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Some prevalent results about strongly monoHölder functions

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Abstract
We study the typical behaviour of strongly monoHölder functions from the prevalence point of view. To this end we first prove wavelet-based criteria for strongly monoHölder functions. We then use the notion of prevalence to show that the functions of $C^\alpha(\boldsymbol{R}^d)$ are almost surely strongly monoHölder with Hölder exponent α. Finally, we prove that for any $\alpha \in (0, 1)$ on a prevalent set of $C^\alpha(\boldsymbol{R}^d)$ the Hausdorff dimension of the graph is equal to $d + 1 - \alpha$.

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1. Introduction

The most popular concept for uniform regularity is uniform Hölder regularity defined from Hölder spaces $C^\alpha(\boldsymbol{R}^d)$. For any $\alpha \in (0, 1)$, a bounded function f belongs to $C^\alpha(\boldsymbol{R}^d)$ if there exist $C_0, r_0 > 0$ such that

$$\forall r \leqslant r_0, \qquad \sup_{|x-y|\leqslant r} |f(x) - f(y)| \leqslant C_0 r^\alpha.$$

This notion can be generalized to exponents greater than one (see section 2). It has been used to study smoothness properties of classical models as trigonometric series (see [16, 28]), sample paths properties of processes as Brownian motion (see [17]) or fractional Brownian motion. The pointwise counterparts of spaces $C^\alpha(\boldsymbol{R}^d)$ are the spaces $C^\alpha(x_0)$. A locally bounded

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function f belongs to $C^\alpha(x_0)$ with $\alpha \in (0, 1)$ (see section 2 for the general case) if there exist $C, R > 0$ such that

$$\sup_{|x-x_0|\leqslant r} |f(x) - f(x_0)| \leqslant Cr^\alpha, \qquad \forall r \leqslant R.$$

In [27], a very natural notion for the pointwise irregularity of a function is obtained by reversing the inequality sign in the definition of the Hölder regularity, when the regularity exponent is lower than 1. This definition is generalized in [5, 6] for any positive exponent. The spaces $I^\alpha(x_0)$ and $I^\alpha(\boldsymbol{R}^d)$ are the irregular analogues of the usual Hölder spaces $C^\alpha(x_0)$ and $C^\alpha(\boldsymbol{R}^d)$, respectively.

In [15], it is showed that most of the historical space-filling functions share the same property: the associated regularity and irregularity exponents are the same, i.e. $f \in C^\alpha(\boldsymbol{R}^d) \cap I^\alpha(\boldsymbol{R}^d)$. Such mappings are said to be strongly monoHölder with exponent α, which is denoted by $f \in SM^\alpha(\boldsymbol{R}^d)$. Increasing interest has been paid to such functions in the case where $\alpha \in (0, 1)$ since the box-counting dimension of their graph on $\boldsymbol{R}^d$, $\Gamma(f) = \{(x, f(x)), x \in \boldsymbol{R}^d\}$ is equal to $d + 1 - \alpha$ (see, e.g., [11]). Let us point out that concerning the Hausdorff dimension of $\Gamma(f)$, it has been proved that the following relationship,

$$\dim_{\mathcal{H}}\Gamma(f) = d + 1 - \alpha, \tag{1}$$

is not true. In [22], McMullen proposed an example of self-affine set which is the graph of a strongly monoHölder function and whose Hausdorff dimension is lower than its box dimension. There are examples where equality (1) holds (see, e.g., [19]). However, even for the case of the classical Weierstraß function W_α ($\alpha \in (0, 1)$) defined on $\boldsymbol{R}$ by

$$W_\alpha(x) = \sum_{k=0}^{+\infty} 2^{-\alpha k} \cos(2\pi 2^k x)$$

equality (1) remains as a conjecture (see, e.g., [3, 21]), although estimates are known. For example, in the more general case of Weierstraß type functions of the form

$$f(x) = \sum_{k=0}^{+\infty} 2^{-k\alpha} g(2^k x),$$

where g is a continuously differentiable function on $\boldsymbol{R}$, there exists a constant $c > 0$ such that

$$2 - \alpha - c/\log b \leqslant \dim_{\mathcal{H}}\Gamma(f) \leqslant 2 - \alpha,$$

for b sufficiently large (see [24]). Some results have also been obtained in the case of Weierstraß functions with random phase added to each term: for such functions, equality (1) holds with probability one (see [13]). For the so-called index α fields studied in [1], the same relation is satisfied.

