of fluids in a natural or anthropogenic environment. Common topics of design for hydraulic engineers include hydraulic structures (dams, channels, canals, levees,etc), water supply networks, water drainage networks, natural rivers and reservoirs, sediment transport, pollutant transfer,… The transport of dispersed phases affects considerably the flow dynamics in most of these cases. “White water” is a famous example of a air-water mixture (CHANSON 1997). Sedimentation and erosion also result from the transport of a dispersed phase (ASCE 2008; DEWALS et al. 2008). Finally, a growing concern about pollution in shallow water flows prompts us to assess the effect of dispersed pollutants on water flows.
As the literature shows, civil and environmental engineers make frequent use of mathematical models to handle such problems. Equations give indeed a reliable basis to design hydraulic structures, assess the impact of river modifications, and even predict the evolution of our natural environment. For this purpose, engineers rely on simplified analytical solutions of the model as well as discretised solutions provided by computers (Computational Fluid Dynamics).
When dealing with the transport of air and sediments, hydraulic engineering mostly recourse to the

Reynolds-averaged Navier-Stokes equations, which is a single-phase 3D model (ISHII and HIBIKI 2006):

$$\begin{cases} \dfrac{\partial \rho_w}{\partial t} + \nabla.\left(\rho_w \mathbf{v}_w\right) = 0 \\ \dfrac{\partial}{\partial t}\left(\rho_w \mathbf{v}_w\right) + \nabla.\left(\rho_w \mathbf{v}_w \otimes \mathbf{v}_w\right) = \nabla.\left(-p_w I + \boldsymbol{\tau}_w + \boldsymbol{\tau}_w^T\right) + \rho_w \mathbf{g} \end{cases} \tag{1}$$

where $\mathbf{v}_w$ is the water velocity, p_w the pressure, ρ_w the fluid density, τ_w the viscous stresses usually given by the Newton law of viscosity, and $\mathbf{g}$ the gravity acceleration. The turbulent stresses $\boldsymbol{\tau}^{\mathbf{T}}_w$ account for the effect of random turbulent fluctuations in fluid momentum. An advection-diffusion equation for passive scalars (GRAF and ALTINAKAR 1998) supplements equations (1) in order to account for the transport of the dispersed phase:

$$\frac{\partial C}{\partial t} + \nabla.\left(C\mathbf{v}\right) = \nabla.\left(\chi_m \nabla C + \mathbf{q}^T\right) + \mathbf{F} \tag{2}$$

where C is the concentration in passive substance, χ_m the coefficient of molecular diffusion, $\mathbf{q}^T$ the turbulent contribution to the flux, and $\mathbf{F}$ the source term in passive substance.

In the one-dimensional framework, equations (1) are area-integrated over a flow cross-section presenting a free surface. It gives the following equations:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\rho_w \Omega\right) + \dfrac{\partial}{\partial x}\left(\rho_w u_w \Omega\right) = 0 \\ \dfrac{\partial}{\partial t}\left(\rho_w u_w \Omega\right) + \dfrac{\partial}{\partial x}\left(\beta \rho_w u_w u_w \Omega + \rho_w g I_1\right) = \rho_w g \Omega\left(S_0 - S_f\right) + \rho_w g I_2 \end{cases} \tag{3}$$

which are the Saint-Venant equations. Pressure terms are defined as:

$$I_1\left(\Omega\right) = \int_{-h_b}^{h_s} \left(h - \xi\right) l\left(x, \xi\right) d\xi \quad \text{and} \quad I_2\left(\Omega\right) = \int_{-h_b}^{h_s} \left(h - \xi\right) \frac{\partial l\left(x, \xi\right)}{\partial x} d\xi \tag{4}$$

Where ξ[m] is the depth integration variable along the vertical axis, $l(x,\xi)$ is the width of the cross section such that $l(x,h)$ is the width of the free surface, Ω[m²] is the flow area, u_w[m³/s] the flow velocity along the main direction of the stream, g[m²/s] the gravity, S_0[-] the bed slope, and S_f[-] the head-loss term computed with empirical resistance law. It integrates the effect of both the turbulence and the friction. Finally, h[m] is the water height, l[m] the free-surface width, h_{fs}[m] the free-surface elevation, and h_b[m] the bottom elevation. The Boussinesq coefficient β accounts for non uniform velocity profiles. Complementing Saint-Venant equations, the one-dimensional advection-diffusion equation for passive scalars describes the transport of the mass concentration C in dispersed phase (GRAF and ALTINAKAR 1998):

$$\frac{\partial}{\partial t}\left(C\Omega\right) + \frac{\partial}{\partial x}\left(C u_w \Omega\right) = \frac{\partial}{\partial x}\left(q^d \Omega + q^T \Omega\right) \tag{5}$$

