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Higgs mechanism in the general Two-Higgs-Doublet Model

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Contents

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Introduction

The next decade is expected to be an exciting period of modern high-energy physics. In some months, the first results from the world's largest and highest-energy particle accelerator, the Large Hadron Collider (LHC), will be available. Its objective is to investigate the properties of the basic constituents of matter together with those of the fundamental interactions in nature in a range of energy never reached, at tera-electronvolt (TeV) energies. The currently consistent theory that provides a framework for describing all observed high-energy phenomena is the standard model (SM) of electroweak and strong interactions. Among the vast list of "Greatest Puzzles" [1], the standard model answers the important question of the origin of mass. Independently of each other, the Scott Peter Higgs and two belgian physicists, François Englert and Robert Brout, proposed in 1964 a mechanism of spontaneous symmetry breaking to give masses to the particles that respects the requirement of renormalizability and gauge invariance [2, 3]. Thanks to this mechanism, it was possible to join electromagnetism with the weak force in a single quantum field theory to obtain a common symmetry, the electroweak symmetry. Deciphering the mechanism that breaks the electroweak symmetry and generates the masses of the known fundamental particles is one of the central challenges of particle physics [4].

The electroweak theory, proposed by Glashow [5], Salam [6] and Weinberg [7], is based on the gauge symmetry group $SU(2)_L \times U(1)_Y$ of weak left–handed isospin and hypercharge. An $SU(2)_L$ doublet of complex scalars fields is introduced and its neutral component develops a non-zero vacuum expectation value. As a consequence the electroweak symmetry is broken to the electromagnetic one. Three of the four degrees of freedom introduced with the doublet confer masses to the weak force carriers $W^\pm$ and Z while the photon remains massless. The remaining one corresponds to the scalar Brout-Englert-Higgs boson. This mechanism can also explain the mass of the quarks and leptons through Yukawa interactions with the scalar field and its conjugate. Combined with Quantum ChromoDynamics (QCD) that is the theory of strong interactions between the colored quarks based on the symmetry group $SU(3)_C$, the standard model describes three of the four interactions in nature and is based on the symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The predictive power of the SM was first demonstrated by the observation of neutral weak currents at the end of the seventies. Then the SM was confirmed by a large amount of well understood experimental data, hundreds of its predictions have been verified with impressive precision in dozens of experiments generating billions of data points at the current energies, hundreds of GeV's [8].

However, despite its great success in explaining all available data, the standard model has serious inadequacies and there are several phenomenological indications that all questions cannot find an answer in the SM. The arguments supporting an extension of the standard model can be classified in three categories. First, it requires the existence of a massive scalar particle, the Brout-Englert-Higgs boson, that is a major ingredient of the electroweak theory and thus of the standard model. The discovery of this new particle appears as the main challenge of the LHC. Most of the particle physicists currently work on experiments which will permit, in a few years, to discover the Higgs boson or to show that it does not exist. In the latter case, the standard model should be strongly modified and a new mechanism would have to be found allowing to explain the origin of mass. Second, the SM may appear as unnatural for various reasons. For example, it does not give an explanation for the fermion mass spectrum, in particular the small neutrino masses and the unnaturally small Higgs boson mass. This is the hierarchy problem. Third, this model does not include gravitation. This is partly because at the level of elementary particles the gravitational force is many orders of magnitude weaker and can be ignored[1]. Moreover, the SM cannot be easily extended to include gravity. These reasons lead physicists to wonder what are the possibilities beyond the standard model. A rich spectrum of theories has been proposed and some of them aimed at the unification of all fundamental forces including gravity. Among these are supersymmetry, extra dimensions, little Higgs, technicolor theories, etc. In general, these theories involve a set of new physical states interacting with the SM particles. Several of these theories predict different kinds of distinct new signals in particle collisions at multi-TeV energies [9]. So, in this case, their masses are in the reach of the LHC, and so these particles are expected to be produced and to decay (or escape) in (from) the detectors. In the large amount of data that will be collected, the challenge will be to extract these new physics traces [10]. In order to detect these new imprints, we should be prepared to discover this expected new physics beyond the SM.

The Two-Higgs-Doublet Model (2HDM) is one of the simplest beyond the standard model extension of the Higgs mechanism of the electroweak symmetry breaking which may arise naturally in the scalar sector of various theories. In this model, we

[1]The quantum effects of gravity become important in describing particle interactions at the Planck scale. This refers to either a very large energy scale ($1.22 \cdot 10^{19}$ GeV) or a very tiny size scale ($1.616 \cdot 10^{-35}$ meters).

introduce two doublets of scalar fields ϕ_1 and ϕ_2 characterized by an appropriately constructed scalar potential. The most general Higgs potential of 2HDM that is invariant and renormalizable contains 14 free parameters. This large number of free parameters makes the analysis of the most general model very complicated. Many studies have considered restricted models, analysing specific cases with only few non-zero free parameters, and have noted that these already lead to a rich spectrum of different and interesting phenomenologies. The resulting phenomenology is specific for the initial choice of free parameters of the Higgs potential. In this work, we are concerned in finding all possibilities the introduction of a second doublet offers and in generating an exhaustive list of possible phenomenologies. Thus, we must consider the most general 2HDM without imposing any special relations among the parameters.

Unfortunately, we cannot analyze the most general 2HDM with straightforward algebra. A problem arises at the first step of the treatment of the most general 2HDM: the minimization of the most general scalar potential leads to coupled equations that cannot be solved. A method to circumvent this computational difficulty is necessary. However, before analyzing the most general 2HDM, we need to review the Higgs mechanism in the SM. This is the subject of the first chapter. This one will review the standard model, focusing on the implementation of the spontaneous symmetry breaking mechanism. In the second chapter, various motivations for extensions of the SM scalar sector are reviewed and two-Higgs-doublet models are introduced as possible extensions. Moreover, we consider some specific cases of 2HDM that lead to interesting characteristics. In the third chapter, we give motivations to study the most general 2HDM and we review a method recently developed [11] that allows one to circumvent the computational difficulties that arise when studying the most general 2HDM. In this approach, one first establishes the structure behind 2HDM and then reformulates the minimization problem. And in the fourth and last chapter, we will present a new calculation deriving a formalism to study the first step in the dynamic of the most general 2HDM : we will develop a method to compute masses of Higgs bosons in any type of vacua and in any 2HDM.

Chapter 1

The Higgs mechanism in the standard model

In order to motivate the study of the two-Higgs-doublet model (2HDM), it is necessary to review the minimal standard model. It is based on two main principles : first the extension of the gauge invariance principle as a local concept [12], and second the spontaneous symmetry breaking mechanism [13]. The introduction of local gauge invariance generates the gauge bosons as well as the interactions of these gauge bosons with fermions, and also, if the gauge group is non abelian, among the gauge bosons themselves. The combination of local gauge invariance with the spontaneous symmetry breaking mechanism leads to the Higgs mechanism which generates the masses of weak vector bosons and fermions. Since the 2HDM is an extension of the symmetry breaking sector, in this chapter, we are going to review the mechanism of electroweak symmetry breaking and focus on the Higgs particle of the standard model.

1.1 Interactions and local gauge invariance

In the standard model, the three interactions, electromagnetic, weak and strong, are derived from a gauge principle similar to that in electromagnetism. The gauge principle can be stated as follow. Consider a matter system that is invariant under a global group G of transformations. Gauging this symmetry consists in enlarging the global group to a local gauge invariance. Such a symmetry requires the existence of a massless vector field for each symmetry generator to which the matter field current becomes coupled. The interactions among these fields are highly restricted by the gauge symmetry. Gauge symmetry is defined by a Lie group, the gauge group [14].

The gauge groups appearing in the SM are the $U(1)$ group of phase transformations that is abelian and has only one generator, and the group $SU(N)$, a subgroup of $U(N)$ where the matrices are not only unitary but also have a determinant 1. This last group is a non-abelian group that has $N^2 - 1$ generators. So, a field theory with local gauge symmetry contains matter fields and gauge fields. We will see later that the representations of the matter fields, that are specified by irreducible representation of the group and which can have different dimensionalities for the $SU(N)$ group, are subtle. Moreover, we will see that the mass of some of these gauge bosons comes from the Higgs mechanism.

Indeed, the standard model is a non-abelian gauge theory with the symmetry group $SU(2) \times U(1) \times SU(3)$. The group of the strong interaction is the $SU(3)_C$ color group of Quantum ChromoDynamics (QCD). Quantum chromodynamics describes the strong interaction between quarks that arises from the exchange of eight massless gluons that couple to the color charge of the fermions. These are the eight gluon fields G^a_μ(a=1,2,...,8) corresponding to the color $SU(3)_C$ group which is not abelian. The electroweak theory, which describes the electromagnetic and weak interactions between quarks and leptons, is based on the electroweak gauge symmetry group $SU(2)_L \times U(1)_Y$ of Glashow, Weinberg, and Salam. In this work we are concerned with the electroweak theory. We are going to review the local gauge invariance concept in more details and apply it to the electroweak theory.

1.1.1 Local gauge invariance

The first interaction described as a gauge theory was Quantum ElectroDynamics (QED). The starting point is that Maxwell's equations are invariant under a *local* gauge transformation of the form:

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x)\,, \tag{1.1}$$

where A_μ is the four-vector potential.

As for the free Dirac lagrangian

$$\mathcal{L}_{Dirac} = \bar{\phi}(i\gamma^\mu \partial_\mu)\phi\,, \tag{1.2}$$

it is invariant under the *global* phase shift $\phi \rightarrow e^{i\delta}\phi$. It is possible to extend this global symmetry to a local one. Such locality is accomplished by replacing the derivative ∂_μ by the covariant derivative D_μ:

$$D_\mu = \partial_\mu + ieQA_\mu \tag{1.3}$$

where eQ is the electric charge of the current coupled to A_μ. When the local transformation

$$\phi \to e^{-ieQ\alpha(x)}\phi \tag{1.4}$$

arises, A_μ is a four vector-field that transforms as (1.1). So, the derivative behaves under a gauge transformation exactly as the field itself. The free Dirac lagrangian becomes

$$\mathcal{L}_{Dirac} = \bar{\phi}(i\gamma^\mu D_\mu)\phi = \bar{\phi}(i\gamma^\mu \partial_\mu)\phi + ieQA_\mu\bar{\phi}\gamma^\mu\phi = \mathcal{L}_{free} - A_\mu J^\mu \,. \tag{1.5}$$

This new lagrangian is invariant under the combined transformations (1.4) and (1.1).

To get the lagrangian of the interaction of the four-vector electromagnetic potential A_μ with a fermionic field ϕ, i.e. the QED lagrangian, we need to add the kinetic term that describes the propagation of free photons, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, which leads to Maxwell's equations, and that is also locally gauge invariant. Therefore, the QED lagrangian is

$$\mathcal{L}_{Dirac} = \bar{\phi}(i\gamma^\mu D_\mu)\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.6}$$

where

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu \,. \tag{1.7}$$

The lagrangian is invariant under local gauge transformations from the group $U(1)$.

We have, though, generated the matter-radiation coupling by imposing the gauge symmetry to be local. To preserve locality we have introduced into the covariant derivative a four-vector field A_μ which is a gauge field. We have also added a parameter Q, that is the electric charge. It acts as a generator for the local group transformations $U(x) = e^{-ieQ\alpha(x)}$. According to the Noether's theorem, because of the continuous symmetry, there is a conserved quantity that is the electric charge. The massless gauge boson associated with the local $U(1)$ symmetry is the photon that couples the electric charge of the fermions. Therefore, QED is an abelian theory with the symmetry group $U(1)$ and one gauge field, the electromagnetic field, with the photon being the gauge boson. Let us use this principle to construct the electroweak theory.

1.1.2 Weak interactions

To describe the electroweak theory we use, in addition to the $U(1)$ group, the $SU(2)$ group that is a non abelian group whose generators obey the Lie algebra of the rotation group in three dimensions $SO(3)$ (i.e. they are isomorphic). An important feature of weak interactions is that they violate parity in the charged-current interactions (by exchange of charged bosons W^+ and W^-). The result is that only left-handed particles are sensitive to charged current weak interactions[1]. So, one must introduce a new quantum number, the weak isospin, I. This number, associated with the group $SU(2)_L$, corresponds to the weak charge. Particles having a weak isospin different from zero are sensitive to W exchange while those whose weak isospin is null are not sensitive to it. One can write the representation of the fermions as the sum of two spinors: one of left helicity P_L and the other of right helicity P_R:

$$\Phi = P_R\Phi + P_L\Phi = \Phi_L + \Phi_R = \frac{1}{2}(1-\gamma_5)\Phi + \frac{1}{2}(1+\gamma_5)\Phi\,. \tag{1.8}$$

The left- and right-handed components of any fermions are assigned to different representations of the $SU(2) \times U(1)$ gauge group. The weak isospin of the fermions of left helicity is $I = \frac{1}{2}$, the third component of weak isospin I_3 can thus take values of $\pm\frac{1}{2}$. One can consequently group the fermions of left helicity in doublets of particles of $SU(2)_L$. Also, one introduces weak hypercharge, noted Y. This quantum number, associated with the group $U(1)_Y$, makes possible to connect the electric charge and the weak isospin by the Gell-Mann–Nishijima formula:

$$Q = I_3 + \frac{1}{2}Y\,, \tag{1.9}$$

such that the electric charge is conserved by electroweak interactions. Therefore, we characterize the $SU(2)_L \times U(1)_Y$ theory by the left-handed quarks :

$$Q_L^1 = \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad Q_L^2 = \begin{pmatrix} c \\ s \end{pmatrix}_L, \quad Q_L^3 = \begin{pmatrix} t \\ b \end{pmatrix}_L, \tag{1.10}$$

with weak isospin $I = \frac{1}{2}$ and weak hypercharge $Y(Q_L) = \frac{1}{3}$; and by the left-handed leptons :

$$L_L^e = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \quad L_L^\mu = \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \quad L_L^\tau = \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \tag{1.11}$$

with weak isospin $I = \frac{1}{2}$ and weak hypercharge $Y(L_L) = -1$.

For the fermions of right helicity, the weak isospin is null. The third component can thus be only 0. That corresponds to singlets of fermions:

$$U_R^{1,2,3} = u_R, c_R, t_R \tag{1.12}$$

[1]We don't know if this parity violation is an asymmetry in the laws in nature , or a left-right symmetry that is hidden.

and

$$D_R^{1,2,3} = d_R, s_R, b_R \tag{1.13}$$

for the right-handed quarks, with weak hypercharge $Y(U_R) = \frac{4}{3}$ and $Y(D_R) = -\frac{2}{3}$ and

$$E_R^{e,\mu,\tau} = e_R^-, \mu_R^-, \tau_R^- \tag{1.14}$$

for the right-handed leptons with hypercharge $Y(E_R) = -2$. Since the doublets and the singlets have a non-zero hypercharge, they have electromagnetic interactions. Moreover, the quarks are triplets under the $SU(3)_C$ group and so can interact via strong interactions, while leptons are color singlets.

Multiplet	Particles Generations	$SU(3)_c \times SU(2)_L \times U(1)_Y$
L_L	$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L , \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$	(1,2,-1)
E_R	$e_R^-,\ \mu_R^-,\ \tau_R^-$	(1,1,-2)
Q_L	$\begin{pmatrix} u \\ d \end{pmatrix}_L , \begin{pmatrix} c \\ s \end{pmatrix}_L , \begin{pmatrix} t \\ b \end{pmatrix}_L$	(3,2,+1/3)
U_R	$u_R,\ c_R,\ t_R$	(3,1,+4/3)
D_R	$d_R,\ s_R,\ b_R$	(3,1,−2/3)

Table 1.1: SM Particles content [15].

So, the matter fields are composed of three generations of fermions : left-handed fermions that are weak $SU(2)_L$ isodoublets, and right-handed fermions that are weak isosinglets. The electroweak theory is based on the electroweak gauge symmetry group $SU(2)_L \times U(1)_Y$ of Glashow, Weinberg, and Salam, where $SU(2)_L$ is the weak isospin symmetry group suggested by the left-handed doublets and $U(1)_Y$ is the weak-hypercharge phase symmetry group. This model gives a unified description of weak and electromagnetic interactions. It is not really a unification but this theory put mathematically in a same gauge symmetry group these two interactions, and therefore, there are two different coupling constants g and g'. These coupling constants describe the intensity of the electroweak interactions. Vectorial fields are associated with each generator of the electroweak gauge group and mediate the corresponding interaction. The field (weak isoscalar) B_μ is related to the generator Y of $U(1)_Y$ and has a coupling constant g', and the three fields $W_\mu^{1,2,3}$ (weak isovectors) are related to the three generators $T_{1,2,3}$ of $SU(2)_L$, and have a coupling constant g.

The electroweak interactions of gauge bosons and of the matter fields are introduced by requiring that the lagrangian of all the free particles be invariant under local gauge transformations from the $SU(2)_L \times U(1)_Y$ gauge group. The gauge transformations from this group are of course different for the left-and right-handed fermions. This invariance imposes the conservation of charges Y and I_3. The lagrangian of the electroweak interaction, without mass terms for fermions and gauge bosons, is

$$L = L_{gauge} + L_{fermions} = -\frac{1}{4} W_a^{\mu\nu} W_{\mu\nu}^a - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} + L_{fermions} \,, \tag{1.15}$$

where

$$L_{fermions} = \bar{\phi}_R i\gamma_\mu \partial^\mu \phi_R + \bar{\phi}_L i\gamma_\mu \partial^\mu \phi_L = L_{leptons} + L_{quarks} \,, \tag{1.16}$$

and with the strength tensors of these fields written as

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\epsilon^{abc} W_\mu^b W_\nu^c \,, \tag{1.17}$$

and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \,. \tag{1.18}$$

It is necessary to replace the partial derivative in $\mathcal{L}_{fermions}$ by the covariant derivative in order to obtain the local gauge invariance:

$$D_\mu = \partial_\mu - igW_\mu^a T_a - ig' \frac{Y}{2} B_\mu \,. \tag{1.19}$$

The matter fields are thus minimally coupled to the gauge fields V_μ through the covariant derivative which leads to unique couplings:

$$-g_i \bar{\phi} V_\mu \gamma^\mu \phi \,. \tag{1.20}$$

The isospin operator, that are equivalent to half of the non-commuting 2-by-2 Pauli matrices, obeys to the commutation relation

$$[T_i, T_j] = i\epsilon_{ijk} T_k \,, \tag{1.21}$$

while, of course, the hypercharge operator obeys

$$[Y, Y] = 0 \,. \tag{1.22}$$

Because the $SU(2)$ group is non-Abelian, there are triple and quartic self-interactions between its gauge bosons. The lagrangian (1.15) is thus invariant under the local $SU(2)_L \times U(1)$ gauge transformations for fermions and gauge fields:

$$\phi_L(x) \to e^{i\alpha_a(x) T^a + i\beta(x)\frac{Y}{2}} \phi_L(x), \quad \phi_R(x) \to e^{i\beta(x)\frac{Y}{2}} \phi_R(x) \,, \tag{1.23}$$

$$\vec{W}_\mu(x) \to \vec{W}_\mu(x) - \frac{1}{g} \partial_\mu \vec{\alpha}(x) - \vec{\alpha}(x) \times \vec{W}_\mu(x) \,,$$

$$B_\mu(x) \rightarrow B_\mu(x) - \frac{1}{g'}\partial_\mu \beta(x)\,,$$

where $\alpha(x)$ and $\beta(x)$ are arbitrary functions of space-time.