Therefore, though relation (1) does not hold in generality, it seems to be satisfied for most of the studied strongly monoHölder models. It is then quite natural to wonder to what point this behaviour is a typical one. Firstly, is 'almost every' function belonging to $C^\alpha(\boldsymbol{R}^d)$ a strongly monoHölder function? Thereafter what can be said about the Hausdorff dimension of the graph of 'almost every' function of $C^\alpha(\boldsymbol{R}^d)$?

We first need to introduce what is meant by 'almost every function'. In a finite dimensional space, we say that a property holds almost everywhere if the set of points where it is not true is of vanishing Lebesgue measure. The Lebesgue measure has here a preponderant role, as it is the only σ-finite and translation invariant measure. Unfortunately, no measure shares those properties in infinite dimensional Banach spaces. A way to recover a natural notion of 'almost every' in infinite dimensional vector spaces is defined by Christensen in [4]. The basic idea is to generalize the well-known characterization of Lebesgue measure zero subsets of $\boldsymbol{R}^d$. In $\boldsymbol{R}^d$,

a Borel set B has measure zero if and only if there exists a compactly supported probability measure μ such that,

$$\forall x \in \boldsymbol{R}^d, \qquad \mu(x + B) = 0.$$

This characterization can be turned into a definition in the infinite dimensional setting and leads to the concept of Haar-null sets. This concept provides the needed analogue of 'Lebesgue measure zero' sets for infinite dimensional spaces.

Definition 1. *Let E be a complete metric vector space. A Borel set $B \subset E$ is Haar-null if there exists a compactly supported Borel probability measure μ such that*

$$\forall x \in E, \qquad \mu(x + B) = 0.$$

A subset S of E is Haar-null if it is included in a Haar-null Borel set. The complement of a Haar-null set is called a prevalent set.

In this paper we study the prevalent behaviour of the functions of $C^\alpha(\boldsymbol{R}^d)$. We first prove that the spaces $SM^\alpha(\boldsymbol{R}^d)$ are prevalent in $C^\alpha(\boldsymbol{R}^d)$.

Theorem 1. *For any $\alpha > 0$, the space $SM^\alpha(\boldsymbol{R}^d)$ is a prevalent subset of $C^\alpha(\boldsymbol{R}^d)$.*

Our second main result proves that, though (1) is not generally satisfied, it is true for a prevalent subset of $C^\alpha(\boldsymbol{R}^d)$:

Theorem 2 (Let $\alpha \in (0, 1)$). *There is a prevalent subset of $C^\alpha(\boldsymbol{R}^d)$ for which the Hausdorff dimension of the graph of its elements is equal to the maximum possible value, $d + 1 - \alpha$, i.e. for any element f of this subset,*

$$\dim_{\mathcal{H}} \Gamma(f) = d + 1 - \alpha.$$

Thus, the classical case where the Hausdorff dimension of the graph of a function is linked to its uniform Hölder exponent corresponds to the typical behaviour of the functions of $C^\alpha(\boldsymbol{R}^d)$.

These two results are proved in section 4 using wavelets. Indeed Jaffard has shown that the wavelet transform in general is a very efficient tool to study the regularity of a function. In particular, the wavelet leaders method most closely characterizes the Hölder regularity of a function (see [14] and references therein). The same pattern is followed for the Hölderian irregularity: this notion is studied under the discrete wavelet lens and several criteria binding the wavelet coefficients with the irregularity exponent are obtained.

Our plan will be as follows. In section 2, we first recall some definitions about pointwise irregularity and strongly monoHölder functions. In section 3, we state and prove our two wavelet criteria. Using these wavelet criteria, we are able in section 4 to prove our two main results: theorem 1 and theorem 2.

2. Hölderian and anti-Hölderian functions

We recall here the definitions related to the Hölderian regularity of a function for exponents greater than one, before introducing the Hölderian irregularity. These considerations also lead to a weaker definition of pointwise smoothness. Finally, we define the strongly monoHölder functions; this notion formalizes the idea of a function which has everywhere the same regularity, in a way as uniform as possible.