where q^d is the diffusive contribution to the flux, and q^T is the turbulent contribution to the flux. Additional constitutive equations close the system by specifying q^d and q^T. In the case of sediment transport, the Exner equation (GRAF and ALTINAKAR 1998) is usually added to account for the mobility of the bed caused by deposition and erosion:

$$\left(1-p\right)\frac{\partial z_b}{\partial t}+\nabla.\mathbf{q}_s=-\mathbf{e}_s \tag{6}$$

where p is the porosity of the mobile bed, z_b the bed elevation, $\mathbf{q_s}$ the sediment flux, and $\mathbf{e_s}$ the net erosion rate. Additional constitutive equations specify the value of the last two terms.

Close scrutiny of equations (1)-(2) (3D model) and equations (3)-(5) (1D model) reveals however weaknesses in the mathematical formulation. First, the definition of the concentration C is wooly, especially its physical meaning at the local level. This lack of definition causes problems in applying turbulence models. Second, primitive unknowns of the Navier-Stokes equations describe only the water flow and fully neglect the velocity of the dispersed phase (which may be different from the water velocity). Third, head losses in single-phase flows can be readily computed with well-established correlations like Manning-Strickler or Colebrook formulations. However, additional head-loss has to be accounted for in two-phase flows. It requires adapting friction correlations. Fourth, the diffusion in the transport equation is very case-specific and focused on molecular (and gravity) diffusion. Such a limitation does not accord with the physic of multiphase phenomena (ISHII and HIBIKI 2006). Finally, interactions between the Exner equation and the water flow are not fully accounted for.

The shortcomings of the standard models have prompted us to investigate how multiphase theories may constitute an adequate alternative to single-phase models. To date, the use of multiphase models in mid- and large-scale civil and environmental engineering has remained circumscribed to a limited number of attempts. As mentioned by Spasojevic and Holly in (ASCE 2008), "the two-phase flow approach seems promising" but "the formulation of governing equations in flow-sediment problems are still in their infancy". In this respect, (GREIMANN et al. 1999) used a two-phase formulation to determine the concentration and velocity profiles of a dilute suspension of particles in a 2D uniform flow. (WU and WANG 2000) reviewed the governing equations of the main two-phase flow models, namely the drift-flux model and the two-fluid models. They present their general theory without developing specific models or validate the approaches.

The objective of this research is thus to rigorously develop a unified mathematical model that describes the transport of a dispersed phase and overcomes the shortcomings of standard methods. Four requirements are sought for the new model. First, it must adequately account for the multiple phases transported in the flow. Second, it must integrate the effect of scale heterogeneities in time and space. Third, it must remain affordable from a computational point of view (1D). Finally, the model must be adapted to industrial issues in civil and environmental engineering. For this purpose, we perform here a theoretical analysis of available multiphase models.

As demonstrated in this paper, the drift-flux model achieves these objectives in many respects and overcomes shortcomings identified in the standard formulation. In particular, an original one-dimensional drift-flux model for free-surface flows is derived to address the specific problems arising in civil and environmental engineering. This approach succeeds in enhancing the mathematical fidelity of models for sediment, air and pollutant transport. From a practical point of view, hydraulic software's based on the drift-flux model may increase in fidelity by considering much more details about the flow structure. Experimental Investigators may gain also new insight into experimental and field data by analyzing them with a multiphase model.

THREE-DIMENSIONAL MULTIPHASE MODEL

Multiphase theories originate mainly from chemical and mechanical engineering. Their application to civil and environmental engineering remains in its infancy. Consequently, establishing a multiphase model for mid-scale free-surface flows requires an important theoretical research. This section proposes a critic analysis standard multiphase models. Evidences suggest the Drift-flux model constitutes a consistent paradigm to meet the objective of this research and overcome shortcomings of standard models.

A rigorous two-phase flow solver should solve at the same time the local instant variables describing the behaviour of each phase. This is the Local Instant Formulation (ISHII and HIBIKI 2006; KERGER et al. 2010). Obtaining a solution this way is however beyond the present computational capability for many engineering applications (ISHII and HIBIKI 2006). Consequently, various researchers derived simplified models by averaging the Local Instant Formulation. Many averaging methods have been developed and used to study two-phase flow systems (ISHII and HIBIKI 2006). In particular, application of Eulerian averaging gives birth to two different models, the Drift-flux model and the Two-fluid model. Both models are extensively discussed in (WU and WANG 2000; ISHII and HIBIKI 2006). Since the drift-flux model describes the multiphase flow as a single-phase mixture flow whose variables refer to the center of mass of the system, it seems more suitable for application in civil and environmental engineering. The motion of the dispersed phase is treated in terms of diffusion through the mixture.