Despite the fact that using local gauge invariance as a dynamical principle we predict the correct particle physics phenomenology, up to now, the gauge fields and the fermions fields have been assumed to be massless. However, in the physical world, the three weak gauge bosons and the fermions are massive. The consequence of the mass of the weak gauge bosons is that fundamental interactions do not have all the same range. Electromagnetism and gravitation are of infinite range, indeed their gauge boson are massless, the photon and the hypothetical graviton. The strong and weak interactions act with a short range. For the weak interaction, this is explained by the fact that the gauge bosons have a mass. For the strong interaction, gluons do not have mass, however one explains the short range by the phenomenon of containment. Therefore, to build a gauge theory which describes the weak interactions, it is necessary to generate a mass term for gauge bosons. However, if we add a mass term, $\frac{1}{2}M^2 W_\mu W^\mu$, for the gauge bosons, this will violate the local $SU(2)_L \times U(1)_Y$ gauge invariance. Moreover, because the two chiralities are in different representations of $SU(2)$, the mass term for the fermions, connecting them, cannot be gauge invariant. A mass term $-m_f \bar{\phi}_f \phi_f$ for each fermion f cannot be added in the lagrangian. Thus the incorporation of mass terms for gauge bosons and for fermions leads to a breakdown of the local $SU(2) \times U(1)$ gauge invariance. The solution is to introduce a new scalar $SU(2)_L$ field ϕ and to choose an appropriately constructed potential for this field such that it is minimized when ϕ has an expectation value v. Then, we can write a Yukawa interaction term coupling left-and right-chiral fermions to this new scalar field. This create a fermion mass term proportional to this expectation value : $v\bar{\phi}_f \phi_f$. Through this mechanism, that is called the Higgs mechanism, the gauge bosons also acquire a mass. The Higgs field ϕ can therefore be described as the source of all masses in the SM. This mechanism is an example of spontaneous symmetry breaking mechanism [16, 17].

1.2 Spontaneous symmetry breaking

A symmetry is spontaneously broken if the lagrangian of a system is invariant under some symmetries whereas the ground state is not. So the ground state doesn't possess the same symmetries as its lagrangian. The term spontaneous comes from the fact that a system tends spontaneously towards its ground state. This spontaneous symmetry breaking leads to the existence of several vacua. One then selects a vacuum on which we base the theory. We will see that the minima of a theory are not necessarily the points of higher symmetries of the potential.

Let us give an example of spontaneously broken symmetry.

1.2.1 Example

Consider a lagrangian density, which dictates how a system will behave, for a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, that is a function of the spatial coordinate x :

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - V(\phi) \tag{1.24}$$

with a potential of the form :

$$V(\phi) = \mu^2 \phi^* \phi + \lambda(\phi^* \phi)^2 = \mu^2 |\phi|^2 + \lambda |\phi|^4 \,. \tag{1.25}$$

This lagrangian has a global $U(1)$ symmetry describing rotations in the complex plane. This means that it is invariant under global gauge transformations

$$\phi \to \tilde{\phi} = e^{i\theta} \phi \,.$$

The ϕ^4-term describes self interaction with intensity λ. The potential cannot have terms ϕ^n with a power larger than four, $n > 4$, because they produce infinities in observables, the theory would be non-renormalizable. The constants μ^2 and λ are both real and λ is positive to make the total field energy bounded from below. In order to analyze the dynamics of the system described by the lagrangian, we have to find minimum or minima of the potential and choose one minimum that will give a main vacuum state of the system, the lowest energy state. The next step is to find an excitation spectrum of the system. For this, we decompose the field ϕ in the vicinity of the main state and find excited states.

We require that the vacuum is invariant under Lorentz transformations and translations, this implies that ϕ(x) is a constant in this vacuum state. According to the parameter μ^2 , two different possibilities can happen for the vacuum state. When $\mu^2 > 0$, the minimum potential energy is reached at $\phi = 0$. This means that the vacuum expectation value for the field $\phi_0 \equiv \langle 0|\phi|0\rangle = 0$. The lagrangian describes a scalar particle of mass μ. If now $\mu^2 < 0$, the minimum energy no longer corresponds to a unique value of ϕ. The field acquires a vacuum expectation value and the minimum of the potential is at

$$|\phi_0|^2 = -\frac{\mu^2}{2\lambda} \,, \tag{1.26}$$

so the energy is degenerate with the minimum as a circle in the complex plane :

$$\phi_0 = \sqrt{\frac{-\mu^2}{2\lambda}} e^{i\theta}, \quad 0 \leq \theta < 2\pi \,. \tag{1.27}$$

There is an infinite number of possible solutions. Without loss of generality, as the lagrangian is invariant under rotations in the complex plane of ϕ, we can set $\theta = 0$ such that

$$\phi_{V_{min}} = \sqrt{\frac{-\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}} \,. \tag{1.28}$$

So, $V(\phi)$ has a minimum values

$$V(\phi)_{min} = -\frac{\lambda v^4}{4} < 0 \tag{1.29}$$

along a circle of radius v in the (ϕ_1, ϕ_2) plane, where $v^2 = -\frac{\mu^2}{\lambda} = \phi_1^2 + \phi_2^2$. The lagrangian no longer describes a particles of mass μ.

Figure 1.1: The potential V in the case $\mu^2 > 0$ (left) and $\mu^2 < 0$ (right) [16].

Quantum field theory demands that the vacuum be unique so that perturbation expansions must be calculated around that point. So, let us choose one minimum on the circle and develop the theory around this minimum. This choice leads us to the breaking of the symmetry. A theory where the vacuum has less symmetry than the lagrangian is called a theory with spontaneous symmetry breaking. By simplicity, we can take the real scalar field with the non-zero expectation value $\phi_1 = v$ and the imaginary part, $\langle 0|\phi_2|0\rangle = 0$. Then, we parametrize ϕ, with two real fields σ and η, as

$$\frac{1}{\sqrt{2}}(v + \sigma(x) + i\eta(x)) \tag{1.30}$$

where $\langle 0|\sigma(x)|0\rangle = 0$ and $\langle 0|\eta(x)|0\rangle = 0$. Now, we expand all the terms in the lagrangian in series in the small parameter $\sigma(x) + i\eta(x)$ around the minimum of the potential. The lagrangian, in terms of σ and η, becomes

$$\mathcal{L} = \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma - \lambda v^2\sigma^2 + \frac{1}{2}\partial^\mu\eta\partial_\mu\eta - \lambda v\sigma^3 - \lambda v\sigma\eta^2 - \frac{1}{4}\lambda(\sigma^2+\eta^2)^2 + \frac{v^4\lambda}{4} \,. \tag{1.31}$$

We can see that σ and η are two real Klein-Gordon fields. The σ (Higgs) field is massive and will have mass $m_\sigma = v\sqrt{2\lambda}$ arising from the σ^2 term, while the η field is massless. The remaining terms in the lagrangian can be treated as interactions among the σ and η particles through perturbation theory.

We have seen in this example that the spontaneous symmetry breaking of the $U(1)$ symmetry that arises from the degenerate energy minimum of the lagrangian can create a perturbative theory with a massive scalar boson, the Higgs boson and a massless field. This is an example of the Golstone theorem.

1.2.2 Mass matrix and Goldstone theorem

The Goldstone theorem has two important consequences for spontaneous global symmetry breaking :

- The lagrangian remains invariant but the state of minimal energy, the vacuum state, is not invariant any more. As the excited states are obtained from the action of generators on the vacuum, symmetry is not manifest any longer in the spectrum of the states.
- There exist some physical massless states. Their properties are connected to those of the generators of the symmetry breaking. These are the Goldstone bosons. There are as many Goldstone bosons than broken generators.

Now we are going to illustrate another formulation of this theorem [18]. Let us consider a theory involving several classical scalar fields $\phi_i(x)$, with a lagrangian of the form

$$\mathcal{L} = K - V(\phi)\,. \tag{1.32}$$

$\phi_{0,i}$ are constant fields that minimizes V, so that

$$\left(\frac{\partial V}{\partial \phi_i}\right)_{\phi_i=\phi_{i,0}} = 0\,. \tag{1.33}$$

These constant fields will minimize the Hamiltonian and therefore the energy. Thus, they define the vacuum state.

If now we expand V about this minimum, we get :

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi-\phi_0)_i(\phi-\phi_0)_j\left(\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right)_{\phi_0} + ... \tag{1.34}$$

Then we redefine the fields: $\varphi_i(x) = \phi_i(x) - \phi_{0,i}$ and we call the quadratic term the mass matrix

$$m_{ij}^2 = \left(\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right)_{\phi_0}\,. \tag{1.35}$$

This matrix is symmetric because derivatives are commuting and the eigenvalues of this matrices give the masses of the field. They are always positive or null since $\phi_{i,0}$ is a minimum. Each continuous symmetry of $\mathcal{L}$ that is not a symmetry of ϕ_0 gives rise to a zero eigenvalue of this mass matrix. This is another formulation the Goldstone theorem which states that for every spontaneously broken continuous symmetry , a massless particle appears.

However, in the SM, the lagrangian possesses a local gauge symmetry. Therefore, the consequences of the symmetry breaking are different. Some Goldstone bosons become new longitudinal polarization states of the gauge bosons, the latter become massive and the former disappear from the spectrum. This phenomenon is called the Higgs mechanism.

1.2.3 Higgs mechanism in an abelian case

Let us first consider an abelian model with a single vector boson, the electromagnetic field A_μ, associated with a $U(1)$ local symmetry and a complex scalar field ϕ :

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi)\,, \tag{1.36}$$

where the potential is the same as in section 1.2.1. The field strength and the covariant derivative are defined in terms of A_μ with a coupling constant e as in (1.7) and (1.3) with $Q = 1$. The lagrangian is therefore renormalizable and invariant under the local $U(1)$ gauge transformation of the fields :

$$\phi(x) \rightarrow e^{i\alpha(x)}\phi(x)\,, \quad A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \tag{1.37}$$

where $\alpha(x)$ is an arbitrary function of space-time.

For $\mu^2 > 0$, one finds the QED lagrangian for a charged scalar particle of mass μ with ϕ^4 self-interactions. For $\mu^2 < 0$, the field $\phi(x)$ acquires a vacuum expectation value (cf. section 1.2.1). As in section 1.2.1, to find energies of the particles, we have to choose one of the minima and develop the lagrangian in the vicinity of it. The complex field ϕ can be parametrized as

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \sigma(x) + i\eta(x)) \equiv \frac{1}{\sqrt{2}}(v + \sigma^{'}(x))e^{\frac{i\eta(x)}{v}}\,. \tag{1.38}$$

We use the freedom of gauge transformations and we choose $\alpha(x) = -\frac{\eta(x)}{v}$. With this particular choice, called the unitary gauge, the gauge transformation becomes:

$$\phi(x) \rightarrow e^{-\frac{\eta(x)}{v}}\phi(x)\,, \quad A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{ev}\partial_\mu\eta(x)\,. \tag{1.39}$$

Therefore the field ϕ takes the parametrized form :

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \sigma(x)), \tag{1.40}$$

and the lagrangian becomes :

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma - \lambda v^2\sigma^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2v^2}{2}A_\mu A^\mu + \frac{e^2}{2}A_\mu A^\mu(2v\sigma + \sigma^2) \\ &\quad -\lambda v\sigma^3 - \frac{1}{4}\lambda\sigma^4 + \frac{v^4\lambda}{4}\,. \end{aligned} \tag{1.41}$$

As in section 1.2.1, the scalar field σ becomes massive, it has positive mass $m_\sigma = v\sqrt{2\lambda}$. We see that $\eta(x)$ disappears from $\mathcal{L}$. In fact, the degree of freedom associated with the would-be Goldstone field $\eta(x)$ can be seen as a longitudinal degree of freedom of the field $A_\mu(x)$, that have now three degrees of freedom. Therefore, A_μ acquires a non zero mass:

$$m_A^2 = e^2v^2\,. \tag{1.42}$$

The $U(1)$ symmetry is not apparent anymore in this lagrangian. Therefore, the symmetry is spontaneously broken, this is the Brout-Englert-Higgs (BEH) mechanism that allows the gauge bosons to acquire a non zero mass. For each massive gauge boson, one Goldstone scalar must disappear from the physical spectrum in order to conserve the number of degrees of freedom.

So, we have illustrated the fact that the Goldstone theorem is generalized when a local gauge symmetry is broken: the would-be Goldstone bosons are absorbed by the massless gauge bosons that become massive and thus remove the massless scalars from the spectrum.

1.2.4 Electroweak theory with spontaneously symmetry breaking

In the non-abelian $SU(2)_L \times U(1)_Y$ electroweak theory, one needs to generate masses for the three gauge bosons $W^\pm$ and Z but the photon must remain massless, and QED must remain an exact symmetry so that the electric charge is conserved [19]. Therefore, one needs at least three degrees of freedom for the scalar field to be absorbed as the longitudinal degrees of freedom of these vector fields. We start with a theory with a $SU(2)_L$ gauge symmetry. To break the symmetry spontaneously, let us introduce a complex $SU(2)_L$ doublet of scalar fields

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}. \tag{1.43}$$

It is the simplest choice to group the four fields into one $SU(2)_L$ representation. However, this theory leads to system with no massless gauge bosons. We therefore introduce an additional U(1) gauge symmetry. We give to the scalar field a weak hypercharge charge $Y = 1$ under this $U(1)$ symmetry.

We add to the lagrangian (1.16) new gauge invariant terms for the interaction and propagation of the scalars :

$$\mathcal{L}_\phi = (D^\mu\phi)^\dagger(D_\mu\phi) - V(\phi^+\phi) \tag{1.44}$$

where the potential has the form

$$V(\phi^\dagger\phi) = \mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2 = \mu^2|\phi|^2 + \lambda|\phi|^4 \tag{1.45}$$

and the product $\phi^\dagger\phi$ can be expressed as:

$$\phi^\dagger\phi = \phi^{+*}\phi^+ + \phi^{0*}\phi^0 = \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2)\,. \tag{1.46}$$

For $\mu^2 > 0$, the state of lower energy corresponds to the annulment of the fields, so there is no spontaneously symmetry breaking. The electroweak symmetry is spontaneously broken if the parameter μ^2 is negative. In this case, the fundamental state is not unique, it corresponds to a circle of degenerated fundamental states as in (1.27). To preserve electric charge conservation, i.e. the $U(1)_{QED}$ symmetry, this nonzero vacuum expectation value cannot be reached in the charged direction. Gauge invariance gives us the freedom to choose the state of minimum energy. The vacuum expectation value can be rotated using a $SU(2)_L \times U(1)_Y$ gauge transformation to take the form :

$$\langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \quad v = \left(-\frac{\mu^2}{\lambda}\right)^{\frac{1}{2}}, \tag{1.47}$$

where the neutral component ϕ_3 of the doublet field develops a nonzero value. This vacuum expectation value will break the gauge symmetry

$$SU(2)_L \times U(1)_Y \longrightarrow U(1)_{EM}. \tag{1.48}$$

In order to develop the theory around this minimum, we write the field ϕ in terms of four fields $\xi_{1,2,3}(x)$ and $H(x)$ as

$$\phi(x) = \begin{pmatrix} \xi_1 + i\xi_2 \\ \frac{1}{\sqrt{2}}(v + H(x)) - i\xi_3 \end{pmatrix} = e^{\left(\frac{i\xi_a T^a}{v}\right)}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}. \tag{1.49}$$

Then, we proceed as in the previous section. We perform an $SU(2)_L$ gauge transformation on this field :

$$\phi(x) \to e^{\left(\frac{-i\xi_a T^a}{v}\right)}\phi(x) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}. \tag{1.50}$$

The three fields ξ_i are three Goldstone bosons that will give masses to the three weak gauge fields. Spontaneous symmetry breaking will lead to a massive boson from the field $H(x)$, the Higgs boson. The gauge boson mass terms come from the square of the covariant derivative

$$D_\mu = \partial_\mu - igW^a_\mu T_a - ig'\frac{1}{2}B_\mu Y\,, \tag{1.51}$$

evaluated at the scalar field expectation value. The relevant terms, since $Y = 1$, are

$$\Delta\mathcal{L} = \frac{1}{2}(0 \quad v)(gW^a_\mu T_a + \frac{1}{2}g'B_\mu)(gW^{\mu b}T_b + \frac{1}{2}g'B^\mu)\begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{1.52}$$

Using $T^a = \frac{\sigma^a}{2}$, the evaluation of the matrix product leads to

$$\Delta\mathcal{L} = \frac{1}{2}\frac{v^2}{4}[g^2(W^1_\mu)^2 + (W^2_\mu)^2 + (-gW^3_\mu + g'B_\mu)^2]\,. \tag{1.53}$$

There are three massive vector bosons, which we will notate as follows:

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^2_\mu), \quad Z^0_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gW^3_\mu - g'B_\mu)\,. \tag{1.54}$$

The fourth vector field, orthogonal to Z^0_μ, remains massless since it doesn't appear in the lagrangian :

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gW^3_\mu + g'B_\mu)\,. \tag{1.55}$$

Mass terms are terms that are bilinear in $W^\pm$, Z, A:

$$m_W = g\frac{v}{2}\,, \quad m_Z = \frac{v}{2}\sqrt{g^2+g'^2}\,, \quad m_A = 0\,. \tag{1.56}$$

Thus, by spontaneously breaking the symmetry $SU(2)_L \times U(1)_Y$ to $U(1)_{QED}$, three Goldstone bosons have been absorbed by the $W^\pm$ and Z bosons to form their longitudinal components and to get their masses. Since the $U(1)_Q$ symmetry is unbroken, the photon, which is associated to its generator, remains massless as should be.

The physical bosons observed in interactions are the photon A and the $W^\pm$ and Z bosons. In fact, $W^\pm$ bosons are mass eigenstates while W^3_μ and B_μ mix to give the two physical bosons A_μ and Z_μ:

$$\begin{pmatrix} Z^0_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos(\theta_W) & -\sin(\theta_W) \\ \sin(\theta_W) & \cos(\theta_W) \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}$$

with[2] $m_A = 0$ and $m_W = m_Z \cos\theta_W$.

The weak mixing angle θ_W to change from the (W^3, B) basis to the (Z^0, A) basis is defined as

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \quad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \tag{1.57}$$

so, that

$$\tan\theta_W = \frac{g'}{g}. \tag{1.58}$$

With the same doublet of scalar fields ϕ, we can also generate the fermion masses. Indeed, we can add $SU(2)_L \times U(1)_y$ gauge-invariant Yukawa interactions between the scalar fields and the fermions which are $SU(2)$ doublets or singlets. For example,

$$\mathcal{L}_{Yukawa} = -\bar{Q}_L \lambda_d \phi D_R - \bar{Q}_L \lambda_u \tilde{\phi} U_R - \bar{L}_L \lambda_l \phi E_R + h.c. \tag{1.59}$$

where the $SU(2)_L$ doublet

$$\tilde{\phi} = iT_2\phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$$

has an hypercharge $Y = -1$ such that the total hypercharge of each term equals zero. After the spontaneous symmetry breaking of the electroweak symmetry, these Yukawa interactions provide mass terms $m_{u,d,l} = \lambda_{u,d,l}\frac{v}{\sqrt{2}}$ to all fermions.

Thus, with the same isodoublet ϕ of scalar fields, we have generated the masses of both the weak vector bosons $W^\pm$, Z and the fermions, while preserving the gauge symmetry in the lagrangian.