The integer part of α will be denoted $[\alpha]$: if $\alpha > 0$, $[\alpha] = \sup\{k \in \boldsymbol{N} : k \leqslant \alpha\}$. We will also use the following notation:

$$B_h(x_0, r) = \{x : [x, x + ([\alpha] + 1)h] \subset B(x_0, r)\}$$

and denote, as usual, the finite differences of arbitrary order as follows:

$$\Delta_h^1 f(x) = f(x + h) - f(x), \qquad \Delta_h^{n+1} f(x) = \Delta_h^n f(x + h) - \Delta_h^n f(x).$$

Definition 2. *Let $f : \mathbf{R}^d \to \mathbf{R}^{d'}$ be a locally bounded function, let $x_0 \in \mathbf{R}^d$ and $\alpha \geqslant 0$; $f \in C^\alpha(x_0)$ if there exist $C, R > 0$ such that*

$$\sup_{|h|\leqslant r} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0,r))} \leqslant Cr^\alpha, \qquad \forall r \leqslant R. \tag{2}$$

Such a function is said to be Hölderian of exponent α at x_0. The lower Hölder exponent of f at x_0 is

$$\underline{h}_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}.$$

A function f is uniformly Hölderian of exponent α ($f \in C^\alpha(\mathbf{R}^d)$) if there exist $C, R > 0$ such that (2) is satisfied for any $x_0 \in \mathbf{R}^d$; f is uniformly Hölderian if there exists $\varepsilon > 0$ such that $f \in C^\varepsilon(\mathbf{R}^d)$.

Condition (2) is satisfied if and only if there exists a polynomial P of degree less than α such that

$$\|f(x) - P(x)\|_{L^\infty(B(x_0,r))} \leqslant Cr^\alpha, \qquad \forall r \leqslant R \tag{3}$$

(see, e.g., [5, 10, 18]). This inequality is more often chosen to define the spaces $C^\alpha(x_0)$. Nevertheless, this last definition cannot directly be linked to our notion of pointwise irregularity, in contrast to definition 2. The lower Hölder exponent is simply denoted Hölder exponent in the literature. However, since we are interested in introducing another concept of pointwise Hölderian regularity, the accustomed notation h is replaced here by $\underline{h}$.

The irregularity of a function can be studied through the notion of anti-Hölderianity.

Definition 3. *Let $f : \mathbf{R}^d \to \mathbf{R}^{d'}$ be a locally bounded function, let $x_0 \in \mathbf{R}^d$ and $\alpha \geqslant 0$; $f \in I^\alpha(x_0)$ if there exist $C, R > 0$ such that*

$$\sup_{|h|\leqslant r} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0,r))} \geqslant Cr^\alpha, \qquad \forall r \leqslant R. \tag{4}$$

Such a function is said to be anti-Hölderian of exponent α at x_0. The upper Hölder exponent (or irregularity exponent) of f at x_0 is

$$\overline{h}_f(x_0) = \inf\{\alpha : f \in I^\alpha(x_0)\}.$$

We will say that f is strongly Hölderian of exponent α at x_0 ($f \in C_s^\alpha(x_0)$) if $f \in C^\alpha(x_0) \cap I^\alpha(x_0)$.

Let us remark that the statement (4) is not equivalent to negating the property $f \in C^\alpha(x_0)$. Indeed, $f \notin C^\alpha(x_0)$ if for any $C > 0$, there exists a sequence $(r_n)_n$ (depending on C) for which

$$\sup_{|h|\leqslant r_n} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0,r_n))} \geqslant Cr_n^\alpha.$$

We are thus naturally led to the following definition.

Definition 4. *Let $f : \mathbf{R}^d \to \mathbf{R}^{d'}$ be a locally bounded function, let $x_0 \in \mathbf{R}^d$ and $\alpha \geqslant 0$; $f \in C_w^\alpha(x_0)$ if $f \notin I^\alpha(x_0)$, i.e. for any $C > 0$ there exists a sequence $(r_n)_n$ strictly decreasing to 0 such that*

$$\sup_{|h|\leqslant r_n} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0,r_n))} \leqslant Cr_n^\alpha, \qquad \forall n \in \mathbf{N}.$$

Such a function is said weakly Hölderian of exponent α at x_0.