Three-dimensional drift-flux model is obtained by Eulerian time averaging of the Local Instant Formulation (ISHII and HIBIKI 2006; KERGER et al. 2010). It results in a continuity equation for the mixture, a diffusion equation expressing the continuity of the dispersed phase and a momentum equation for the mixture flow:

$$\begin{cases} \dfrac{\partial \rho_m}{\partial t} + \nabla\left(\rho_m \mathbf{v}_m\right) = 0 \\ \dfrac{\partial}{\partial t}\left(\alpha_d \rho_d\right) + \nabla.\left(\alpha_d \rho_d \mathbf{v}_m\right) + \nabla.\left(\alpha_d \rho_d \mathbf{V}_{dm}\right) = \Gamma_d \\ \dfrac{\partial}{\partial t}\left(\rho_m \mathbf{v}_m\right) + \nabla.\left(\rho_m \mathbf{v}_m \otimes \mathbf{v}_m\right) \quad = -\nabla.\left(p_m \mathbf{I}\right) + \nabla.\left(\tau_m + \tau^T + \tau^D\right) + \rho_m \mathbf{g} + \mathbf{M}_m \end{cases} \tag{7}$$

The momentum equation for the dispersed phase is neglected in favor of a constitutive equation for the relative velocity between the center of mass and each phase. The parameter α_d is the void fraction of the dispersed phase, ρ_m the mixture density, and $\mathbf{v}_m$ the velocity of the centre of mass of the system, which is different from both the water velocity and the dispersed phase velocity. The drift velocity $\mathbf{V}_{dm}$ is defined as the relative velocity with respect to the mass center of the mixture. The mass source term Γ_d accounts for the exchange of mass between the water and the active dispersed phase. It is usually given by a case-specific correlation. The mixture pressure p_m is a primitive unknown. The mixture momentum equation includes three kinds of stresses: the classical

Newtonian viscous stresses τ_m, turbulent stresses τ^T and diffusion stresses τ^D due to the relative velocity between phases. Finally, the mixture momentum source term $\mathbf{M}_m$ represents the effect of the surface tension on the mixture momentum.

Comparison of the drift-flux model (7) with the standard model (1)-(2) underlines many similarities. Both models consist in three partial differential equations expressing the mass and momentum conservation. On the contrary, several discrepancies appear between both models. Since it depends on the void fraction, the mixture density ρ_m is no longer constant even if both phases are incompressible:

$$\rho_m = (1-\alpha_d)\rho_w + \alpha_d \rho_d \tag{8}$$

where ρ_d is the dispersed phase density. Similarly, the mixture velocity $\mathbf{v}_m$ is neither the velocity of the water $\mathbf{v}_w$ nor the velocity of the dispersed phase $\mathbf{v}_d$. It is the velocity of the center of mass of the mixture:

$$\mathbf{v}_m \triangleq \frac{(1-\alpha_d)\rho_w \mathbf{v}_w + \alpha_d \rho_d \mathbf{v}_d}{\rho_m} \tag{9}$$

The mixture continuity does not ensure the continuity of the water, but the continuity of both the water and the dispersed phase taken as a whole.

Diffusion equation accounts for the conservation of the dispersed phase only and governs the evolution of the void fraction α_d. This parameter is rigorously defined as the probability to find the dispersed phase at a given point $\mathbf{x}_0$ (ISHII and HIBIKI 2006; KERGER et al. 2010)

$$\alpha_k = \frac{1}{\Delta t}\int_{[\Delta t]} M_d \, dt = \frac{[\Delta t]_d}{[\Delta t]} \tag{10}$$

where M_d is the density function, the value of which is one in presence of the dispersed phase and zero in any other case. The time interval is broken down into $[\Delta t] = [\Delta t]_w + [\Delta t]_d$ where $[\Delta t]_w$ is the set of time intervals in which the characteristics of the water dominate and $[\Delta t]_d$ is the set of time intervals in which the characteristics of the dispersed phase dominate. Void fraction may be considered as a concentration in dispersed phase. What is more, drift-flux model accounts for all form of diffusion and relative velocities thanks to the notion of diffusion velocity:

$$\mathbf{V}_{dm} \triangleq \mathbf{v}_d - \mathbf{v}_m \tag{11}$$

Diffusion velocity can have any form and must be given by a case-specific constitutive equation. The diffusion equation does not enclose a turbulent flux. It is a key point and a direct consequence from the definition of the void fraction, which is an averaged value that appears naturally in the time-integration of the local Instant Formulation. The well-defined concentration does not generate turbulent terms in the drift-flux theory!