1.2.5 The Higgs boson

The mass and self–interaction parts of the Higgs lagrangian come from the scalar potential $V(\phi)$ and from the kinetic part. Using the relation $v^2 = -\frac{\mu^2}{\lambda}$, the lagrangian of the Higgs fields is thus

$$\mathcal{L} = \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{\lambda}{4}H^4. \tag{1.60}$$

Therefore, the Higgs boson mass is

$$m_H^2 = 2\lambda v^2 = -2\mu^2 \tag{1.61}$$

[2] An important parameter of the SM is $\rho = \left(\frac{m_W}{cos\theta_W m_Z}\right)$ and it was experimentally confirmed to be very close to 1.

where λ is the Higgs self-coupling parameter. Since λ is unknown at present, the value of the standard model Higgs boson mass is not predicted. However, theoretical considerations place constraints on the Higgs boson mass. For example, a lower limit on m_H can be derived from the requirement of vacuum stability. Similarly, an upper bound on m_H is obtained from the requirement that the perturbative description of the theory remains valid.

From this lagrangian, we can derive the Feynman rules for the Higgs self–interaction vertices. These are:

$$g_{HHH} = 3!i\lambda v = 3i\frac{m_H^2}{v}, \quad g_{HHHH} = 4!i\frac{\lambda}{4} = 3i\frac{m_H^2}{v^2}. \tag{1.62}$$

The Higgs boson couplings to gauge bosons and fermions are:

$$g_{Hf\bar{f}} = i\frac{m_f}{v}, \quad g_{HVV} = -2i\frac{m_V^2}{v}, \quad g_{HHVV} = -2i\frac{m_H^2}{v^2}\,. \tag{1.63}$$

where $V = W$ or Z. The Higgs couplings to fermions (bosons) are thus predicted to be proportional to the corresponding particle masses (squared-masses). So, in Higgs production and decay processes, the dominant mechanisms involve the coupling of the Higgs boson to the heaviest particles that are $W^\pm$, Z and the third generation quarks and leptons. This characteristic coupling of the Higgs is a useful tool in searching for it. The vacuum expectation value v is fixed in terms of the W mass determined by the value of the Fermi constant G_F that appears in the old four-fermion theory of weak interactions:

$$m_W = \frac{1}{2}gv = \left(\frac{\sqrt{2}g^2}{8G_\mu}\right)^{\frac{1}{2}}\,. \tag{1.64}$$

Muon decay, that occurs through gauge interactions mediated by W boson exchange, is a particular process through which G_F is measured very accurately. The actually accepted value is $v \approx 246$ GeV.

Despite the great successes of the LEP, SLC and Tevatron colliders in verifying many detailed aspects of the standard model, the Higgs boson is still missing. Very precise experiments, which allow a sensitivity to quantum corrections, have been made in the last fifteen years and at the same time, a large theoretical effort has been devoted to the calculation of the radiative corrections to the electroweak observables. Indeed, except for the Higgs mass, all the parameters of the SM have been determined experimentally with a great accuracy. Using these parameters, one can in principle calculate any physical observable and compare the result with experiments. For example, the couplings of quarks and leptons to the weak gauge bosons $W^\pm$ and Z are indeed precisely those prescribed by gauge symmetry. The triple gauge vertices γWW and ZWW have also been found in agreement with the specific predictions of the electroweak gauge theory. For more details of all these results, see [19].

The big remaining questions are about the nature and the properties of the Higgs particle. In fact, the Higgs boson has not been observed yet and we don't know its mass. Nevertheless, the precision electroweak data impose some strong constraints, and seem to provide strong support for the standard model with a weakly-coupled Higgs boson. The radiative corrections computed in the SM when compared to the data on precision electroweak tests lead to a clear indication for a light Higgs. Indeed, the present experimental information on the Higgs sector can be summarized as follows. The relation $m_W^2 = m_Z^2 \cos\theta_W^2$ has been experimentally proved. Direct recent measurements at LEP-II have excluded a SM Higgs boson with a mass $m_H < 114.4$ GeV at 95% C.L. This lower bound has been derived from the failure to see the Higgs boson being produced at the LEP machine. The present experimental upper limit on m_H is: $m_H < 190$ GeV. This upper bound is a constraint from the required agreement between theorical calculations of electroweak observables including the Higgs as a virtual state, and precision measurements. Moreover, recently, for the first time the Tevatron excludes a SM Higgs boson mass range [160-170] GeV beyond the LEP limit at 95% CL [20].

The quest for the Higgs boson is the main priority of the LHC. There are several ways, depending on the mass of this boson, to observe it. One way the Higgs boson may be produced at the LHC is when two gluons split into a top/anti-top pair which then combine to make a neutral Higgs. One other possible way of formation of a neutral Higgs boson would be to start from two quarks that exchange electroweak bosons. The Higgs boson has also a number of indirect effects. For example, Higgs loops result in tiny corrections to W and Z masses. Further details of the basic properties of the SM Higgs boson and of its decay modes and main production mechanisms are given in the Higgs hunter's guide [21].

All these properties and these experimental possibilities and limitations discussed aboved refer only to the minimal SM Higgs boson. In the next chapter, we will consider more complicated Higgs sectors whose results will be different. However, these high-precision electroweak data are a useful tool in the search for indirect effects, through possible small deviations of the experimental results from the theoretical predictions of the minimal SM, and constitute an excellent probe of its still untested scalar sector, as well as a probe of new physics beyond the SM, [19].

Chapter 2

The Two-Higgs-Doublet-Model

Despite the success of the standard model in describing all known experimental data available today and in particular of the electroweak theory with one $SU(2)_L$ Higgs doublet, which provides a successful description for the observed electroweak phenomena, the necessity for new physics is not recent [22, 23]. Observed neutrino oscillation phenomenon [24], implying that the neutrino has a non-zero mass, which is not part of the original standard model of particle physics, confirmed that the SM had inadequacies that can only find an explanation in some extensions. Moreover, the Higgs sector is still unknown since no Higss boson has been discovered yet. Therefore, there is no reason to assume that the Higgs sector contains only one doublet.

In this chapter, we are going to review some shortbacks of the SM so that the necessity to expand it will appear clearly. Then we will detail several motivations to extend the Higgs sector and especially motivations for a two-Higgs-doublet model (2HDM) that is the simplest extension compatible with the gauge invariance of the minimal Higgs sector. We will introduce this model and detail some of its properties.

2.1 Inadequacies of the standard model

It is very surprising that the standard model describes so well the observed phenomena whereas it doesn't explain the quantum numbers of the particles, it doesn't predict the mass spectrum, it doesn't include gravity, it doesn't explain the three generations of fermions, etc. Let us detail some reasons the SM cannot be the whole theory describing Nature.

2.1.1 Neutrinos

Neutrinos, that only participate to weak processes through the exchange of W and Z bosons, were assumed to be massless and only left-handed. However, there is experimental evidence that neutrinos are massive through the observation of the neutrino oscillation phenomenon [25]. Neutrino oscillations come from the fact that eigenstates of mass are not eigenstates of flavour which can happen if the mass terms involve mixing. So, the neutrino oscillation phenomenon in which a neutrino of one flavour spontaneously converts itself into another one of different flavour during free propagation is possible only if neutrinos are massive. Those masses are extremely tiny (sub-eV) compared to the masses of the other particles.

The tiny values the neutrino masses and the fact that only left-handed neutrinos are natural in the SM suggest that the neutrino masses have an origin different from those of other elementary fermions, the quarks and charged leptons.

2.1.2 Dark matter

In 1932, astrophysicists discovered that there was a large contribution of non–baryonic and non–luminous matter to the critical density of the Universe [26]. They called it the dark matter. They observed that the density of known matter, i.e. baryonic matter, represents less than 4 % of the total energy density of the universe. Now, it is known that nearly a quarter of the energy content of the Universe is due to a form of distributed matter, i.e. the dark matter, that has gravitational interaction but that is non-luminous and electrically neutral. The remainder is completely unknown for us and one attributes it to a kind of dark energy.

A plausible explanation would be that the dark matter is due to the dominant presence of some new kind of stable or very long-lived electrically neutral particle(s) without strong[1] and electromagnetic interactions. Moreover, it is know that this dark matter is cold [27], this means that it has a non-relativistic velocity distribution. The SM does not include any candidate particle to account for such a dark matter component. Its presence would therefore require an extension of the model. If it has weak interactions, it should be producible at the LHC.

[1]If dark matter can interact via strong interactions, that would have interfered with the nucleosynthesis process.

2.1.3 Hierarchy problem

We have seen in the previous chapter that in the standard model, the electroweak theory is an $SU(2)_L \times U(1)_Y$ theory that is spontaneously broken to the $U(1)_{QED}$ gauge symmetry. The electroweak symmetry breaking is described by the Higgs mechanism which provides masses for gauge bosons, and respect gauge invariance and renormalizability. Its realization is based on a single complex $SU(2)_L$ doublet of scalar fields that will acquire a non-zero vacuum expectation value. Three of these scalar fields are absorbed by the three weak gauge bosons that acquire masses and the remaining one, the physical Higgs boson, couples to the gauge and matter fields and self-interacts via the quartic scalar potential. At the classical level, this above statement makes good sense. However, a problem arises when one tries to take quantum loop corrections into account. In fact, contrarily to fermions masses that are protected by chiral symmetry, masses of elementary scalar fields are unstable under quantum corrections. In fact, the squared physical renormalized Higgs-boson mass m_H^2 that is expected to be of the order of the squared vacuum expectation value v^2 of the Higgs field, receives large quantum corrections that quadratically depend on the cutoff Λ:

$$\delta m_H^2 \propto \mathcal{O}\left(\frac{\alpha}{\pi}\right)\Lambda^2\,, \tag{2.1}$$

α being the fine-structure constant. Thus, if one makes the reasonable requirement that the quantum correction to the mass should not be too large as compared with the original mass, one is forced to keep a cutoff within an order of magnitude of the mass, which in the case of the electroweak theory is about 1 TeV. Indeed, if we extend the theory to the Plank scale, upper scale beyond which we cannot neglect quantum gravity, then the natural value of the Higgs mass will be the Planck scale, an energy scale around 1.22×10^{28} eV. This is an other question the SM doesn't answer: why are there 17 orders of magnitude which separate the electroweak scale from the Planck scale ? Of course, we can choose the renormalization such that the Higgs boson mass is fixed to the range of values in which it is experimentally required to be. Such a fine tuning is considered unnatural. Moreover, this procedure of renormalization would need to be repeated order by order in perturbation theory making the theory somewhat unnatural. Since the Higgs boson mass is around 100 Gev, this problem leads to say that the SM is only valid within an order of magnitude of that. An other hierarchy problem is about masses: why is the top quark 4×10^5 times heavier than the electron?

2.2 Physics beyond the standard model and motivations for a 2HDM

During the last decades there were numerous attempts to find physics beyond the SM. Some of them are based on an extended Higgs sector. Indeed, the scalar sector of the standard model described in the previous chapter is minimal, this means that the associated Higgs representation[2] is the simplest possibility allowing for non vanishing mass terms for weak bosons and fermions after spontaneous symmetry breaking. Nevertheless, the mechanism of electroweak symmetry breaking with one doublet is still not confirmed. Therefore, one may consider larger or additional representations. Motivations to enlarge the Higgs sector [28] can be related to the requirements of higher scale symmetries, like supersymmetry, grand unification theories, etc, or they can be justified by phenomenological arguments, such as the possibility of new sources of CP violation.

In this work we are concerned with the simplest extension of the scalar sector: the two-Higgs-doublet models (2HDM) that require the introduction of a second Higgs doublet. Let us concentrate on the motivations for such an extension of the Higgs sector.

2.2.1 Theories behind the SM with a higher symmetry

Supersymmetry

Supersymmetry is often considered as the most attractive extension of the standard model as it solves some of the problems mentioned above [15]. It predicts the existence of a partner to every known particle which differs by $\frac{1}{2}$ unit of spin, so the supersymmetry is a symmetry that transforms fermions into bosons and vice-versa. In this model, the SM must therefore be extended by adding a new elementary particle, called a superpartner or a sparticle, for every known particle. However, if supersymmetry were exact, these would have the same masses as the ordinary particles. That is in contradiction with observations as we know that the selectron mass is different from the electron mass, and therefore supersymmetry is a broken symmetry and various scenarios have been proposed.

Supersymmmetry provides a solution to the hierarchy problem, provides unification of the three interactions at the grand unification theory scale and has a possible candidate for the dark matter.

[2]The smallest representation containing the three would-be Goldstone fields associated with the three massive gauge bosons and the Higgs scalar is a single $SU(2)_L$ doublet.

The simplest supersymmetric extension of the standard model is the Minimal Supersymmetric Standard Model (MSSM), which contains at least two fundamental Higgs fields that are responsible for the generation of masses. This model requires a second Higgs doublet to preserve the cancellation of gauge anomalies. So, the Higgs sector of this model is a 2HDM which contains two chiral Higgs supermultiplets that are distinguished by the sign of their hypercharge. The theoretical structure of the MSSM Higgs sector is constrained by supersymmetry, leading to numerous relations among Higgs masses and couplings.

Grand Unification theories

Grand Unified Theories (GUTs) aim at the ambitious task of unifying strong and electroweak gauge interactions in a non-abelian gauge theory based on a single compact Lie group that contains the standard model group $SU(3)_C \times SU(2)_L \times U(1)_Y$. Several Lie groups have been considered, from the simplest, $SU(5)$ [29], $SO(10)$, to larger groups as E_6 models. In general, unification theories have some common and interesting characteristics. For example, they include gauge couplings unification at very high energy. However, due to the presence of new very massive gauge bosons, they also allow, in general, proton decay which is the most dramatic prediction coming from unification.

We need to break the Lie group to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ group. To do so, different Higgs representation are required. In general, this implies the presence of an extended scalar sector at the electroweak scale. For example, a two-Higgs-doublet model in needed to break $SO(10)$ to the SM group .

2.2.2 Phenomenology arguments

Sources of CP violation in the Higgs sector

One of the reasons for introducing an extended Higgs sector with two Higgs doublets was to implement CP violation via the Higgs sector [30]. The phenomenon of CP violation plays an important role both in the study of weak interactions in particle physics and also in cosmology. Indeed, it is a necessary ingredient to explain the dominance of matter over antimatter in the present Universe. In the SM, hermiticity requires that the parameters of the scalar potential are real. consequently, the resulting bosonic sector of the electroweak theory is CP-conserving. CP violation arises through a non vanishing complex phase in the CKM matrix. However, this mechanism cannot account by itself for the observed baryon asymmetry. The main reason is that the CP violating effects associated with the three generations CKM matrix are too small.

Extensions of the SM scalar sector can provide a solution to this problem. Indeed, in the two-Higgs-doublet models, for example, the CP symmetry can be violated explicitly in the scalar sector. CP violation can appear if some of the coefficients in the 2HDM potential are complex. However, the requirement of neutral flavour conservation restricts the possibilities for CP violation.

Dark matter

A potential candidate for dark matter is the lightest supersymmetric particle (LSP), in most cases a neutralino or a gravitino.

Moreover, a specific 2HDM called the inert model [31], requiring a two-Higgs-doublet extension of the SM scalar sector, could be a candidate to explain dark matter. The main feature of this model is a Z_2 symmetry which remains unbroken, imposing one of the doublet to acquire a null vacuum expectation value. This doublet, called the inert Higgs doublet, has neither vacuum expectation value nor couplings to quarks and leptons but it has a non zero mass. After mixing, the lightest particle might compose the dark matter while the usual Higgs boson is heavy ($>$ 400 GeV) and does not contradict the precision EW tests.

Yukawa couplings and Fermion mass spectrum

The Standard model doesn't explain the fermion mass spectrum, which is related to the Yukawa couplings between the Higgs field and the fermions. These couplings are completely arbitrary, and so cannot explain why there are three generations of particles or the large spectrum of masses.

For example, the top and bottom quarks of the third generation have very different masses , $m_{top} \approx 174$ GeV while $m_{bottom} \approx 5$ GeV. In a model with one doublet, all quarks receive their masses from the same doublet. In a model with two doublets, the Yukawa coupling can be more natural because it is possible to generate Yukawa couplings such that the bottom quark receives its mass from one doublet and the top from another doublet.

2.3 The two-Higgs-Doublet Model

The two-Higgs-Doublet Model (2HDM) is the simplest extension of the standard model (SM) with one extra scalar doublet which contains more physical neutral and charged Higgs fields. Therefore, this model contains two complex doublets of scalar fields, ϕ_1 and ϕ_2:

$$\phi_i = \begin{pmatrix} \phi_i^+ \\ \phi_i^0 \end{pmatrix} \tag{2.2}$$

with i=1,2. Hence, there are eight degrees of freedom that will be used to give masses to the gauge bosons. In some cases, after symmetry breaking, three Goldstone bosons provide the longitudinal modes of the bosons $W^\pm$ and Z, that become massive. And there will remain five physical Higgs bosons : three neutral ones h_1, h_2, h_3 and two charged ones $H^\pm$.

2.3.1 Lagrangian for the 2HDM

The spontaneous electroweak symmetry breaking via the Higgs mechanism is described by the most general $SU(2)_L \times U(1)_Y$ invariant lagrangian for the 2HDM that can be written as

$$\mathcal{L}_{2HDM} = \mathcal{L}_\phi + \mathcal{L}_{Yukawa} + \mathcal{L}_{SM} \,. \tag{2.3}$$

$\mathcal{L}_{SM}$ describes the $SU(2)_L \times U(1)_Y$ standard model interactions of gauge bosons and fermions, $\mathcal{L}_{Yukawa}$ describes the Yukawa interactions of fermions with Higgs scalars. These two terms will not be discussed here because they are not relevant in our analysis. The Higgs scalar lagrangian $\mathcal{L}_\phi$ is

$$\mathcal{L}_\phi = \sum_{i=1,2} (D_\mu \phi_i)^+ (D^\mu \phi_i) - V_H(\phi_1, \phi_2) \,. \tag{2.4}$$

These terms replace the kinetic term and the Higgs potential in the standard model lagrangian with the same covariant derivative:

$$D_\mu = \partial_\mu - igW_\mu^a T_a - ig' \frac{Y}{2} B_\mu \,, \tag{2.5}$$

where T_a and Y are the generator of weak-isospin and weak-hypercharge transformations.

In order to keep, for the quantity $\rho = \left(\frac{m_W}{cos\theta_W m_Z}\right)$, the value $\rho = 1$ at tree level, both Higgs fields should be weak isodoublets ($I = 1/2$) with hypercharges $Y = \pm 1$. Here, we use $Y = +1$ for both doublets. In the MSSM, $Y_1 = 1, Y_2 = -1$, but our results hold up to redefinitions.

The 2HDM potential

The most general gauge invariant and renormalizable potential $V(\phi_1, \phi_2)$ for the 2HDM is defined in the 8-dimensional space of Higgs field and is a hermitian combination of the electroweak-invariant combinations $(\phi_1^\dagger \phi_1)$, $(\phi_2^\dagger \phi_2)$, $(\phi_1^\dagger \phi_2)$, $(\phi_2^\dagger \phi_1)$. $(\phi_i^+ \phi_j)$, $i, j = 1, 2$. In models of electroweak interactions with spontaneously broken gauge invariance, renormalizability limits to four the degree of the Higgs potential, terms of order greater than four have to be excluded because they are not renormalizable, therefore, the maximum power of the combination $(\phi_i^\dagger \phi_j)$ is 2. The most general two-Higgs-doublet potential is conventionally parametrized in a generic basis as :

$$V_H = V_2 + V_4$$

where

$$\begin{aligned}
V_2 &= -\frac{1}{2}[m_{11}^2(\phi_1^\dagger\phi_1) + m_{22}^2(\phi_2^\dagger\phi_2) + m_{12}^2(\phi_1^\dagger\phi_2) + m_{12}^{2*}(\phi_2^\dagger\phi_1)] \\
V_4 &= \frac{\lambda_1}{2}(\phi_1^\dagger\phi_1)^2 + \frac{\lambda_2}{2}(\phi_2^\dagger\phi_2)^2 + \frac{\lambda_3}{2}(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \\
&\quad \frac{\lambda_4}{2}(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) + \frac{1}{2}[\lambda_5(\phi_1^\dagger\phi_2)^2 + \lambda_5^*(\phi_2^\dagger\phi_1)^2] + \\
&\quad \{[\lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)](\phi_1^\dagger\phi_2) + h.c.\}\,. \qquad (2.6)
\end{aligned}$$

This general potential with all quadratic and quartic terms contains 14 real free parameters (in contrast to only two real parameters for one doublet) : the real parameters (by hermiticity of the potential) $m_{11}^2, m_{22}^2, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and the complex parameters $m_{12}^2, \lambda_5, \lambda_6, \lambda_7$.