Roughly speaking, a function is weakly Hölderian of exponent α at x_0 if for any $C > 0$, one can bound the oscillation of f over $B(x_0, r_n)$ by Cr_n^α for a remarkable decreasing subsequence

$(r_n)_n$ of scales, whereas for an Hölderian function, the oscillation of f over $B(x_0, r)$ has to be bounded at each scale $r > 0$ by Cr^α, for some $C > 0$.

Strongly monoHölderian functions naturally arise in the study of the regularity of mappings such as Weierstraß-type functions, space-filling functions or random processes (see, e.g., [9, 12, 15]). Indeed, many results only hold for such mappings.

Definition 5. *A function $f : \boldsymbol{R}^d \to \boldsymbol{R}^{d'}$ is strongly monoHölderian of exponent α ($f \in SM^\alpha(\boldsymbol{R}^d)$) if $f \in C^\alpha(\boldsymbol{R}^d) \cap I^\alpha(\boldsymbol{R}^d)$, i.e. if there exist $C, R > 0$ such that, for any $x_0 \in \boldsymbol{R}^d$,*

$$r^\alpha/C \leqslant \sup_{|h|\leqslant r} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0,r))} \leqslant Cr^\alpha \qquad \forall r \leqslant R.$$

3. Wavelet criteria for pointwise irregularity

Results binding the lower Hölder exponent of a function with its wavelet leaders are well known (see, e.g., [14]). In this section, we show that the Hölder irregularity of a function can also be studied through the wavelet leaders method. However, for the Hölder irregularity, only weaker results hold.

3.1. Wavelets and usual pointwise regularity

Let us briefly recall some definitions and notations (for more precisions, see, e.g., [7, 20, 23]). Under some general assumptions, there exist a function ϕ and $2^d - 1$ functions $(\psi^{(i)})_{1\leqslant i<2^d}$, called wavelets, such that $\{\phi(x-k)\}_{k\in\boldsymbol{Z}^d} \cup \{\psi^{(i)}(2^j x-k) : 1 \leqslant i < 2^d, k \in \boldsymbol{Z}^d, j \in \boldsymbol{N}\}$ form an orthogonal basis of $L^2(\boldsymbol{R}^d)$. Any function $f \in L^2(\boldsymbol{R}^d)$ can be decomposed as follows:

$$f(x) = \sum_{k\in\boldsymbol{Z}^d} C_k \phi(x-k) + \sum_{j=1}^{+\infty} \sum_{k\in\boldsymbol{Z}^d} \sum_{1\leqslant i<2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int_{\boldsymbol{R}^d} f(x)\psi^{(i)}(2^j x - k)\,\mathrm{d}x,$$

and

$$C_k = \int_{\boldsymbol{R}^d} f(x)\phi(x-k)\,\mathrm{d}x.$$

Let us remark that we do not choose the $L^2(\boldsymbol{R}^d)$ normalization for the wavelets, but rather an L^∞ normalization, which is better fitted to the study of the Hölderian regularity. Hereafter, the wavelets are always supposed to belong to $C^\gamma(\boldsymbol{R}^d)$ with $\gamma > \alpha$, and the functions $\{\partial^s\phi\}_{|s|\leqslant\gamma}$, $\{\partial^s\psi^{(i)}\}_{|s|\leqslant\gamma}$ are assumed to have fast decay.

A dyadic cube of scale j is a cube of the form

$$\lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right),$$

where $k = (k_1, \ldots, k_d) \in \boldsymbol{Z}^d$. From now on, wavelets and wavelet coefficients will be indexed with dyadic cubes λ. Since i takes $2^d - 1$ values, we can assume that it takes values in $\{0, 1\}^d - (0, \ldots, 0)$; we will use the following notations:

- $\lambda = \lambda(i, j, k) = \dfrac{k}{2^j} + \dfrac{i}{2^{j+1}} + \left[0, \dfrac{1}{2^{j+1}}\right)^d$,
- $c_\lambda = c_{j,k}^{(i)}$,
- $\psi_\lambda = \psi_{j,k}^{(i)} = \psi^{(i)}(2^j \cdot -k)$.

The pointwise Hölderian regularity of a function is closely related to the decay rate of its wavelet leaders.