Finally, the mixture momentum equation differs from Navier-Stokes model by resorting to the mixture parameters. In particular, the mixture pressure p_m describes both phases. What is more, mixture momentum equation integrates the contribution of three kinds of stresses, namely the time-averaged viscous stresses τ_m, the stresses τ_D originating from the difference in velocity between both phases, and the turbulent stress τ^T due to the macroscopic effect of the local instant variations

in both phases. All of them are given by constitutive equations and closure model .A new head loss that appear in the drift-flux model is the mixture momentum source term $\mathbf{M}_m$. It represents the effect of the surface tension on the momentum, which is totally neglected in the standard model.
In conclusion, 3D drift-flux model constitutes a process oriented alternative to 3D RANS model for modelling transport phenomena in mid-scale applications. This model overcomes most of the shortcomings identified in the introduction.

AREA-INTEGRATED DRIFT-FLUX MODEL

Computation of three-dimensional models requires in many cases a prohibitive computational effort. If the flow is essentially one-dimensional (river, channel, pipe,…), hydraulic engineers usually simplify 3D models by area-integrating them over the flow cross-section (CUNGE et al. 1980). It results in one-dimensional model easier to solve. In this respect, 1D drift-flux models that we found in the literature only describe pressurized flows. Such models are not sufficient for many practical cases in civil and environmental engineering. Consequently, we derive here an original 1D free-surface drift flux model.
For this purpose, we consider an integration domain defined by a general cross-section presenting a free-surface (Fig.1). The integration takes into account vertical variations of the flow parameter by considering a multi-layer integration domain (Fig.1). The integration gives the following set of 3 partial differential equations for each layer:

- The mixture continuity equation:

$$\frac{\partial}{\partial t}\left(\langle\rho_m\rangle_l \Omega_l\right) + \frac{\partial}{\partial x}\left(\langle\rho_m\rangle_l \tilde{u}_{m,l}\Omega_l\right) = 0 \quad (12)$$

- The diffusion equation for the dispersed phase:

$$\frac{\partial}{\partial t}\left(\langle\alpha_d\rangle_l \Omega_l\right) + \frac{\partial}{\partial x}\left(\langle\alpha_d\rangle_l \tilde{u}_{m,l}\Omega_l + \langle\alpha_d\rangle_l \frac{\rho_w}{\langle\rho_m\rangle_l}\tilde{U}_{dj,l}\Omega_l\right) = \left\langle \Gamma_d / \rho_d \right\rangle_l \Omega_l \quad (13)$$

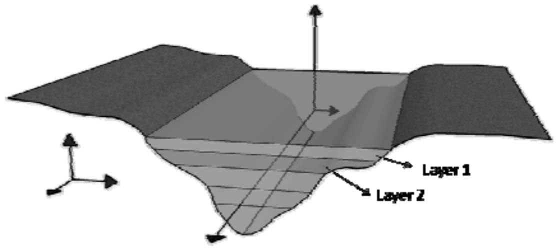
- The mixture momentum equation:

$$\frac{\partial}{\partial t}\left(\langle\rho_m\rangle_l \tilde{u}_{m,l}\Omega_l\right) + \frac{\partial}{\partial x}\left(\beta_l \langle\rho_m\rangle_l \tilde{u}_{m,l}\tilde{u}_{m,l}\Omega_l + gP_{\Omega,l}\right) + \frac{\partial}{\partial x}\left(\frac{\langle\alpha_d\rangle_l}{1-\langle\alpha_d\rangle_l}\frac{\rho_d\rho_w}{\langle\rho_m\rangle_l}\tilde{U}_{dj,l}\tilde{U}_{dj,l}\Omega_l\right)$$
$$= \langle\rho_m\rangle_l g\Omega_l\left(S_{0,l} - S_{F,l}\right) + \langle\rho_m\rangle_l gP_{\partial\Omega,l} - \frac{\partial}{\partial x}\left(p_{s,l}\Omega_l\right) \quad (14)$$

where the subscript l indicates the layer considered, Ω_l is the area and $\langle\ \rangle_l$ the area average:

$$\langle f\rangle_l \triangleq \frac{1}{\Omega_l}\int_{\Omega_l} f\,dA \quad (15)$$

Fig. 1 – The domain of integration includes a free-surface and several layers.