This potential is responsible for the stability and the symmetry-breaking of the model. After the electroweak symmetry breaking, it is responsible, with the interaction terms from the kinetic terms, for the generation of gauge bosons masses. But the large number of free parameters makes the characterization of the symmetry breaking for different regions in parameter space very complicated. Moreover, we will see that the potential of the 2HDM can have extrema with different physical properties.

In contrast with the standard model, the potential is not unique. Each set of parameters will lead to different mass eigenstates, interactions, Feynman rules, etc. Therefore, the 2HDM is governed by the choice of the Higgs potential parameters and moreover by the Yukawa couplings of the two scalar doublets to the three generations of quarks and leptons.

The parameters depend on the choice of the $\phi_1 - \phi_2$ basis. In writing (2.6), we have implicitly chosen a basis in the two-dimensional space of the scalar fields. In order to allow for other bases, we will see later that it is convenient rewrite (2.6) in a covariant form and to analyze the properties of the potential, such as its stability and its spontaneous symmetry breaking, in terms of covariant quantities.

2.4 Constraints on the Higgs Langrangian

2.4.1 Positivity constraints

These are conditions on the parameters from the requirement of a stable vacuum. To have a stable vacuum, the potential must be positive at large quasiclassical values of fields ϕ_1 and ϕ_2 for any directions in the (ϕ_1, ϕ_2) plane.

2.4.2 Minimum constraints

The minimum constraints are the conditions ensuring that the extremum is a minimum for all directions in the (ϕ_1, ϕ_2) plane, except in the direction of the Goldstone modes. This condition is realized if the mass matrix of the Higgs fields is definite positive.

2.4.3 Perturbativity and tree-level unitarity constraints

The quartic terms of the Higgs potential (λ_i) are transformed to the quartic self-couplings of the physical Higgs bosons. In fact, the tree-level amplitudes[3] for the scattering of longitudinal gauge bosons at high energy can be related to the corresponding amplitudes in which the longitudinal gauge bosons are replaced by Goldstone bosons and the latter can be computed in terms of quartic coupling λ_i. They lead, at tree level, to the s-wave couplings Higgs-Higgs and $W_L W_L$ and $W_L H$, etc. By imposing tree-level unitarity constraints on these amplitudes, one can derive upper bounds on the values of certain combinations of Higgs quartic couplings. Tree-level unitarity constraints were obtained in the 2HDM with CP-violation in [32].

The perturbativity condition for the validity of a tree-level approximation in the description of interactions of the lightest Higgs boson may be less restrictive than the unitarity constraints.

[3]Tree level approximation means that we do not consider loops in calculation.

2.5 Particular cases of 2HDM

In many applications, we don't need to consider the most general 2HDM with the 14 parameters in the scalar potential. There exist different versions of 2HDM with different choices of the parameters that have interesting characteristics.

For example, as studied in the first article about the 2HDM [30], if V is Z_2 symmetric under the transformations: $\phi_1 \to -\phi_1, \quad \phi_2 \to \phi_2$, then there is no $\phi_1 \longleftrightarrow \phi_2$ transition (or vice versa) and this implies $\lambda_6 = \lambda_7 = m_{12}^2 = 0$. Then, in this case all coefficients of the potential are real and it describes the theory without CP violation in the Higgs sector. However, if it contains some complex coefficients, these make CP violation in Higgs sector possible. For a detail discussion, see [33].

Another important illustration of 2HDM is the minimal supersymmetric model. This is based on a specific version of 2HDM with $\lambda_1 = \lambda_2 = -2\lambda_3$ and $\lambda_5 = \lambda_6 = \lambda_7 = 0$, to break the electroweak symmetry.

There also exists, as we have mentioned above, a specific 2HDM model which could explain dark matter, the inert model.

To show the new phenomenologies of extended scalar models, we are going to analyze a very simple case with two complex fields. We will see first of all that the analysis of this simple case leads to rather cumbersome calculations, such as the minimization of the potential, or the computation of mass matrices, but also that this model already displays interesting and novel characteristics.

2.5.1 A simple case of 2HDM

For simplicity, let ϕ_1 and ϕ_2 denote two scalar Higgs fields (and not doublets) and let us consider a case where $\lambda_1 = \lambda_2$, and $\lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = m_{12}^2 = 0$, so that

$$\begin{aligned} V_H &= -\frac{1}{2}[m_{11}^2(\phi_1^+\phi_1) + m_{22}^2(\phi_2^+\phi_2)] \\ &\quad +\frac{\lambda}{2}(\phi_1^+\phi_1)^2 + \frac{\lambda}{2}(\phi_2^+\phi_2)^2 + \frac{\lambda_3}{2}(\phi_1^+\phi_1)(\phi_2^+\phi_2). \end{aligned} \tag{2.7}$$

We choose, among different forms of the lagrangian describing the same physical reality, a specific one in which the vacuum expectation values of both Higgs fields are real. This will simplify the analysis of this problem. This lagrangian, obtained by an appropriate $U(1)_Y$ transformation, is such that the scalar field vacuum expectation values are of the form :

$$\langle\phi_1\rangle = \frac{1}{\sqrt{2}}v_1\,, \quad \langle\phi_2\rangle = \frac{1}{\sqrt{2}}v_2$$

with v_1 and v_2 real and positive.

Positivity conditions

In order to have a stable vacuum, the potential has to be bounded from below. Let us derive the constraints on the parameters λ_i such that it is satisfied. It is sufficient to look at the quartic terms :

$$\frac{\lambda}{2}(\phi_1^+\phi_1)^2+\frac{\lambda}{2}(\phi_2^+\phi_2)^2+\frac{\lambda_3}{2}(\phi_1^+\phi_1)(\phi_2^+\phi_2) = \frac{\lambda}{2}((\phi_1^+\phi_1)-(\phi_2^+\phi_2))^2+(\lambda+\lambda_3)((\phi_1^+\phi_1)(\phi_2^+\phi_2)).$$

The positivity conditions are :

$$\lambda > 0\,, \quad \lambda+\lambda_3 > 0.$$

Parameters of the potential are also constrained by demanding that the extremum is a minimum. These constraints will be derived later.

Extrema of the Higgs potential

Ground states are described by the constant fields that minimize the potential. So, to find the excitation spectrum of the system, we need the extrema of the potential. The conditions for extrema of the scalar potential

$$\left(\frac{\partial V}{\partial \phi_1}\right)_{\phi_1=\langle\phi_1\rangle,\phi_2=\langle\phi_2\rangle} = 0\,, \quad \left(\frac{\partial V}{\partial \phi_2}\right)_{\phi_1=\langle\phi_1\rangle,\phi_2=\langle\phi_2\rangle} = 0 \qquad (2.8)$$

define the vacuum expectation values $\langle\phi_1\rangle\,,\langle\phi_2\rangle$.

In our case, the conditions are

$$\frac{v_1}{2}(-m_{11}^2+\lambda v_1^2+\lambda_3 v_2^2) = 0\,, \quad \frac{v_2}{2}(-m_{22}^2+\lambda v_2^2+\lambda_3 v_1^2) = 0\,. \qquad (2.9)$$

Therefore, there are four possible phases :

- the phase A with $v_1 = 0$ and $v_2 = 0$,
- the phase B_1 with $v_1 \neq 0$ and $v_2 = 0$,
- the phase B_2 with $v_1 = 0$ and $v_2 \neq 0$,
- the phase C with $v_1 \neq 0$ and $v_1 \neq 0$,

These phases lead to different symmetries. The conditions to have a minimum for the four phases are derived in the appendix.

The phase A corresponds to the case without symmetry breaking since the potential has one global extremum at the origin that is a minimum if the Higgs parameters obey conditions spelled out in the appendix.

Electroweak symmetry breaking arises if the minimum of the potential occurs for nonzero expectation values of the scalars fields. This happen in the phases B_1 , B_2 and C.

The phases B_1 and B_2 correspond to the case where the potential acquires two extrema in one direction in the (ϕ_1, ϕ_2) plane that are symmetric compared to the origin and have the same depth. In the phase B_1 (B_2), the potential has two global minima in the ϕ_1 (ϕ_2) direction, one for $\phi_1 = \frac{v_1}{\sqrt{2}}$ ($\phi_2 = \frac{v_2}{\sqrt{2}}$) and one for $\phi_1 = -\frac{v_1}{\sqrt{2}}$ ($\phi_2 = -\frac{v_2}{\sqrt{2}}$) as the potential is a polynomial function of $(\phi_i^* \phi_i)$, $i = 1, 2$. Then, we have to choose among these two minima one of them, that breaks the symmetry, and develop the theory in the vicinity of this minimum of the potential. In this case, only one of the scalar fields will couple to fermions as in the inert model.

The phase C happens when the potential acquires four minima, the first two in the ϕ_1 direction corresponding to $\phi_1 = \frac{v_1}{\sqrt{2}}$ and $\phi_1 = -\frac{v_1}{\sqrt{2}}$ that are symmetric relative to the origin. And the second two in the ϕ_2 direction corresponding to $\phi_2 = \frac{v_2}{\sqrt{2}}$ and $\phi_2 = -\frac{v_2}{\sqrt{2}}$ that are symmetric relative to the origin too. In this phase, the two Higgs fields can couple to fermions in different ways.

Therefore, this model is governed by the choice of the Higgs potential and the Yukawa couplings of the two scalar fields to the three generations of fermions.

Phase diagrams

We have three phase diagrams in function of the values of the parameters λ and λ_3. We observe phase transitions (change of minima) upon continuous change of parameters m_{ij}^2.

1. $\lambda > \lambda_3 > 0$

Figure 2.1: The phase diagram for $\lambda > \lambda_3 > 0$

In this case, depending on the values of the parameters m_{ij}^2, the four different phases are possible. Upon crossing the line separated this phases, phase transitions take place.

When the two mass parameters m_{11}^2 and m_{22}^2 are both negative, the potential has one global minimum at the origin and so it is stable (phase A). When m_{11}^2 or m_{22}^2 become zero, phase transition takes place :

- If m_{11}^2 becomes zero and then positive, the potential transits to the phase B_2. Then, if m_{22}^2 becomes also positive, when $m_{22}^2 = \frac{\lambda}{\lambda_3} m_{11}^2$, a new transition takes place; the potential is then in the phase C.
- If m_{22}^2 becomes zero and then positive, the potential transits to the phase B_1. Then, if m_{11}^2 becomes also positive, when $m_{22}^2 = \frac{\lambda_3}{\lambda} m_{11}^2$, a new transition takes place; the potential is then in the phase C.

2. $\lambda_3 > \lambda > 0$

Figure 2.2: The phase diagram for $\lambda_3 > \lambda > 0$

With this choice of values for the parameters λ and λ_3, we can see that the phase C is not possible and that the phases B_1 and B_2 coexist between $m_{22}^2 = \frac{\lambda_3}{\lambda} m_{11}^2$ and $m_{22}^2 = \frac{\lambda_3}{\lambda} m_{11}^2$. In this zone, where the parameters m_{11}^2 and m_{22}^2 are positive and such that $\frac{\lambda}{\lambda_3} m_{11}^2 < m_{22}^2 < \frac{\lambda_3}{\lambda} m_{11}^2$, above the bisectrix $m_{11}^2 = m_{22}^2$, the minima of the phase B_2 are the global ones while those of the phase B_1 are local, whereas below it, the minima of the phase B_1 are global. Moreover, as in the previous case, when the two mass parameters m_{11}^2 and m_{22}^2 are both negative, the potential is in the phase A, and transition to the phase B_1 or B_2 takes place when one of these two parameters becomes zero.

3. $0 > \lambda_3 > -\lambda$

Figure 2.3: The phase diagram for $0 > \lambda_3 > -\lambda$

The approach here is similar to the one used to analyze the first phase diagram (Figure 2.1). Depending on the values of the parameters m_{11}^2 and m_{22}^2, the four phases, A, C, B_1, B_2, are possible.

In this very simple case, we note that interesting phenomena happen : depending on the choice of parameters, different phases and different Yukawa couplings for the two fields are possible.

Chapter 3

The most general two Higgs doublet model

3.1 Motivations

In this Chapter we study in detail the 2HDM with the most general form of the potential. Before we come to the calculations, we first give several motivations.

First, there are a lot of possible models containing two Higgs doublets with different choices of the parameters in the potential. It has been observed that sometimes they lead to different phenomenologies, and sometime to very similar ones. This leads to the conjecture that there exist some classes of models with different parametrizations but similar physics. So, a general question arises: what are the relations among various particular realizations of 2HDM?

Another observation is that most of the specific models studied in literature had quite simple potentials, which can be studied with usual algebra. This simplicity is always the result of some additional symmetry in 2HDM. Usually, one studies Z_2 symmetry, $U(1)$ symmetry or CP-symmetry of the Higgs potential. It has been noted that such symmetries can be implemented in seemingly different ways, but still leading to similar physics. So, it is desirable to establish a complete list of possible symmetries and their phenomenological consequences. This was the main theme of several recent works on so-called generalized CP-transformations [34, 35, 36]. This list will be useful for building further models with predefined symmetries.

It is also interesting to see what happens if these symmetries are broken [37], because restrictions coming from experimental data can also accommodate this situation.

In addition, the general 2HDM can be useful for the Minimal Supersymmetric standard model, MSSM. At tree level this model uses a very simple variant of 2HDM, but the Higgs potential can receive loop corrections from new supersymmetric particles. These loop corrections introduce effective couplings among the Higgs fields, which were absent at tree level. Therefore, as stated in [38], one can describe the Higgs-sector of the (broken) MSSM by an effective field theory consisting of the most general two-Higgs-doublet model.

All this shows that it is very useful to study the most general Higgs potential of 2HDM without imposing any special relation among the parameters. So, we want to describe the whole list of physical possibilities, which are offered by the introduction of the second doublet.

3.2 Difficulties with a straightforward approach to the most general 2HDM

Let us start by introducing the following notation:

$$\phi_a = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \tag{3.1}$$

Note that here both ϕ_1 and ϕ_2 are electroweak doublets themselves, so that although $a = 1, 2$, ϕ_a effectively incorporates 4 complex fields. So, one should remember that in the definition of ϕ_a there is a hidden electroweak index. Let us write the general Higgs potential in the following form:

$$V_H = Y_{ab}(\phi_a^\dagger \phi_b) + \frac{1}{2} Z_{abcd}(\phi_a^\dagger \phi_b)(\phi_c^\dagger \phi_d) . \tag{3.2}$$

where Y_{ab} and Z_{abcd}, constructed from the Higgs potential parameters, represent mass terms and quartic couplings respectively and the indices run from 1 to 2. The basic objects used here $(\phi_a^\dagger \phi_b)$ are electroweak scalar products of doublets ϕ_a and ϕ_b so that $(\phi_a^\dagger \phi_b)$ is a 2-by-2 matrix and has no electroweak indices. The number of free parameters in tensors is restricted by the symmetry

$$Z_{abcd} = Z_{cdab} . \tag{3.3}$$

Hermiticity of V also implies

$$Y_{ab} = (Y_{ba})^*, \quad Z_{abcd} = (Z_{badc})^* . \tag{3.4}$$

These relations reduce the number of parameters: Y_{ab} has 4 free parameters and Z_{abcd} has 10. In total, as we have seen in the previous chapter, the potential contains

14 free parameters. These can match the standard 2HDM notation given in (2.6), see [38].

The first step towards phenomenology is to find the minimum of the potential. For example, following [38], we assume that the vacuum respects the electromagnetic gauge symmetry. That is, the expectation value of ϕ_1 and ϕ_2 are assumed to be aligned in $SU(2)_L$ space, and we follow the standard convention, after using the appropriate $SU(2)_L$ transformation, only neutral components of the doublets acquire non-zero expectation values:

$$\langle\phi_1^0\rangle = \frac{v}{\sqrt{2}}\hat{v_1} \quad \langle\phi_2^0\rangle = \frac{v}{\sqrt{2}}\hat{v_2} \tag{3.5}$$

where $\hat{v_a} = (\hat{v_1}, \hat{v_2})$ is a vector of unit norm in the space of doublets. Taking the derivative of eq.(3.2) with respect to ϕ_b and setting $\langle\phi_a^0\rangle = \frac{v}{\sqrt{2}}\hat{v_a}$, we find the covariant form for the scalar potential extremum conditions:

$$\hat{v_a}^*(Y_{ab} + \frac{1}{2}v^2 Z_{abcd}\hat{v_c}^*\hat{v_b}) = 0\,. \tag{3.6}$$

Unfortunately, these coupled equations cannot be solved explicitly. That is, we cannot find $\langle\phi_a\rangle$ is terms of Y_{ab} and Z_{abcd}. It means that we can do nothing in the general 2HDM with straightforward algebra.

One can try, of course, to analyze the general 2HDM numerically. However, with 14 free parameters it becomes a very complicated task. There are indeed some papers where this approach is used to study the symmetries of the vacuum [39] or the masses of the Higgs bosons [40], but the results are very inconclusive.

Another method has been suggested in [11, 41], that allows one to analyze many characteristics of the most general 2HDM without the need to compute the exact position of the global minimum of the potential. In this formalism, we first establish the structure behind 2HDM and then, we reformulate the problem of minimization in geometric terms. This method cannot, of course, give the explicit solutions to the minimization problem, but using it one can obtain some information about the number of minima of the potential, symmetries of the minima, prove several coexistence theorems, and give a full description of the phase diagram of the model. We will explain the essence of this method in this chapter and we will use it in the next chapter to study the masses of the general 2HDM.

3.3 Reparametrization symmetry

The starting point of this method is the reparametrization symmetry of the 2HDM potential.

In the most general two-Higgs-doublet model (2HDM), the Higgs potential depends on the Higgs fields, ϕ_1 and ϕ_2, and on coupling constants λ_i, m_{ij}^2: $V(\lambda_i, m_{ij}^2, \phi_\alpha)$. The physical observables, such as vacuum expectation values and masses of the physical bosons, are functions of these coupling constants but not of the fields themselves.

Consider now the same Higgs potential but with another set of parameters: $V(\tilde{\lambda}_i, \tilde{m}_{ij}^2, \phi_\alpha)$. If we perform a linear transformation of the Higgs fields

$$\phi_\alpha \longrightarrow \tilde{\phi}_\alpha = R_{\alpha\beta}\phi_\beta \,, \tag{3.7}$$

then the potential will keep its generic form, but with redefined parameters. Sometimes it is possible to find such transformation of the fields, which brings the potential back to the form with original coefficients $V(\lambda_i, m_{ij}^2, \tilde{\phi}_\alpha)$. When it is the case, the physics encoded in $V(\lambda_i, m_{ij}^2, \phi_\alpha)$ and in $V(\tilde{\lambda}_i, \tilde{m}_{ij}^2, \phi_\alpha) = V(\lambda_i, m_{ij}^2, \tilde{\phi}_\alpha)$ is the same, because it does not depend on the fields. Therefore, by construction, the potential with coefficients λ_i, m_{ij}^2 and that with coefficients $\tilde{\lambda}_i$, $\tilde{m}_{ij}^2$ describe the same physical reality. In this case, we say that the theory has reparametrization invariance. It is easy to prove that such transformations form a group, which we call reparametrization group.

This is a very general property. We want to find its explicit realization in 2HDM. So, we first find the full reparametrization group, then find the best way to implement it, and then establish to which representations of this group the fields and the parameters correspond. The physical observables must correspond to the singlet representation of this group; so they must be obtained by a complete convolution of a given representation of parameters.

We start by establishing the reparametrization group. In the 2HDM we have two Higgs doublets with the same quantum numbers. Therefore, these two different Higgs doublets can also be viewed as two components of a single complex 2-vector Φ :

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \tag{3.8}$$

As we said, the Higgs potential keeps its generic form under the action of the group of invertible linear transformations of a complex-valued 2-vector Φ. This group contains all complex 2-by-2 matrices with non-zero determinant. This group is called the general linear group $GL(2, C)$. This is an 8-dimensional Lie group that can be written as

$$GL(2, C) = C^* \times SL(2, C) \,, \tag{3.9}$$

where C^* is the subgroup of multiplication of Φ by non-zero complex numbers. Multiplication of all the fields by the same real nonzero constant can be compensated by a rescaling of all the observables, whereas $SL(2, C)$ embraces all non-trivial transformations. This is the reparametrization group of our problem.

The fields are in the fundamental representation of this group, while the coefficients are in tensorial representations, as seen in (3.2). So, we have constructed, in principle, a reparametrization-covariant description of the potential, in which all the objects are some specific representations of the reparametrization group. We showed also that the physical observables must be reparametrization-invariant. So, we want to establish a reparametrization-covariant method of calculating these physical observables. Reparametrization-covariant methods are useful, because one can use the freedom to choose a convenient Higgs-field basis for the description of the Higgs potential, prove something in this basis, rewrite it in a covariant way, and then switch to another basis, if necessary.

However, before we do this, we introduce a much more transparent way to study the reparametrization properties of the potential, which we are now going to describe.

3.4 The orbit space of the 2HDM potential

The Higgs potential of the 2HDM is gauge invariant. It means that it remains absolutely the same if we make an $SU(2)_L \times U(1)_Y$ transformation inside each doublet. The electroweak gauge group has 4 generators. Therefore, if we start with a specific point in the 8-dimensional space of Higgs fields and apply the gauge transformations, we will get a 4-dimensional hypersurface. This surface is called a gauge orbit.

The Higgs potential is equal for any points inside any chosen orbit. Points within the same orbit are indistinguishable for the Higgs potential and orbits never intersect. Therefore, one can simplify the problem by introducing the space of gauge orbits and think that the Higgs potential is defined in this orbit space instead of the 8-dimensional space of Higgs fields. By counting dimensions we see that the orbit space must itself be 4-dimensional.

To specify the orbit space, let us note that the Higgs potential depends on the fields via $SU(2)_L \times U(1)_Y$ invariant scalar products $(\phi_1^\dagger \phi_1)$, $(\phi_2^\dagger \phi_2)$, $(\phi_1^\dagger \phi_2)$, $(\phi_2^\dagger \phi_1)$. Therefore, the values of all combinations $(\phi_i^\dagger \phi_j)$ can be used to define the orbit.

3.4.1 The orbit space

Let us introduce a four-vector

$$r^\mu = (r_0, r_i) = (\phi^\dagger \sigma^\mu \phi), \tag{3.10}$$

$\mu = 0, 1, 2, 3$ with

$$r_0 = (\Phi^\dagger \Phi) = (\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) \tag{3.11}$$

and

$$r_i = (\Phi^\dagger \sigma_i \Phi) = \begin{pmatrix} (\phi_2^\dagger \phi_1) + (\phi_1^\dagger \phi_2) \\ -i((\phi_1^\dagger \phi_2) - (\phi_2^\dagger \phi_1)) \\ (\phi_1^\dagger \phi_1) - (\phi_2^\dagger \phi_2) \end{pmatrix} = \begin{pmatrix} 2Re(\phi_1^\dagger \phi_2) \\ 2Im(\phi_1^\dagger \phi_2) \\ |\phi_1|^2 - |\phi_2|^2 \end{pmatrix} \tag{3.12}$$

where σ_i are the Pauli matrices. When Φ is transformed by a $GL(2, C)$ transformation, r_0 and r_i transform as a single 4-vector $r^\mu = (r_0, r_i)$. So, the scalars r_0 and r_i are parts of a single irreducible representation of $SO(3, 1)$. This four vector is gauge invariant and parametrizes the gauge orbits in the space of the Higgs fields.

Since potentials are invariant under the global phase rotations of both doublets $\phi_i \longrightarrow e^{-i\alpha_0}$ with common phase α_0, the same set of observables can be described by a class of lagrangians that differ from each other by independent phase rotations for each doublet, accompanied by compensating phase rotations of parameters of the lagrangian. This is a particular case of reparametrization invariance, a rephasing invariance. Therefore, if we multiply ϕ_1 and ϕ_2 by a common phase factor, this does not change r^μ, therefore, each r^μ parametrizes a $U(1)$ orbit in the 8-dimensional space of fields. The potential being also $U(1)$-invariant can be defined in this $1+3$-dimensional orbit space. The SL(2,C) group of transformation of Φ induces the proper Lorentz group $SO(3, 1)$ of transformation of r^μ.

This group includes 3D rotations of the vectors r_i induced by $SU(2)$ and also boosts along the three axes that mix r_0 and r_i. Thus, the orbit space composed by all possible vectors r^μ, is equipped with the Minkowski space structure with metric $g_{\mu\nu} = diag(1, -1, -1, -1)$ that relates covariant and contravariant vectors.

In fact, the orbit space in 2HDM is not the entire Minkowski space. The square of the four-vector r^μ is invariant under any proper Lorentz transformation and since ϕ_1 and ϕ_2 are complex, we can show that it is non-negative:

$$r^2 \equiv r^\mu r_\mu = r_0^2 - r_i^2 = 4[(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) - (\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1)] \tag{3.13}$$

If we express

$$\phi_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \tag{3.14}$$

and if we use angle notation to represent a complex number with amplitude and phase

$$a_1 = |a_1|e^{i\phi_1},\ b_1 = |b_1|e^{i\theta_1},\ a_2 = |a_2|e^{i\phi_2},\ b_2 = |b_2|e^{i\theta_2} \tag{3.15}$$

we find

$$\begin{aligned}(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - (\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) &= (|a_1||b_2| - |b_1||a_2|)^2 \\ &\quad +2|a_1||b_1||a_2||b_2|[1 - \cos(\theta_1 - \theta_2 - \phi_1 + \phi_2)]\,.\end{aligned} \tag{3.16}$$

This means that $r^\mu r_\mu \geq 0$. We can notice that $r^\mu r_\mu = 0$ if

$$\frac{|a_2|}{|a_1|} = \frac{|b_2|}{|b_1|} = C, \quad \theta_1 - \theta_2 = \phi_1 - \phi_2 = \alpha \tag{3.17}$$

In this case,

$$\phi_1 = \begin{pmatrix} |a_1|e^{i\phi_1} \\ |b_1|e^{i\theta_1} \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} C|a_1|e^{i(\phi_1-\alpha)} \\ C'|b_1|e^{i(\theta_1-\alpha)} \end{pmatrix} = Ce^{-i\alpha}\begin{pmatrix} |a_1|e^{i\phi_1} \\ |b_1|e^{i\theta_1} \end{pmatrix}. \tag{3.18}$$

So, the surface of the future light-cone, $r^\mu r_\mu = 0$, corresponds to the situation when the two Higgs fields ϕ_1 and ϕ_2 are proportional to each other.

Therefore, since $r^0 > 0$ and the value of r^μ are not restricted from above, the orbit space lies inside and on the border of the future light-cone (LC^+) in the Minkowski space. And the reparametrization group in the orbit space, $SO(3,1)$, as expected, leaves the orbit space invariant.

Now, the Higgs potential can be rewritten in the orbit space as :

$$V_H = -B_\mu r^\mu + \frac{1}{2}\Lambda_{\mu\nu}r^\mu r^\nu \tag{3.19}$$

where

$$B^\mu = \frac{1}{4}(m_{11}^2 + m_{22}^2, -2Re\, m_{12}^2, 2Im\, m_{12}^2, -m_{11}^2 + m_{22}^2), \tag{3.20}$$

$$\Lambda_{\mu\nu} = \frac{1}{2}\begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} + \lambda_3 & -Re(\lambda_6+\lambda_7) & Im(\lambda_6+\lambda_7) & -\frac{\lambda_1-\lambda_2}{2} \\ -Re(\lambda_6+\lambda_7) & \lambda_4 + Re\lambda_5 & -Im\lambda_5 & Re(\lambda_6-\lambda_7) \\ Im(\lambda_6+\lambda_7) & -Im\lambda_5 & \lambda_4 - Re\lambda_5 & -Im(\lambda_6-\lambda_7) \\ -\frac{\lambda_1-\lambda_2}{2} & Re(\lambda_6-\lambda_7) & -Im(\lambda_6-\lambda_7) & \frac{\lambda_1+\lambda_2}{2} - \lambda_3 \end{pmatrix} \tag{3.21}$$

We obtained a more compact reparametrization-covariant description of the potential. The reparametrization group is now $SO(1,3)$, the two fields are grouped in the fundamental representation of this group, and the coefficients are grouped into fundamental and rank-2 tensorial representations.

3.4.2 Reparametrization and kinetic term

The canonically normalized gauge covariant kinetic energy terms of the scalar fields are invariant under arbitrary global $U(2)$ transformations in the complex two-dimensional space spanned by the two fields. Thus, we are free to redefine our two scalar fields by making an arbitrary $U(2)$ transformation.

However, transformations from the reparametrization group $SL(2, C)$ modify the Higgs kinetic term. Therefore, in order to restore a canonical form of the kinetic term a field renormalization is needed in addition to the transformation. The kinetic term can be written in the reparametrization covariant form

$$K = K_\mu \rho^\mu \tag{3.22}$$

with

$$\rho^\mu = (\partial_\alpha \Phi)^\dagger \sigma^\mu (\partial^\alpha \Phi) \tag{3.23}$$

where α indicates the true space-time coordinates and μ the coordinate in the Higgs orbit space. The reparametrization transformation laws of ρ^μ are the same as for r^μ. In a generic frame, $K_\mu = (1, 0, 0, 0)$. Upon $SL(2, C)$ transformations of the Higgs fields, spacelike components of K_μ become non-zero still keeping $K_\mu K^\mu = 1$ because Lorentz transformations from $SO(3, 1)$ group preserve the quadratic form. So, the full Higgs lagrangian includes also the kinetic term, which can be off-diagonal in a general case (that is, doublets ϕ_1 and ϕ_2 can mix).

Note that there is a very general physical argument that says that in any basis the four-vector K^μ always lies inside the future light-cone. In fact, the requirement that the energy density must be positive, implies that K_μ lies inside the future lightcone: $K_0 > 0$, $K^\mu K_\mu > 0$. This condition remains true under an arbitrary $SO(1, 3)$ transformation.

The kinetic term contains the terms with covariant derivatives and through these, the scalars have couplings to the gauge bosons. So, these coupling are also not reparametrization invariant, but should be rather described in a reparametrization-covariant way. We will not discuss these couplings here.

The quantities K^μ, B^μ transform as four-vectors, and $\Lambda_{\mu\nu}$ transforms as a four-tensor. These three objects give a complete description of the Higgs lagrangian. Therefore, the physical observables must appear as complete convolutions of K^μ, B^μ and $\Lambda_{\mu\nu}$.

We do not need the four-vector K_μ in the search for the minimum of the potential. The structure of this term does not affect the position of the stationary points. However it affects the mass matrix at this minimum.

3.4.3 Positivity conditions

The tensor $\Lambda_{\mu\nu}$ has an important property, which originates from the positivity condition.

In order to have a stable vacuum, the potential has to be bounded from below in the entire ϕ_i-space. This means that the potential must be positive at large values of fields for any direction in the $(\phi_1\phi_2)$ plane. Therefore, the potential is stable if its quartic part V_4 increases in all directions in the entire ϕ_i-space. In other words, the positivity condition leads to inequalities between the coefficients λ.

Such conditions were found explicitly only for simple potentials, for example for potentials with $\lambda_6 = \lambda_7 = 0$ [42]. For the most general case, nobody has found these explicit inequalities, because the most general case is algebraically very complicated.

We are working in the orbit space, so that these conditions are imposed on $\Lambda_{\mu\nu}$. In the orbit space, V_4 is positive definite if $\Lambda_{\mu\nu}$ is positive definite on and inside the future light-cone, that is, $\Lambda_{\mu\nu}r^\mu r^\nu > 0$ on and inside the LC^+. As was proved in [43], this is equivalent to the statement that $\Lambda_{\mu\nu}$ is diagonalizable by an $SO(3,1)$ transformation and that after diagonalization it has form

$$\Lambda_{\mu\nu} = \begin{pmatrix} \Lambda_0 & 0 & 0 & 0 \\ 0 & -\Lambda_1 & 0 & 0 \\ 0 & 0 & -\Lambda_2 & 0 \\ 0 & 0 & 0 & -\Lambda_3 \end{pmatrix} \tag{3.24}$$

with $\Lambda_0 > \Lambda_1, \Lambda_2, \Lambda_3$. It is obvious that if $\Lambda_{\mu\nu}$ satisfies this condition, the positive definiteness is assured. It has been proved in [43] that this condition follows from the positive definiteness of $\Lambda_{\mu\nu}$. Finding the eigenvalues explicitly in terms of λ_i requires to solve the characteristic equation of the fourth order, which constitutes one of the computational difficulties of the straightforward algebra. We will never need these explicit expressions, because we will always compute other properties using only the eigenvalues.

3.5 Vacua in 2HDM

We are interested, in principle, in dynamics of the 2HDM. But first we would like to understand the vacuum structure of this model. As we already mentioned, a problem already arises in this first step : the minimization of the 2HDM potential leads to a system of coupled equations that cannot be solved. We are going to use the formalism developed in the previous section and rewrite this minimization problem in the orbit space as in [11]. So, in this section, we are interested in the vacuum expectation value of the four-vector r^μ, $\langle r^\mu \rangle$. In the next chapter, we will study the first step of the dynamic of the most general 2HDM by computing a new formalism to get masses of Higgs bosons.

Indeed, the problem of minimization of some group-invariant potential is simplified if one switches from the space of Higgs fields to the orbit space. Thanks to this formalism, we can learn about the ground state of the 2HDM without finding its location explicitly. We will also note that different kinds of vacuum states with various physical properties are possible depending on interrelation among the parameters of the potential, [44].

3.5.1 Extrema of the 2HDM

The extrema of the potential define the vacuum expectation values $\langle \phi_i \rangle$, $i = 1, 2$, of the fields ϕ_i via :

$$\left(\frac{\partial V}{\partial \phi_i}\right)_{\phi_i = \langle \phi_i \rangle} = 0\,, \quad \left(\frac{\partial V}{\partial \phi_i^\dagger}\right)_{\phi_i = \langle \phi_i \rangle} = 0\,. \tag{3.25}$$

Depending on the parameters, these equations could have the electroweak symmetry conserving solution $\langle \phi_i \rangle = 0$ or they could have several electroweak symmetry breaking solutions.

If we use the phase freedom of the lagrangian and if we choose appropriately the 3 axis in the isospin space, the most general electroweak symmetry breaking vacuum expectation can be written in the form :

$$\langle \phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle \phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ v_2 e^{i\xi} \end{pmatrix} \tag{3.26}$$

where v_1 and v_2 are real.

The two Higgs doublets have an hypercharge one and the upper component of these doublets corresponds to a 3-component of the isospin $I_3 = \frac{1}{2}$ while the down-component to $I_3 = \frac{1}{2}$. Using the Gell-Mann–Nishijima formula that relates isospin and electric charge, we deduce that the $u \neq 0$ solution corresponds to a **charged vacuum** while the $u = 0$ solution, which corresponds to the situation where the two Higgs fields are proportional, is a **neutral vacuum** since it remains invariant under $U(1)_{QED}$ transformations.

Let us express the minimization problem in the orbit space. The latter is a region with border in the Minkowski space, so several possibilities must be considered when searching for the stationary point of the potential (it would better to say stationary orbit).

The first possibility is when $\langle r^\mu \rangle = 0$. This extremum is located at the origin of the future light-cone LC^+. This corresponds to the point $\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$. This is thus the electroweak symmetry conserving extremum point. The breaking vacuum is located on the surface ($r^\mu r_\mu = 0$) or inside ($r^\mu r_\mu \neq 0$) the future light cone LC^+. In this case, it has been proved in [11] that the 2HDM potential bounded from below cannot have nontrivial maxima, so all nontrivial extrema are either minima or saddle points. When it lies inside LC^+, a stationary orbit in the Higgs field space corresponds also to a stationary point in the orbit space. This is the charged breaking vacuum. When it lies on the surface, a stationary orbit doesn't necessarily correspond to a stationary point since now there are conditions on r^μ because it must lie on the surface of the future lightcone. As we mentioned just above, the neutral minimum arises when the two Higgs fields are proportional. In the orbit space, this condition is satisfied when $\langle r^\mu \rangle \langle r_\mu \rangle = 0$.

Therefore, the minimum of (3.19) corresponding to the vacuum expectation value $\langle r_\mu \rangle$ can be of the following three types:

- $\langle r^\mu \rangle = 0$ corresponds to $\langle \phi_1 \rangle = 0$ and $\langle \phi_2 \rangle = 0$. It is the electroweak conserving minimum.
- $\langle r^\mu \rangle \neq 0$ and $\langle r^\mu \rangle \langle r_\mu \rangle = 0$ corresponds to the neutral vacuum.
- $\langle r^\mu \rangle \neq 0$ and $\langle r^\mu \rangle \langle r_\mu \rangle > 0$. In this case one cannot set to zero the upper components in both doublets $\langle \phi_i \rangle$ simultaneously, and this corresponds to the charge-breaking minimum.

And these three possibilities correspond to the following parts of the orbit space: the origin, the surface, and the interior of the future lightcone.

3.5.2 Electroweak symmetry conserving minimum

The point $\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$ is the electroweak symmetry conserving extremum point. Its nature depends on m_{ij}^2:

- It is a minimum if $\det|m_{ij}^2| \geq 0$ and $m_{11}^2 < 0, m_{22}^2 < 0$.
- It is a maximum if $\det|m_{ij}^2| \geq 0$ and $m_{11}^2 > 0, m_{22}^2 > 0$.
- It is a saddle point if $\det|m_{ij}^2| < 0$.

No other extremum can be a maximum of the Higgs potential in the 2HDM, as is proved in [11].