Like for 3D models, comparison of the original drift-flux model (12)-(14) with the standard model (3)-(6) underlines both similarities and discrepancies. Both models consist in three partial differential equations for each layer.
The mixture continuity equation has a very similar form to the Saint-Venant equation. However, the new model describes the evolution of mixture parameters instead of water variables. It is thus expressed in terms of mean density of the mixture $\langle \rho_m \rangle_l$ and mean mixture velocity $\tilde{u}_{m,l}$:

$$\langle \rho_m \rangle_l \triangleq \langle \alpha_d \rangle_l \rho_d + \left(1 - \langle \alpha_d \rangle_l\right) \rho_w \quad \text{and} \quad \tilde{u}_{m,l} \triangleq \frac{\langle \rho_m u_m \rangle_l}{\langle \rho_m \rangle_l} \tag{16}$$

The continuity equation ensures the conservation of both phases as a whole but not separately.
The diffusion equation (13) differs more deeply from the advection-diffusion equation (5). The concentration is now precisely defined as the mean value of the void fraction $c \triangleq \langle \alpha_d \rangle_l$. The diffusive q^d and turbulent q^T contributions to the flux are replaced by a single term depending on the area-averaged drift velocity $\tilde{U}_{dj,l}$. This term accounts for the relative velocity between both phases:

$$\tilde{U}_{dj,l} \triangleq \langle\langle u_d \rangle\rangle_l - \langle j \rangle_l \quad \text{with} \quad \langle\langle u_d \rangle\rangle_l \triangleq \frac{\langle \alpha_d u_d \rangle_l}{\langle \alpha_d \rangle_l} \tag{17}$$

where $\langle j \rangle$ is the area-averaged volumetric flux defined as follows:

$$\langle j \rangle_l = \langle \alpha_d \rangle_l \langle\langle u_d \rangle\rangle_l + \left(1 - \langle \alpha_d \rangle_l\right) \langle\langle u_w \rangle\rangle_l \tag{18}$$

Drift-velocity is intimately linked to diffusion velocities but constitutes a most convenient parameter to determine. This formulation of the diffusion is obviously more rigorous and consistent than the determination of the diffusive flux in a transport equation.
Finally, mixture momentum equation (14) also differs widely from Saint-Venant equations because it relies on the mixture parameters instead of water variables. What is more, new pressure terms appear and are defined as:

$$P_{\Omega_l} = \int_{\Omega_l} \left[\langle \rho_m \rangle\Big|_z^{h_{s,l}} \left(h_{s,l}(x,t) - z\right) \right] dA \quad \text{and} \quad P_{\partial\Omega_l} = \int_{-h_{b,l}}^{h_{s,l}} \left(\langle \rho_m \rangle\Big|_z^{h_{s,l}} \left(h_{s,l}(x,t) - z\right) \frac{\partial l_w}{\partial x} \right) dz \tag{19}$$

where the means value $\langle \rho_m \rangle\Big|_z^{h_{s,l}}$ is defined as:

$$\langle \rho_m \rangle\Big|_z^{h_{s,l}} = \frac{1}{h_{s,l} - z} \int_z^{h_{s,l}} \rho_m(x,z,t)\, dz \tag{20}$$

Consequently, new pressure terms take into account the two-phase configurations of the flow. On the other hand, the pressure $p_{s,l}$ exerted at the interface between l^{th} and $(l+1)^{th}$ layer accounts for the link between the various layers. In addition, the effect of the relative velocity between phases appears explicitly under the form of a diffusion stress. Its value depends only on the drift-velocity $\tilde{U}_{dj,l}$. Finally, remaining head losses terms appearing in the integration are conflated into friction slope $S_{F,l}$. It integrates contributions of turbulence, surface tension at interfaces, and friction with the river bed. Its value must be computed thanks to a friction correlation (Homogeneous Colebrook, Martinelli-Lockhart,…) that takes into account multiphase interactions.

CONCLUSION

Comparison of the standard model (3D RANS equations + advection-diffusion equation) with the 3D drift-flux model underlines the advantages offered by the multiphase approach. In particular, the drift-flux theory addresses all the identified shortcomings in the standard model. This conclusion is confirmed in deriving an original one-dimensional free-surface drift flux model. The new model presents a more accurate description of the interactions between the water and the dispersed phase. It also covers a wide range of potential applications by unifying the mathematical description of sediment, air and pollutants transport.

From a practical point of view, hydraulic software's based on the drift-flux model may increase in fidelity by considering much more details about the flow process. It also provides a unified framework to create single software for sediment, air and pollutants transport. Investigators may also gain new insight into experimental and field data by analyzing them with a multiphase model.

Present results pave the way for further research in theoretical fluid dynamics and applied hydraulics. The approach presented here should be extended to two-dimensional problem and to pressurized flows as well. Experimental research should supply the model in case-specific constitutive laws and validation cases.

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