3.5.3 Electroweak symmetry breaking minimum

The stationary orbit is composed of stationary points and the conditions for the stationary point of the Higgs potential are:

$$\frac{\partial V}{\partial \phi_i^\dagger} = \frac{\partial r^\mu}{\partial \phi_i^\dagger}\frac{\partial V}{\partial r^\mu} = d_i^\mu \xi_\mu = 0\,, \quad i = 1, 2\,, \tag{3.27}$$

where

$$d_i^\mu = \sigma_{ij}^\mu \phi_j\,, \quad \xi_\mu = -B_\mu + \Lambda_{\mu\nu} r^\nu\,, \tag{3.28}$$

with

$$\sigma^\mu = (\sigma^0, \sigma^i) = (I, \sigma^i)\,. \tag{3.29}$$

Since both real and imaginary parts have to be set to zero, this gives a system of four equations. We introduce the light-cone vectors

$$n_+^\mu = (1, 0, 0, 1)\,, \quad n_-^\mu = (1, 0, 0, -1)\,, \tag{3.30}$$

and the transverse unit vectors

$$e_1^\mu = (0, 1, 0, 0)\,, \quad e_2^\mu = (0, 0, 1, 0)\,. \tag{3.31}$$

Therefore,

$$d_1^\mu = \sigma_{11}^\mu \phi_1 + \sigma_{12}^\mu \phi_2 = (\phi_1, \phi_2, -i\phi_2, \phi_1) = n_+^\mu \phi_1 + (e_1^\mu - i e_2^\mu)\phi_2\,, \tag{3.32}$$

and

$$d_2^\mu = \sigma_{21}^\mu \phi_1 + \sigma_{22}^\mu \phi_2 = (\phi_2, \phi_1, i\phi_1, -\phi_2) = n_-^\mu \phi_2 + (e_1^\mu + i e_2^\mu)\phi_1\,. \tag{3.33}$$

Now, we can rewrite the equations for the minimization of the potential as

$$\phi_1 n_+^\mu \xi_\mu + \phi_2 (e_1^\mu - i e_2^\mu)\xi_\mu = 0\,, \tag{3.34}$$

$$\phi_2 n_-^\mu \xi_\mu + \phi_1 (e_1^\mu + i e_2^\mu)\xi_\mu = 0\,. \tag{3.35}$$

Charged-breaking extremum

In this extremum, the interaction of gauge bosons with fermions will not preserve the electric charge and the photon become massive. Certainly, this case is not realized in our world.

In this case, the minimum lies inside LC^+, the two doublets are nonzero and are not proportional to each other. Therefore, the equations for the minimization splits into a pair of conditions that correspond to vanishing coefficients in front of ϕ_1 and ϕ_2 separately. We obtain

$$\xi_\mu n_+^\mu = 0\,, \quad \xi_\mu n_-^\mu = 0\,, \tag{3.36}$$

$$\xi_\mu e_1^\mu = 0\,, \quad \xi_\mu e_2^\mu = 0\,, \tag{3.37}$$

from which we get $\xi^\mu = 0$. Therefore

$$\Lambda^{\mu\nu}\langle r_\nu\rangle = B^\mu\,. \tag{3.38}$$

This is an inhomogeneous system of linear equations.

- If all eigenvalues of $\Lambda_{\mu\nu}$ are nonzero, $\Lambda_{\mu\nu}$ is an invertible operator and a solution always exists and is unique. The solution is then

  $$\langle r^\mu\rangle = (\Lambda^{-1})^\mu_\nu B^\nu\,. \tag{3.39}$$

  However, the requirement that $\langle r^\mu\rangle$ must lie inside the future lightcone leads to conditions on B^μ: a physical solution exists if B^μ lies inside a specific cone with the apex at the origin. In this case, the theorem of non-coexistence of charge-breaking and neutral minima asserts that all neutral extrema are saddle points. Moreover, we can rewrite the potential as:

  $$\begin{aligned} V &= -B_\mu r^\mu + \frac{1}{2}\Lambda_{\mu\nu}r^\mu r^\nu = \frac{1}{2}\Lambda_{\mu\nu}(r^\mu - (\Lambda^{-1})^\mu_\nu B^\nu)(r^\nu - (\Lambda^{-1})^\nu_\mu B^\mu) + C \\ &= \frac{1}{2}\Lambda_{\mu\nu}(r^\mu - b^\mu)(r^\nu - b^\nu) + C \end{aligned} \tag{3.40}$$

  where C is a constant, $b^\mu = (\Lambda^{-1})^\mu_\nu B^\nu$ and we have shifted the minimum to zero. Therefore, in order for the potential to have a minimum, the condition

  $$\Lambda_{\mu\nu}(r^\mu - b^\mu)(r^\nu - b^\nu) > 0, \quad \forall(r^\nu - b^\nu)$$

  has to be satisfied. This means that the charge-breaking stationary point is a minimum if and only if $\Lambda_{\mu\nu}$ is definite positive in the entire Minkowski space, i.e. if all $\Lambda_i < 0$, $i = 1, 2, 3$.

- If at least one eigenvalues of $\Lambda_{\mu\nu}$ is zero, then $\Lambda_{\mu\nu}$ is not invertible. In this case, the specific cone in which B_μ has to lie in order to have a physical solution is of smaller dimension

Neutral extremum

This solution obeys the $U(1)$ symmetry of electromagnetism. The neutral vacuum solutions are located on the border of the orbit space $r^\mu r_\mu = 0$. We have mentioned that this situation happens when the two Higgs doublets ϕ_1 and ϕ_2 are proportional to each other and that a neutral stationary orbit of the potential in the space of Higgs fields does not necessarily correspond to a stationary point in the orbit space. This is because we limit the region where we search minima with a border that is the surface of the LC^+. We call, to simplify, a neutral stationary point, a point in the orbit space that corresponds to the neutral stationary orbit in the space of Higgs fields.

In order to get conditions for the extrema lying on the surface of the LC^+, one needs a lagrangian multiplier ξ

$$\frac{\partial}{\partial r^\mu}(V - \frac{\xi}{2} r^\mu r_\mu) = 0 \Rightarrow \Lambda^{\mu\nu}\langle r_\nu\rangle - \xi \cdot \langle r^\mu\rangle = B^\mu, \quad \langle r_\mu\rangle\langle r^\mu\rangle = 0\,. \tag{3.41}$$

We can simplify the analysis. We know that LC^+ is invariant under $SO(3,1)$ transformations. Therefore, we can perform a boost to align the timelike principal axis of $\Lambda^{\mu\nu}$ with the future line of LC^+ such as $\Lambda^{0i} = 0$. Moreover, we can take a point $r^\mu = \frac{1}{2}v^2(1, \vec{n})$ lying on LC^+ and perform a 3D rotation that makes $r^\mu = \frac{1}{2}v^2 n_+^\mu$, which corresponds to setting $\phi_2 = 0$. Therefore, as $\xi_\mu \neq 0$ and r^μ is proportional to n_+^μ, we get from the four equations (3.35)

$$\xi_\mu n_-^\mu \neq 0\,. \tag{3.42}$$

We can choose $\xi^\mu = \xi n_+^\mu$ Therefore the equation for the minimization becomes

$$\frac{1}{2}\Lambda^{\mu\nu} v^2 n_{+\nu} - \xi n_+^\mu = B^\mu\,. \tag{3.43}$$

From this equation, depending on the position of B^μ, different possibilities exist. This system can have up to six solutions, among which there are at most two local minima, while the other are saddle points.

This formalism, based on the reparametrization property, turns out to be a powerful tool in the analysis of the existence and number of extrema of the scalar potential and their classification according to whether the extremum is a minimum or a saddle point and of neutral or charge breaking nature. So, the key point is that we don't need to manipulate high-order algebraic equations in order to learn about the general structure of the 2HDM vacuum. In this approach geometric constructions appear naturally in the orbit space and allow to prove various theorems concerning the number, the coexistence and the nature of the extrema, and also to find conditions for which the symmetry is broken and establish the phase diagram of the scalar sector of 2HDM, see [11].

Chapter 4

Dynamics of the general 2HDM

When we study field theories with spontaneous symmetry breaking, we first find the vacuum state and then we describe excitations above this vacuum. The vacuum state is given by the minimum or minima of the Higgs potential. This was the subject of the previous chapter []. The next step is to analyze the dynamics of the system described by the 2HDM. It includes, in particular, the following tasks:

- finding the excitations of the scalar sector of 2HDM and calculating their mass spectrum;
- finding the interaction among these excitations;
- establishing the interaction of these scalar particles with fermions and gauge bosons.

In short, one needs to calculate the dynamics of 2HDM.

Let us focus here only on the scalar sector of the theory. The scalar lagrangian of the most general 2HDM, including a generic kinetic term, can be written in terms of the two complex doublets ϕ_1 and ϕ_2 of scalar fields:

$$\mathcal{L} = K - V_2 - V_4 = (\partial_\alpha \phi_i)^\dagger K_{ij} (\partial^\alpha \phi_j) - \phi_i^+ B_{ij} \phi_j + \frac{1}{2} Z_{ijkl} (\phi_i^\dagger \phi_j)(\phi_k^\dagger \phi_l). \qquad (4.1)$$

Here all indices run from 1 to 2. After spontaneous symmetry breaking these doublets acquire non-trivial vacuum expectation values, so that one can introduce the physical scalar bosons φ_i:

$$\phi_i = \langle \phi_i \rangle + \varphi_i \,.$$

If we know the vacuum expectation values, we could rewrite this lagrangian using the physical scalar fields:

$$\mathcal{L}(\phi_i, \partial_\alpha \phi_i) \to \mathcal{L}(\langle \phi_i \rangle + \varphi_i, \partial_\alpha \varphi_i) \,.$$

Then, using this lagrangian one can in principle calculate various n-point correlation functions of physical scalar bosons:

$$V_n(i_1, i_2, ..., i_n) = \langle 0|T\varphi_{i_1}(x_{i_1})...\varphi_{i_n}(x_{i_n})|0\rangle \,. \tag{4.2}$$

This notation is rather schematic because the n indices $i_1, ..., i_n$ not only give the number of doublets, but also specify the field. Besides, one can consider similar correlation functions with conjugate fields.

The most important among them is the two-point correlation function V_2, that is, the propagator. Three-point and four-point correlation function V_3 and V_4 are given, at the tree level, by the couplings of the potential, while even higher order correlation functions are given by some specific convolutions of these. With V_2, V_3, V_4, one can formulate Feynman rules of the model and one can, in principle, calculate any scattering process in the scalar sector.

These correlation functions, V_n, however, have the same important problem as the lagrangian itself: their general form is redundant. Indeed, each V_n contains elements which depend on the basis for representing the Higgs fields, so they do not reflect the physical content of the model, but just depend on the way we describe it. These elements, which are not basis-invariant, are unphysical. On the other hand, there are several basis-invariant combinations in each V_n, which are truly important for the physical content of the most general 2HDM. It is these invariant quantities that we want to find.

Unfortunately, it is very difficult to compute them working in the space of the scalar fields. We know from the previous chapter that this task is simplified if we switch from the space of Higgs fields to the orbit space. In this formalism, all basis-invariant quantities can be represented via full contractions of K_μ, $\Lambda_{\mu\nu}$ and B_μ and invariant tensors $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$. Therefore, a natural task is to find these basis-invariant quantities.

This is an outline of a long and complicated program. In this work, we will focus only on one particular task: calculating the basis-invariant elements in V_2. These elements are the **masses** of the physical Higgs bosons.

4.1 Basis-invariant calculation of masses in a free theory with two scalars

In order to illustrate the essence of this task, let us first consider a very simple model: a free scalar theory with two complex scalars.

4.1.1 Mass Matrix

Let us first illustrate an important fact: the mass matrix itself is a basis-dependent quantity.

Consider the lagrangian for two complex scalar fields ϕ_1 and ϕ_2, in matrix form:

$$\mathcal{L}_\phi = K - V = \sum_{i=1,2} \left[(\partial_\mu \phi_i)^\dagger (\partial^\mu \phi_i) - m_i^2 \phi_i^\dagger \phi_i \right] . \tag{4.3}$$

In this case we have no interaction between the fields ϕ_1 and ϕ_2, so that m_1^2 and m_2^2 represent the masses squared of these particles.

As we have seen in previous papers, transformations from the reparametrization group $GL(2, C)$ of the Higgs fields modify the Higgs kinetic term. The full Higgs lagrangian can therefore includes kinetic terms which can be off-diagonal in the general case. Therefore, we need to deal with non-diagonal kinetic terms and consider a lagrangian in the following general form:

$$\mathcal{L} = (\partial_\alpha \phi_i)^\dagger K_{ij} (\partial^\alpha \phi_j) - \phi_i^\dagger B_{ij} \phi_j . \tag{4.4}$$

where the potential $V = \phi_i^\dagger B_{ij} \phi_j$ has an extremum at $\langle \phi_i \rangle = 0$.

In this simple case, each index i in the n-point correlation function takes the values 1 or 2. For the two-point correlation function one gets the following form

$$V_2(i,j)(p) = \int \frac{d^4x}{(2\pi)^4} e^{ipx} \langle 0 | T \phi_i^\dagger(x) \phi_j(0) | 0 \rangle = i \left[K_{ij} p^2 - B_{ij} \right]^{-1} , \tag{4.5}$$

where K_{ij} and B_{ij} are the same as in the lagrangian.

This expression can be derived in the following way. We start with the path integral formulation of our model.

$$Z(J) \equiv \langle 0|0 \rangle_J = \int \mathcal{D}\phi e^{i \int d^4x [\mathcal{L} + J\phi]} \tag{4.6}$$

where $\mathcal{L}$ is the lagrangian (4.4) and $\mathcal{D}\phi \equiv \prod_i \mathcal{D}\phi_i$ is the functional measure in our case, and $J\phi = J_i^\dagger(x)\phi_i(x) + \phi_i^\dagger(x) J_i(x)$. Then we proceeding in the standard way, see for example [45], but we keep track of non-diagonal K_{ij} and B_{ij}. We get for the action

$$S = \int \frac{d^4p}{(2\pi)^4} \left[\tilde{\phi}_i^\dagger(p) (K_{ij} p^2 - B_{ij}) \tilde{\phi}_j(-p) + \tilde{J}_i^\dagger(p) \tilde{\phi}_i(-p) + \tilde{J}_i(-p) \tilde{\phi}_i^\dagger(p) \right] , \tag{4.7}$$

where tilde indicates Fourier transformed quantities. Now as usual we shift the fields:

$$\tilde{\xi}_i(p) = \tilde{\phi}_i(p) - (Kp^2 + B)^{-1}_{ij} \tilde{J}_j , \quad \tilde{\xi}_i^\dagger(p) = \tilde{\phi}_i^\dagger(p) - \tilde{J}_j^\dagger (Kp^2 + B)^{-1}_{ji} , \tag{4.8}$$

then use $\langle 0|0\rangle_{J=0} = 1$ and obtain

$$
\begin{aligned}
Z_J &= \exp\left[i \int \frac{d^4p}{(2\pi)^4} \tilde{J}_i^\dagger(p)(Kp^2 - B)^{-1}_{ij} \tilde{J}_j(-p) \right] &(4.9)\\
&= \exp\left[i \int d^4x d^4x' J_i^\dagger(x) \Delta_{ij}(x - x') J_j(x') \right] . &(4.10)
\end{aligned}
$$

We have defined the propagator

$$
\Delta_{ij}(x - x') = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} (Kp^2 - B)^{-1}_{ij} \,, \tag{4.11}
$$

which proves (4.5).

Note that the propagator is also the Green's function of the equation of motion, see Eq. (4.14) below:

$$
(K\Box + B)_{ij} \Delta_{jk}(x - x') = \delta_{ik} \delta^4(x - x'). \tag{4.12}
$$

This can be seen directly by plugging (4.11) into (4.12).

Now let us derive the mass matrix in two specific bases and show that they are different.

First, let us calculate the Lagrange equations from lagrangian (4.4):

$$
\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i)^\dagger} - \frac{\partial \mathcal{L}}{\partial \phi_i^\dagger} = 0 \,. \tag{4.13}
$$

We get the following equations of motion:

$$
K_{ij} \Box \phi_j + B_{ij} \phi_j = 0 \,. \tag{4.14}
$$

If we multiply this equation by K^{-1}, we get

$$
\Box \phi_j + (K^{-1}B)_{ij} \phi_j = 0 \,. \tag{4.15}
$$

Therefore, we get a set of Klein-Gordon equations for fields ϕ_i. Their *mass matrix*, in this basis, is

$$
M_{ij} = (K^{-1}B)_{ij} \,. \tag{4.16}
$$

Let us now calculate the mass matrix in another way, by performing a transformation of the fields which makes the kinetic term of the lagrangian (4.4) diagonal. So, we perform a transformation on the fields:

$$
\phi_i \longrightarrow \tilde{\phi}_i = T_{ii'} \phi_{i'} \,, \quad \phi_i^\dagger \longrightarrow \tilde{\phi}_i^\dagger = \phi_{i'}^\dagger T_{i'i}^\dagger \,, \tag{4.17}
$$

where T is some invertible 2-by-2 matrix, i.e. $T \in GL(2, C)$.

The lagrangian becomes:

$$\mathcal{L} = (\partial_\alpha \phi_{i'})^\dagger T^\dagger_{i'i} K_{ij} T_{jj'} (\partial^\alpha \phi_{j'}) - \phi^\dagger_{i'} T^\dagger_{i'i} B_{ij} T_{jj'} \phi_{j'} \, . \tag{4.18}$$

We choose T so that the kinetic term becomes diagonal, that is, $T^\dagger K T = I$. Therefore

$$K = (T^\dagger)^{-1}(T)^{-1} = (TT^\dagger)^{-1} \quad \rightarrow \quad TT^\dagger = K^{-1} \, . \tag{4.19}$$

Therefore, in this new basis, *the mass matrix* becomes:

$$M'_{ij} = (T^\dagger B T)_{ij} \, . \tag{4.20}$$

Since matrices T and B do not necessarily commute, one can see that

$$M_{ij} \neq M'_{ij} \, , \tag{4.21}$$

which confirms that the mass matrix is a basis dependent quantity. As a consequence, V_2 is also a basis-dependent quantity.

Nevertheless, we can easily prove that the masses of the physical scalar bosons, m_1 and m_2, being the eigenvalues of the mass matrices, are the same for these two bases. Indeed, these masses can be calculated from the traces of powers of the mass matrix. We have

$$Tr(T^\dagger B T) = Tr(TT^\dagger B) = Tr(K^{-1}B) \, . \tag{4.22}$$

Here we used (4.19) and the cyclic property of the trace. So,

$$m_1^2 + m_2^2 = Tr(M) = Tr(M') \, . \tag{4.23}$$

We can also calculate the trace of the square of the mass matrix:

$$Tr(T^\dagger B T T^\dagger B T) = Tr(TT^\dagger B TT^\dagger B) = Tr(K^{-1} B K^{-1} B) \, , \tag{4.24}$$

so that

$$m_1^4 + m_2^4 = Tr(M^2) = Tr(M^{'2}) \, . \tag{4.25}$$

The same equality is also valid for any power of the mass matrices. We note that what is basis independent is the eigenvalues of the mass matrix. We can also calculate the determinant of the mass matrix via the traces,

$$\begin{aligned} 2Det(M) &= [Tr(M)]^2 - Tr(M^2) = [Tr(K^{-1}B)]^2 - Tr(K^{-1}BK^{-1}B) \\ &= [Tr(M')]^2 - Tr(M^{'2}) \, . \end{aligned} \tag{4.26}$$

Therefore,

$$Det(M) = Det(M') = m_1^2 m_2^2 \, . \tag{4.27}$$

4.1.2 Masses in the four-vector formalism

Anticipating work with 2HDM, we should also show how the masses in this simple problem can be calculated in the four-vector formalism.

The lagrangian (4.4) can be written, in the orbit space in a compact form as

$$\mathcal{L} = K_\mu \rho^\mu - B_\mu r^\mu \,. \tag{4.28}$$

The minimum of the potential is at $\langle r^\mu \rangle = 0$, since $r^\mu = (\Phi^\dagger \sigma^\mu \Phi)$. The kinetic term written in the reparametrization covariant form is :

$$K = (\partial_\alpha \phi_i)^\dagger K_{ij} (\partial^\alpha \phi_j) = K_\mu \rho^\mu, \quad \rho^\mu = (\partial_\alpha \Phi)^\dagger \sigma^\mu (\partial^\alpha \Phi) \,. \tag{4.29}$$

In the usual basis, with the diagonal kinetic term, the kinetic four-vector $K_\mu = (1, 0, 0, 0)$. Upon an $GL(C, 2)$ transformation of the fields, which corresponds to an $SO(3, 1)$ transformation in the orbit space, spacelike components of K_μ become non-zero, boosts make K_μ a nontrivial vector. But, since Lorentz transformations conserve quadratic forms, K_μ always obeys $K_\mu K^\mu = 1$ and always lies inside the future light cone.

One can reformulate the theory using quantities defined in the orbit space. In particular, we have the following correspondence :

$$\begin{array}{ccccc} \text{degrees of freedom} & (\phi_i, \phi_j) & \longrightarrow & r_\mu \\ \text{reparametrization group} & SL(2, C) & \longrightarrow & SO(3, 1) \\ \text{parameters} & K_{ij}, B_{ij} & \longrightarrow & K_\mu, B_\mu \end{array} \tag{4.30}$$

What we want to calculate is the analogous table for the masses:

$$\begin{array}{cccc} m_1^2 + m_2^2 & : & Tr(K_{ij}^{-1} B_{jm}) & \longrightarrow & ? \\ m_1^4 + m_2^4 & : & Tr(K_{ij}^{-1} B_{jk} K_{kl}^{-1} B_{lm}) & \longrightarrow & ? \end{array} \tag{4.31}$$

The matrix K_{ij} and B_{ij} are 2-by-2 hermitian matrices, so we can express them in the covariant notation, using $\sigma_\mu = (\sigma_0, \sigma_a)$ with $\sigma_0 \equiv I$ and σ_a the Pauli matrices with $a = 1, 2, 3$, as

$$K_{ij} = K_\mu \sigma^\mu = K_0 \sigma_0 - K_a \sigma_a \,, \tag{4.32}$$

$$B_{ij} = B_\mu \sigma^\mu = B_0 \sigma_0 - B_a \sigma_a \tag{4.33}$$

where $K_\mu = (K_0, K_a)$ and $B_\mu = (B_0, B_a)$.

We will need the inverse matrix of K_{ij}. This inverse exists and can be written, using $\bar{\sigma_\mu} = (I, -\sigma_a)$, as

$$(K^{-1})_{ij} = \frac{(K^\mu \bar{\sigma}_\mu)_{ij}}{K^2} \,, \tag{4.34}$$

where $K^2 = K_\mu K^\mu = 1$. Let us derive this relation. We must have $K^{-1}_{ii'} K_{i'j} = I_{ij}$. Let us look at the quantity $K_\mu K_\nu \bar{\sigma^\mu} \sigma^\nu$. Since $K_\mu K_\nu$ is a symmetric Lorentz tensor, we need the symmetric part of $\bar{\sigma^\mu}\sigma^\nu$. Let us compute it

$$\begin{array}{c|ccc} \mu\nu & & & \\ \hline & & & \\ 00 & \bar{\sigma^0}\sigma^0 & = & I \\ 0i & \bar{\sigma^0}\sigma^i & = & -\sigma^i \\ i0 & \bar{\sigma^i}\sigma^0 & = & \sigma^i \\ ij & \bar{\sigma^i}\sigma^j & = & -\delta^{ij} I - i\epsilon^{ijk}\sigma^k \end{array} \tag{4.35}$$

Since, δ_{ij} is symmetric and ϵ^{ijk} is antisymmetric, the symmetric part of $\bar{\sigma^\mu}\sigma^\nu$ is

$$\frac{1}{2}(\bar{\sigma^\mu}\sigma^\nu + \bar{\sigma^\nu}\sigma^\mu) = g^{\mu\nu} \,. \tag{4.36}$$

We now contract $g^{\mu\nu}$ with $K_\mu K_\nu$ and we find that

$$\frac{K_\mu K_\nu \bar{\sigma^\mu}\sigma^\nu}{K_\mu K^\mu} = \frac{K_\mu K^\mu}{K^\mu K_\mu} = I, \tag{4.37}$$

which proves (4.34).

We are now ready to calculate the traces of the mass matrix in the four-vector formalism.

$$Tr(M) = Tr(K^{-1}B) = Tr\left(\frac{K^\mu \bar{\sigma}_\mu}{K^2} B^\nu \sigma_\nu\right) = \frac{K^\mu B^\nu}{K^2} Tr(\bar{\sigma}_\mu \sigma_\nu) \,. \tag{4.38}$$

Using $Tr(\bar{\sigma}_\mu\sigma_\nu) = 2g_{\mu\nu}$ and $K^2 = 1$, we finally get:

$$Tr(M) = m_1^2 + m_2^2 = 2K^\mu B_\mu = 2(KB) \,. \tag{4.39}$$

In order to calculate masses, one needs to derive $Tr(M^2)$.

$$\begin{aligned} Tr(M^2) &= Tr(K^{-1}BK^{-1}B) = Tr(K^\mu \bar{\sigma}_\mu B^\nu \sigma_\nu K^\tau \bar{\sigma}_\tau B^\delta \sigma_\delta) \\ &= K^\mu B^\nu K^\tau B^\delta Tr(\bar{\sigma}_\mu \sigma_\nu \bar{\sigma}_\tau \sigma_\delta) \,. \end{aligned} \tag{4.40}$$

Let us show that

$$Tr(\bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\tau \sigma^\delta) = 2(g^{\mu\nu}g^{\tau\delta} - g^{\mu\tau}g^{\nu\delta} + g^{\mu\delta}g^{\nu\tau} - i\epsilon^{\mu\nu\tau\delta}) \,. \tag{4.41}$$

First, we can express this trace in terms of its symmetric and antisymmetric parts. We have shown that the symmetric part of $\bar{\sigma}^\mu\sigma^\nu$ is $g^{\mu\nu}$, therefore

$$\bar{\sigma}^\mu \sigma^\nu = g^{\mu\nu} + \Pi^{\mu\nu} \tag{4.42}$$

with $\Pi^{\mu\nu} = -\Pi^{\nu\mu}$. Using the relation between the sigma matrices $\sigma^a\sigma^b = i\epsilon^{abc}\sigma^c + \delta^{ab}$, we get

$$\Pi^{0a} = \sigma^a, \quad \Pi^{ab} = -i\epsilon^{abc}\sigma^c \tag{4.43}$$

Therefore, since $Tr(\sigma^a) = 0$, we get

$$Tr(\bar{\sigma}_\mu\sigma_\nu\bar{\sigma}_\tau\sigma_\delta) = Tr(g^{\mu\nu}g^{\tau\delta}\hat{\mathrm{I}}) + Tr(\Pi^{\mu\nu}\Pi^{\tau\delta})\,. \tag{4.44}$$

Direct calculation shows that

$$Tr(\Pi^{\mu\nu}\Pi^{\tau\delta}) = 2(-g^{\mu\tau}g^{\nu\delta} + g^{\mu\delta}g^{\nu\tau} - i\epsilon^{\mu\nu\tau\delta}) \tag{4.45}$$

where we have used the fact that $\epsilon^{abc} = \epsilon^{0abc} = \epsilon^{0cab}$ and $g^{ab} = -\delta^{ab}$. So we have proved (1.41).

Eventually, one gets

$$Tr(M^2) = m_1^4 + m_2^4 = 2(KB)(KB) - K^2B^2\,, \tag{4.46}$$

where $(KB) \equiv K_\mu B^\mu$.

Now, we can calculate the determinant

$$Det(M) = m_1^2 m_2^2 = \frac{1}{2}\left[[Tr(M)]^2 - Tr(M^2)\right] = B^2\,. \tag{4.47}$$

Now, as we know the trace and the determinant of the mass matrix, we are able to calculate the masses themselves, m_1^2 and m_2^2. To this end, we need to solve the following system of equations

$$m_1^2 m_2^2 = B^2, \quad m_1^2 + m_2^2 = 2(KB)\,. \tag{4.48}$$

We get that

$$m_1^2,\ m_2^2 = (KB) \pm \sqrt{(KB)^2 - B^2}\,. \tag{4.49}$$

Let us make a simple analysis of these results.

First, the quantity $(KB)^2 - B^2$ is always positive because K_μ lies inside the future light-cone. Indeed, this quantity is Lorentz-invariant, we can calculate it in a particular frame, in which $K_\mu = (1, 0, 0, 0)$. In this frame, this quantity becomes $B_1^2 + B_2^2 + B_3^2 \equiv |\vec{B}|^2$, which is of course positive. Therefore, in this frame the masses take a simple form:

$$m_1^2,\ m_2^2 = B_0 \pm |\vec{B}|\,. \tag{4.50}$$

Now let us find conditions in terms of B^μ for which this is the minimum of the potential. An extremum is a minimum if all the eigenvalues of the mass matrix, the squares of the particles masses, are positive. Therefore, we have the minimum if $DetM > 0$ and $TrM > 0$. Looking at (4.50), we see that both m_i^2 are positive if $B_0 > 0$ and $B_0 > |\vec{B}|$. It means that B^μ lies inside the future light-cone. This is a basis-invariant statement.

The phase transition happens when one of the eigenvalues becomes zero, that is $DetM = 0$. So, the phase transition takes place if

$$B_0^2 = B_1^2 + B_2^2 + B_3^2 \,. \tag{4.51}$$

Therefore, the phase transition takes place on the surface of the forward light-cone.

4.1.3 Mass matrices for the interacting field theory

In an interacting theory, after spontaneous symmetry breaking, the mass terms can depend on $\varphi\varphi, \varphi^\dagger\varphi^\dagger, \varphi^\dagger\varphi, \varphi\varphi^\dagger$. So it is convenient to decompose ϕ into its real and imaginary parts:

$$\phi_a = (Re\phi, Im\phi),$$

where $a = 1, 2$.

Next, let us introduce two new quantities.

First, we define an effective matrix B_{ab} as

$$B_{ab}^{(eff)} = \frac{1}{2}\frac{\partial^2 V}{\partial\phi_a\partial\phi_b} \,. \tag{4.52}$$

This matrix is:

$$\begin{pmatrix} \frac{\partial^2 V}{\partial\varphi\partial\varphi} + \frac{\partial^2 V}{\partial\varphi^\dagger\partial\varphi^\dagger} + 2\frac{\partial^2 V}{\partial\varphi\partial\varphi^\dagger} & i\left(\frac{\partial^2 V}{\partial\varphi\partial\varphi} - \frac{\partial^2 V}{\partial\varphi^\dagger\partial\varphi^\dagger}\right) \\ i\left(\frac{\partial^2 V}{\partial\varphi\partial\varphi} - \frac{\partial^2 V}{\partial\varphi^\dagger\partial\varphi^\dagger}\right) & -\frac{\partial^2 V}{\partial\varphi\partial\varphi} - \frac{\partial^2 V}{\partial\varphi^\dagger\partial\varphi^\dagger} + 2\frac{\partial^2 V}{\partial\varphi\partial\varphi^\dagger} \end{pmatrix} .$$

This second derivative matrix should be taken at the minimum.

Indeed, if $\phi_{0,a}$ is a constant field that minimizes the Higgs potential and if we expand the potential around this minimum, we get :

$$V(\phi) = V(\phi_0) + (\phi-\phi_0)_a(\phi-\phi_0)_b\left(\frac{\partial^2 V}{\partial\phi_a\partial\phi_b}\right)_{\phi_0} + ... \tag{4.53}$$

The coefficient of the quadratic term is a symmetric matrix whose eigenvalues, if the kinetic term is diagonal, give the masses of the fields. They are always positive or null since ϕ_0 is a minimum. Then, as we have seen that a reprametrization of the Higgs fields modifies the kinetic term which may become non-diagonal, we also introduce an effective kinetic matrix K_{ab}:

$$K_{ab}^{(eff)} = \frac{\partial^2 L}{\partial(\partial_\alpha \phi_a)\partial(\partial_\alpha \phi_b)} \,. \tag{4.54}$$

Then we use the same formalism as before with the mass matrix

$$(M)_{ab} = (K_{ac}^{(eff)})^{-1} B_{cb}^{(eff)} \,. \tag{4.55}$$

4.2 Trace of the mass matrices in 2HDM

Let us apply this approach to the two Higgs doublet model. As we have seen, in the r^μ space, the lagrangian of the most general 2HDM can be written in a covariant way as

$$L_\phi = K_\mu \rho^\mu + B_\mu r^\mu - \frac{1}{2}\Lambda_{\mu\nu} r^\mu r^\nu \,. \tag{4.56}$$

As we have mentioned, we have three phases in 2HDM : electroweak symmetric, charge breaking, and neutral[1].

We are going to consider each of these vacua, but before doing this let us extend the formalism we have developed in the previous section to the case of two doublets of scalar fields.

4.2.1 Two complex doublets

We are going to use another notation and consider another vector ϕ_a with $a = 1, 2, 3, 4, 5, 6, 7, 8$ defined as

$$\phi_a = (Re\phi_{1,\uparrow}, Im\phi_{1,\uparrow}, Re\phi_{2,\uparrow}, Im\phi_{2,\uparrow}, Re\phi_{1,\downarrow}, Im\phi_{1,\downarrow}, Re\phi_{2,\downarrow}, Im\phi_{2,\downarrow}). \tag{4.57}$$

With this notation r^μ becomes

$$r^\mu = \Phi^\dagger \sigma_\mu \Phi = \phi_a \Sigma^\mu_{ab} \phi_b \,, \tag{4.58}$$

where we introduced a new set of real symmetric 8-by-8 Σ-matrices. Σ^0 is just the unit matrix, while explicit form of Σ^i can be immediately reconstructed from

[1]For each type of minimum, there are always four different masses.

the definitions (see Appendix B). Since the upper and lower components of the doublets are not mixed by the Higgs potential, matrices Σ^i have a block-diagonal form, composed of identical 4-by-4 matrices. Below, we will often deal with these 4-by-4 matrices, denoting them by the same letter Σ^i. Which set of matrices is being used, 4-by-4 or 8-by-8, should be clear from the context.

Here some properties of these Σ-matrices:

$$\begin{aligned} \Sigma^\mu \bar{\Sigma}^\nu + \bar{\Sigma}^\nu \Sigma^\mu &= 2g^{\mu\nu} \cdot I && (4.59) \\ \frac{1}{2}(\Sigma^\nu \bar{\Sigma}^\rho \Sigma^\mu + \Sigma^\mu \bar{\Sigma}^\rho \Sigma^\nu) &= g^{\nu\rho}\Sigma^\mu + g^{\mu\rho}\Sigma^\nu - g^{\mu\nu}\Sigma^\rho && (4.60) \\ & && (4.61) \end{aligned}$$

They also share with σ^i an important property:

$$\{\Sigma^i, \Sigma^j\} = 2\delta^{ij} \cdot I \,,$$

where brackets denote the anticommutator. In contrast to σ^i, the matrices Σ^i do not form a closed algebra, but they belong to a larger algebra (Σ^i,Π^i), described in the Appendix B.

With these Σ-matrices, we can use the same formalism as before.

4.2.2 The electroweak symmetric vacuum

In this phase the minimum of the potential is at $\langle r^\mu \rangle = 0$, which corresponds to $\langle \phi_i \rangle = 0$. The electroweak symmetry is not broken, so this is the EW-conserving minimum.

The four complex fields have only two different masses, because up and down components enter the expressions in the same way. We have the same results as before except for an extra factor 2:

$$Tr(M) = 2m_1^2 + 2m_2^2 = -4(KB)\,, \tag{4.62}$$

$$Tr(M^2) = 2m_1^4 + 2m_2^4 = 4[2(KB)(KB) - B^2]\,. \tag{4.63}$$

The relation between the determinant and the traces is now different:

$$Det(M) = m_1^4 m_2^4 = \left(\frac{1}{8}\{[Tr(M)]^2 - 2Tr(M^2)\}\right)^2 . \tag{4.64}$$

So,

$$Det(M) = (B^2)^2\,. \tag{4.65}$$

And the masses are

$$m_1^2,\ m_2^2 = -(KB) \pm \sqrt{(KB)^2 - B^2}\,. \tag{4.66}$$

The analysis of these results is similar to the one we did in the section. 4.1.2. In the same particular frame, in which $K_\mu = (1, 0, 0, 0)$, the masses take the simple form:

$$m_1^2,\ m_2^2 = -B_0 \pm |\vec{B}|\,. \tag{4.67}$$

These masses can be calculated in terms of $m_{11}^2, m_{22}^2, m_{12}^2$. Let us remind the reader of the expression of B^μ:

$$B^\mu = \frac{1}{4}(m_{11}^2 + m_{22}^2, -2Re\, m_{12}^2, 2Im\, m_{12}^2, -m_{11}^2 + m_{22}^2)\,. \tag{4.68}$$

So,

$$|\vec{B}| = \frac{1}{4}(4|m_{12}^2|^2 + m_{11}^4 + m_{22}^4 - 2m_{11}^2 m_{22}^2)^{\frac{1}{2}}\,, \quad B_0 = \frac{1}{4}(m_{11}^2 + m_{22}^2)\,.$$

Therefore the masses become:

$$m_1^2,\ m_2^2 = -\frac{1}{4}(m_{11}^2 + m_{22}^2) \pm \frac{1}{4}(4|m_{12}^2|^2 + m_{11}^4 + m_{22}^4 - 2m_{11}^2 m_{22}^2)^{\frac{1}{2}}. \tag{4.69}$$

Now let us find conditions in terms of B^μ, when this is the minimum of the potential. We have the minimum if $DetM > 0$ and $TrM > 0$. Looking at (4.67), we see that both m_i^2 are positive if $B_0 < 0$ and $|B_0| > |\vec{B}|$. This means that B_μ lies inside the past light cone, LC^-.

4.2.3 Charged vacuum

We know that for the charged vacuum, the minimum of the potential

$$V = -B_\mu r^\mu + \frac{1}{2}\Lambda_{\mu\nu} r^\mu r^\nu \tag{4.70}$$

is at

$$\langle r^\mu \rangle = (\Lambda^{-1})^\mu_\nu B^\nu \tag{4.71}$$

if $\Lambda^{\mu\nu}$ is not singular.

Now let us calculate the mass matrix. According to (4.1.3) , we first compute the second-derivative matrix of the potential:

$$\frac{\partial r^\mu}{\partial \phi_a} = 2\Sigma^\mu_{ab}\phi_b, \tag{4.72}$$

$$\frac{\partial V}{\partial \phi_a} = \frac{\partial r^\mu}{\partial \phi_a}\frac{\partial V}{\partial r^\mu} = \frac{\partial r^\mu}{\partial \phi_a} \cdot \xi_\mu \,. \tag{4.73}$$

where $\xi_\mu \equiv (-B_\mu + \Lambda_{\mu\nu} r^\nu)$. Therefore,

$$\left(\frac{\partial^2 V}{\partial \phi_a \partial \phi_b}\right)_{ch} = \left(\frac{\partial^2 V}{\partial \phi_a \partial \phi_b}\right)_{\xi_\mu = 0} = \frac{\partial r^\mu}{\partial \phi_a}\frac{\partial r^\nu}{\partial \phi_b}\Lambda_{\mu\nu} \,. \tag{4.74}$$

Thus the second derivative matrix of the potential is

$$\frac{1}{2}\left(\frac{\partial^2 V}{\partial \phi_a \partial \phi_b}\right)_{ch} = 2\Sigma^\mu_{aa'}\phi_{a'}\phi_{b'}\Sigma^\nu_{b'b}\Lambda_{\mu\nu} \equiv B^{(eff)}_{ab} \,. \tag{4.75}$$

Then, we compute the second derivative of the kinetic term, which gives the same result as before:

$$\frac{\partial^2 L}{\partial(\partial_\alpha \phi_a)\partial(\partial_\alpha \phi_b)} = K_\mu \Sigma^\mu_{ab} = K_{ab} \,. \tag{4.76}$$

Now, having found $B^{(eff)}_{ab}$ and K_{ab}, we can proceed with the calculation of the trace of the mass matrix. However, there is one tricky issue. We should remember that these matrices must be calculated at the point of the minimum. However, we do not know the values of $\langle \phi_a \rangle$ at the minimum; we know only $\langle r^\mu \rangle$ at this point. Therefore, we will need to convert $\phi_{a'}\phi_{b'}$ into r^μ.

We have

$$Tr(M) = 2Tr(K^{-1}\Sigma^\mu_{aa'}\phi_{a'}\phi_{b'}\Sigma^\nu_{b'b}\Lambda_{\mu\nu}) \tag{4.77}$$

where $K^{-1} = K_\mu \bar{\Sigma}^\mu$ as in the previous section. Thus

$$Tr(M) = 2K_\rho \Lambda_{\mu\nu} Tr(\bar{\Sigma}^\rho_{ca}\Sigma^\mu_{aa'}\phi_{a'}\phi_{b'}\Sigma^\nu_{b'b}) \,. \tag{4.78}$$

As the trace is cyclic

$$\begin{aligned} Tr(M) &= 2K_\rho \Lambda_{\mu\nu} Tr(\phi_{b'}\Sigma^\nu_{b'b}\bar{\Sigma}^\rho_{ba}\Sigma^\mu_{aa'}\phi_{a'}) \\ &= 2K_\rho \Lambda_{\mu\nu}\phi \Sigma^\nu \bar{\Sigma}^\rho \Sigma^\mu \phi \\ &= 2K_\rho \Lambda_{\mu\nu}\phi(g^{\nu\rho}\Sigma^\mu + g^{\rho\mu}\Sigma^\nu - g^{\mu\nu}\Sigma^\rho)\phi \end{aligned} \tag{4.79}$$

where we have used the properties of the Σ matrices and neglected the antisymmetric part of $\Sigma^\mu \bar{\Sigma}^\rho \Sigma^\mu$ since $\Lambda_{\mu\nu}$ is symmetric. We see that now everything is expressed in terms of $r^\mu = \phi^\dagger \Sigma^\mu \phi$, which we now replace with $b^\mu \equiv (\Lambda^{-1})^\mu_\nu B^\nu$. Finally we get

$$Tr(M) = 2[2(KB) - (Kb)Tr(\Lambda)] \,. \tag{4.80}$$

where $(Kb) \equiv K_\mu b^\mu$ and $Tr\Lambda = \Lambda_{\mu\nu} g^{\mu\nu}$.

Let us make a cross-check. Consider a simple case when B^μ is proportional to K^μ. Then, we can switch to the basis where

$$K^\mu = (1,0,0,0), \quad B^\mu = (B^0,0,0,0)\,, \tag{4.81}$$

while $\Lambda_{\mu\nu}$ can be arbitrary. In this frame, the expression giving the trace is simplified to

$$Tr(M) = 2[2\Lambda^{00} b_0 - b^0 Tr(\Lambda)]\,. \tag{4.82}$$

Using $Tr(\Lambda) = \Lambda^{\mu\nu} g_{\mu\nu} = \Lambda_{00} - \Lambda_{11} - \Lambda_{22} - \Lambda_{33}$, we find that

$$Tr(M) = 2b_0(\Lambda_{00} + \sum \Lambda_{ii})\,. \tag{4.83}$$

This is consistent with results found in [11].

We can also in principle calculate the trace of any power of the mass matrix. For example,

$$\begin{aligned} Tr(M^2) \;=\; & 2[2(KB)(KB) + 2(K\Lambda K)(B\Lambda^{-1}B) - 4(K\Lambda^{-1}B)(K\Lambda B) \\ & +(K\Lambda^{-1}B)(K\Lambda^{-1}B)Tr(\Lambda^2)]\,. \end{aligned} \tag{4.84}$$

We could also calculate in principle the determinant of the mass matrix.

4.2.4 Neutral vacuum

The neutral minimum is characterized by non-zero $\xi_\mu \neq 0$. Therefore, the second-derivative matrix is

$$\frac{1}{2}\left(\frac{\partial^2 V}{\partial\phi_a \partial\phi_b}\right)_{n.} = \xi_\mu \Sigma^\mu_{ab} + 2\Sigma^\mu_{aa'}\phi_{a'}\phi^\dagger_{b'}\Sigma^\nu_{b'b}\Lambda_{\mu\nu} \tag{4.85}$$

while the second-derivative matrix for the kinetic term remains the same. The trace of the mass matrix then becomes

$$Tr(M) = Tr(K_\rho \bar{\Sigma}^\rho \xi_\mu \Sigma^\mu) + 2Tr(K_\rho \bar{\Sigma}^\rho \Sigma^\mu \phi\phi^\dagger \Sigma^\nu \Lambda_{\mu\nu})\,, \tag{4.86}$$

where for simplicity we omitted all field indices. Using the properties of the Σ matrices, we get

$$Tr(M) = 4K_\mu \xi^\mu + 4K_\mu \Lambda^\mu_\nu \langle r^\nu\rangle - 2K_\mu \langle r^\mu\rangle Tr(\Lambda) \tag{4.87}$$

$$= 4K_\mu(-B^\mu + \Lambda^{\mu\nu}\langle r_\nu\rangle) + 4K_\mu \Lambda^\mu_\nu \langle r^\nu\rangle - 2K_\mu \langle r^\mu\rangle Tr(\Lambda) \tag{4.88}$$

since $\xi_\mu = (-B_\mu + \Lambda_{\mu\nu} r^\nu)$ and $r^\mu = \phi_a \Sigma^\mu_{ab}\phi_b$. Finally,

$$Tr(M) = -4(KB) + 8(K\Lambda\langle r\rangle) - 2(K\langle r\rangle)Tr(\Lambda) \tag{4.89}$$

This is a universal formula for all neutral vacua, even if there are two minima of the potential with different $\langle r^\mu \rangle$.

Again, the formalism that we have developed allows in principle to calculate the trace of any power of the mass matrix, and finally to find the masses themselves of absolutely any 2HDM.

Conclusion

The end of the twentieth century celebrated the triumph of the standard model of the electroweak and strong interactions of elementary particles. The electroweak theory, describing the electromagnetic and weak interactions between quarks and leptons, combined with quantum chromodynamics, the theory of strong interactions between quarks, provides a unified framework to describe three of the four forces in nature. We have seen in the first chapter that the Higgs mechanism, introduced in the electroweak theory to give mass to all particles of the SM, leads to the introduction of a new scalar particle, the Higgs boson. The discovery of this particle is of great importance. Indeed, the various experiments carried out during the last decades established that the SM is the correct effective theory of the strong and electroweak interactions at presently accessible energies. However, these experiments failed to find the new type of particle that is the Higgs boson. Only constraints on its mass have been inferred from the high-precision data. Finding this particle is the main goal of the present high-energy accelerators: the Tevatron and the CERN Large Hadron Collider.

Moreover, during this period, a large number of theoretical advances on the properties of the Higgs boson were performed. This was done both in the framework of the SM and of its various possible extensions. Indeed, in the second part, we have detailed some motivations to extend the SM. Furthermore, the mechanism of spontaneous electroweak symmetry breaking in the SM generates the weak vector boson masses in a way that is minimal, and there is no reason to assume that the Higgs mechanism is minimal. In several theories beyond the SM, the Higgs sector involves larger and/or additional representations. There are various theories going beyond the standard model, however one has no hint from experimental data supporting one of these theories. We know from the corrections to the Higgs mass that new physics could be at a scale of one or few TeV. It would be possible then to see effects beyond the standard model (SM) at the LHC. The Large Hadron Collider was indeed built with the intent of testing various predictions of high-energy physics, as the existence of the Higgs boson and of the large family of new particles predicted by supersymmetry. It is why we are entering a decisive period in high-energy physics.

Is this new physics supersymmetry, or does it involve extra dimensions, or something else ? Therefore, we should be prepared to discover this expected new physics beyond the SM.

That is why in the course of this work we have examined one of the simplest extensions of the Higgs mechanism: the two-higgs-doublet model which is a common structure of various theories beyond the SM. This extension can satisfy existing theoretical and experimental constraints and give rise to interesting phenomenologies at high-energy colliders. We have presented its main properties and shown with a specific example that a model with two Higgs doublets leads to a new phenomenology that was not possible with one doublet. However, if experimental data reveal the existence of a Higgs sector beyond that of the standard model, it will be crucial to test whether the observed scalar spectrum is consistent with a 2HDM interpretation. In order to be completely general within this framework, one should allow for the most general 2HDM when confronting the data. So, it is necessary to study the most general two-higgs-doublet model. However, as we have noted we cannot analyze the most general 2HDM with straightforward algebra. So, to circumvent these difficulties, we have used a method recently developed that allows one to analyze many characteristics of the most general 2HDM without the need to compute the exact position of the global minimum of the potential. In this approach, we first establish the structure behind 2HDM : the space of gauge orbits of the Higgs potential has 1 + 3-dimensional Minkowski space structure. Then, we reformulate the problem of minimization in geometric terms. In the last part, we have carried out the first step towards an understanding of the dynamics of 2HDM. We worked out the formalism to compute the traces of any power of the mass matrix in any type of minimum in a general 2HDM. Then, one can in principle get the masses themselves. The algebraical calculation is very difficult, though, because one would need to solve equations of fourth order. However, there might exist a geometric formulation of these mass matrices. Indeed, it is known that surfaces of phase transitions in the space of B_μ have a rather simple shape. But these surfaces correspond to one of the masses going to zero. Therefore, there should indeed be some simple expression for the determinant of the mass matrix. Maybe with a geometric formulation one can advance further and study the spectrum of the scalar sector of 2HDM in even greater detail.

Appendix A: Simple case of 2HDM

Here, we derive the conditions to have a minimum for the four possible phases in the particular simple case with two complex scalar fields of section 2.5.1 leading to the three phase diagrams 2.1, 2.2 and 2.3.

Phase A

The phase A appears when the vacuum expectation values of the two fields are zero: $v_1 = 0$ and $v_2 = 0$. In this case there always exists an extremum. Let us construct the mass matrix. We need the second derivative of the potential :

$$\left(\frac{\partial^2 V}{\partial\phi_1\partial\phi_1}\right)_{v_1=0,v_2=0} = -\frac{m_{11}^2}{2}\,, \qquad \left(\frac{\partial^2 V}{\partial\phi_2\partial\phi_2}\right)_{v_1=0,v_2=0} = -\frac{m_{22}^2}{2}$$
$$\left(\frac{\partial^2 V}{\partial\phi_1\partial\phi_2}\right)_{v_1=0,v_2=0} = 0\,, \qquad \left(\frac{\partial^2 V}{\partial\phi_2\partial\phi_1}\right)_{v_1=0,v_2=0} = 0$$

and we obtain the following mass matrix :

$$\begin{pmatrix} -\frac{m_{11}^2}{2} & 0 \\ 0 & -\frac{m_{22}^2}{2} \end{pmatrix}.$$

The conditions ensuring that above extremum is a minimum are realized if the mass matrix is positive definite, so if the eigenvalues (the physical mass squared) are positive. So, if m_{11}^2 and m_{11}^2 are negative, the potential has a minimum $\forall\lambda$ and λ_3 with $\lambda > 0$ and $\lambda + \lambda_3 > 0$. This is the global minimum at the origin.

Phase B_1

The phase B_1 appears when $v_1 \neq 0$ and $v_2 = 0$. In this case, the conditions for the stationary points of the Higgs potential are:

$$\left(\frac{\partial V}{\partial \phi_1}\right)_{v_1 \neq 0, v_2 = 0} = 0$$

if

$$-m_{11}^2 + \lambda v_1^2 = 0 \Rightarrow \lambda v_1^2 = m_{11}^2 \Rightarrow m_{11}^2 > 0$$

and

$$\left(\frac{\partial V}{\partial \phi_2}\right)_{v_1 \neq 0, v_2 = 0} = 0$$

is always possible as $v_2 = 0$.

The mass matrix is

$$\begin{pmatrix} v_1^2 \lambda & 0 \\ 0 & -\frac{1}{2}(-m_{22}^2 + \lambda_3 v_1^2) \end{pmatrix} \tag{4.90}$$

. The potential has a minimum if the eigenvalues are positive,

$$\Rightarrow -m_{22}^2 + \lambda_3 v_1^2 > 0 \,.$$

Using the first condition of the minimization, we replace v_1^2 by $\frac{m_{11}^2}{\lambda}$

$$\Rightarrow -m_{22}^2 + \frac{\lambda_3}{\lambda} m_{11}^2 > 0 \,.$$

We distinguish three cases depending on the values of the parameters λ and λ_3:

1. $\lambda > \lambda_3 > 0$
2. $\lambda_3 > \lambda > 0$
3. $0 > \lambda_3 > -\lambda$.

Phase B_2

The phase B_2 appears when $v_1 = 0$ and $v_2 \neq 0$. This case is very similar to the previous one.

In this case, there always exists a extremum in the ϕ_1 direction as $v_1 = 0$.

In the ϕ_2 direction, the condition for extrema is:

$$-m_{22}^2 + \lambda v_2^2 = 0 \Rightarrow \lambda v_2^2 = m_{22}^2 \Rightarrow m_{22}^2 > 0\,.$$

The mass matrix is

$$\begin{pmatrix} -\frac{1}{2}(-m_{11}^2 + \lambda_3 v_2^2) & 0 \\ 0 & v_2^2\lambda \end{pmatrix}.$$

We have a minimum if the eigenvalues are positive, i.e. if

$$-m_{11}^2 + \frac{\lambda_3}{\lambda} m_{22}^2 > 0\,.$$

Phase C

The phase C appears when $v_1 \neq 0$ and $v_2 \neq 0$. The extrema appear for

$$v_2^2 = \frac{m_{22}^2\lambda - m_{11}^2\lambda_3}{\lambda^2 - \lambda_3^2} \tag{4.91}$$

and

$$v_1^2 = \frac{m_{11}^2\lambda - m_{22}^2\lambda_3}{\lambda^2 - \lambda_3^2}\,. \tag{4.92}$$

As previously, we distinguish three cases depending on the values of the parameters λ and λ_3.

1. $\lambda > \lambda_3 > 0$

   In this case, $\lambda^2 - \lambda_3^2 > 0$ and so $v_1^2 > 0$ and $v_2^2 > 0$ if

   $$\lambda m_{11}^2 - \lambda_3 m_{22}^2 > 0$$

   and

   $$\lambda m_{22}^2 - \lambda_3 m_{11}^2 > 0\,.$$

   The second derivatives of the potential in this case are :

$$\left(\frac{\partial V}{\partial \phi_1}\right)_{v_1\neq 0, v_2\neq 0} = -\frac{1}{2}(-m_{11}^2 + \lambda v_1^2 + \lambda_3 v_2^2) + \frac{1}{2}v_1 2\lambda v_1 = \lambda v_1^2\,,$$
$$\left(\frac{\partial V}{\partial \phi_2}\right)_{v_1\neq 0, v_2\neq 0} = -\frac{1}{2}(-m_{22}^2 + \lambda v_2^2 + \lambda_3 v_1^2) + \frac{1}{2}v_2^2\lambda v_2 = \lambda v_2^2\,. \tag{4.93}$$

Because from the extrema conditions : $-m_{11}^2 + \lambda v_1^2 + \lambda_3 v_2^2 = 0$. And

$$\left(\frac{\partial^2 V}{\partial\phi_1 \partial\phi_2}\right)_{v_1\neq 0, v_2\neq 0} = \left(\frac{\partial^2 V}{\partial\phi_2 \partial\phi_1}\right)_{v_1\neq 0, v_2\neq 0} = \lambda_3 v_1 v_2 \,.$$

We get the following mass matrix :

$$\begin{pmatrix} \lambda v_1^2 & \lambda_3 v_1 v_2 \\ \lambda_3 v_1 v_2 & \lambda v_2^2 \end{pmatrix} .$$

Let us derive the conditions for the eigenvalues to be positive. The eigenvalues α are given by the characteristic equation :

$$\alpha^2 - \alpha\lambda(v_1^2 + v_2^2) + v_1^2 v_2^2(\lambda^2 - \lambda_3^2) = 0 \,,$$

$$\lambda(v_1^2 + v_2^2) > 0$$

and

$$v_1^2 v_2^2(\lambda^2 - \lambda_3^2) > 0 \,.$$

Therefore, the eigenvalues are positive. The potential has four minima, the first two corresponding to v_1 and $-v_1$ with the same depth and the second two corresponding to v_2 and $-v_2$ with the same depth too.

2. $\lambda_3 > \lambda > 0$

   In this case, $\lambda_3^2 - \lambda^2 < 0$ and so $v_1^2 > 0$ and $v_2^2 > 0$ if

$$\lambda m_{11}^2 - \lambda_3 m_{22}^2 < 0$$

and

$$\lambda m_{22}^2 - \lambda_3 m_{11}^2 < 0 \,.$$

We get the same mass matrix as in the previous case and so the same characteristic equation but

$$\lambda(v_1^2 + v_2^2) > 0$$

and

$$v_1^2 v_2^2(\lambda^2 - \lambda_3^2) < 0 \,,$$

and so, the eigenvalues are not both positive and the potential has no minimum.

3. $0 > \lambda_3 > -\lambda$

   In this case, $\lambda^2 - \lambda_3^2 > 0$ and $v_1^2 > 0$ ans $v_2^2 > 0$ if

$$\lambda m_{11}^2 - \lambda_3 m_{22}^2 > 0$$

   and

$$\lambda m_{22}^2 - \lambda_3 m_{11}^2 > 0\,.$$

   With the same argument as in the previous case, the eigenvalues are both positive and the potential has four minima.

Appendix B: Algebra of matrices σ^μ and Π^μ

The full 8-by-8 matrices Σ^i have block-diagonal form and are built from two identical 4-by-4 matrices, which we also denote by the same letter Σ's and whose properties we describe here. Σ^0 is just the unit matrix, while the explicit expressions of Σ^i are:

$$\Sigma^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

They satisfy an important property:

$$\{\Sigma^i \Sigma^j\} = 2\delta^{ij} I\,,$$

The set of Σ's is not closed under taking commutators. Instead, they can be expressed via real antisymmetric matrices Π^i:

$$\Pi^i \equiv \Pi^0 \Sigma^i\,, where \quad \Pi^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The π^0 matrix has the following properties:

$$\pi^0 \pi^0 = -I \tag{4.94}$$
$$\Sigma^i = -\pi^0 \pi^i \tag{4.95}$$
$$\phi_a \pi^\mu \phi_b = 0^\mu \tag{4.96}$$

Moreover, the atrix Π^0 commutes with all Σ^i.

The set of matrices Σ^i and Π^i now forms the algebra:

$$[\Sigma^i, \Sigma^j] = 2\epsilon^{ijk}\Pi^k\,, \quad [\Sigma^i, \Pi^j] = -2\epsilon^{ijk}\Sigma^k\,, \quad [\Pi^i, \Pi^j] = -2\epsilon^{ijk}\Pi^k\,. \tag{4.97}$$

Note that Π^i do form a closed algebra. The algebra of Σ^i and Π^i is isomorphic to the usual Poincaré algebra of the generators of boosts and rotations.

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