Definition 6. *The wavelet leaders are defined by*

$$d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|.$$

Two dyadic cubes λ *and* λ' *are adjacent if they are at the same scale and if* $\operatorname{dist}(\lambda, \lambda') = 0$. *We denote by* 3λ *the set of* 3^d *dyadic cubes adjacent to* λ *and by* $\lambda_j(x_0)$ *the dyadic cube of side of length* 2^{-j} *containing* x_0. *Then*

$$d_j(x_0) = \sup_{\lambda \subset 3\lambda_j(x_0)} d_\lambda.$$

The following theorem [14] allows to 'nearly' characterize the Hölderian regularity by a decay condition on d_j as j goes to infinity.

Theorem 3. *Let* $\alpha > 0$; *if* $f \in C^\alpha(x_0)$, *then there exists* $C > 0$ *such that*

$$d_j(x_0) \leqslant C 2^{-\alpha j}, \qquad \forall j \geqslant 0. \tag{5}$$

Conversely, if (5) holds and if f *is uniformly Hölderian, then there exist* $C, R > 0$ *and a polynomial* P *of degree less than* α *such that*

$$\|f(x) - P(x)\|_{L^\infty(B(x_0,r))} \leqslant C r^\alpha \log \frac{1}{r}, \qquad \forall r \leqslant R.$$

In particular, if f is uniformly Hölderian, the usual Hölder exponent, denoted here $\underline{h}_f(x_0)$, can be estimated from a log–log regression of the wavelet leaders:

Corollary 1. *Assume that* f *is uniformly Hölderian. One has*

$$\underline{h}_f(x_0) = \liminf_{j \to \infty} \frac{\log d_j(x_0)}{-j \log 2}.$$

From now on, we will suppose that the wavelets are compactly supported; such wavelets are constructed in [7].

3.2. A polynomial characterization of the weak pointwise regularity

Proposition 1 will be useful to obtain criteria for pointwise irregularity. We first need the following easy lemma, allowing us to work on dyadic scales.

Lemma 1. *A locally bounded function* f *belongs to* $C_w^\alpha(x_0)$ *if and only if, for any* $C > 0$, *there exists a strictly increasing sequence of integers* $(j_n)_n$ *such that, for any* $j \geqslant j_n$,

$$\sup_{|h| \leqslant 2^{-j}} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0, 2^{-j}))} \leqslant C 2^{-j_n \alpha}.$$

Proof. The 'if' part is trivial; let us show the converse. Let $C > 0$; since $f \in C_w^\alpha(x_0)$, there exists a sequence $(r_n)_n$ strictly decreasing to 0 such that

$$\sup_{|h| \leqslant r_n} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0, r_n))} \leqslant C 2^{-\alpha} r_n^\alpha.$$

Now, for $n \in \mathbb{N}$, let $j_n = [-\log r_n / \log 2] + 1$. Since $r_n/2 \leqslant 2^{-j_n} < r_n$, one has

$$\sup_{|h| \leqslant 2^{-j_n}} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0, 2^{-j_n}))} \leqslant C 2^{-\alpha} r_n^\alpha \leqslant C 2^{-\alpha} 2^{(-j_n+1)\alpha} = C 2^{-j_n \alpha},$$

which allows to conclude.

Proposition 1. *The two following properties are equivalent:*

(1) $f \in C_w^\alpha(x_0)$,

(2) For any $C > 0$, *there exist a strictly increasing sequence of integers* $(j_n)_n$ *and a sequence of polynomials* $(P_n)_n$ *with degree less than* $[\alpha]$ *such that,* $\forall j \geqslant j_n$,

$$\|f - P_n\|_{L^\infty(B(x_0,2^{-j}))} \leqslant C2^{-j_n\alpha}. \tag{6}$$

Proof. Let f be a locally bounded function and assume that $f \in C_w^\alpha(x_0)$. The Whitney theorem (see, e.g., [26]) asserts that there exists a constant $C_0 > 0$ depending only on α and d such that, for any $x_0 \in \boldsymbol{R}^d$ and any integer j,

$$\inf_{\deg(P)\leqslant[\alpha]} \|f - P\|_{L^\infty(B(x_0,2^{-j}))} \leqslant C_0 \sup_{|h|\leqslant 2^{-j}} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0,2^{-j}))}.$$

This shows that inequality (6) is satisfied.

Let us prove the converse assertion. Assume that (6) is satisfied. Let $h \in \boldsymbol{R}^d$ such that $|h| \leqslant 2^{-j}$ and $x \in \boldsymbol{R}^d$ such that $[x, x + ([\alpha]+1)h] \subset B(x_0, r)$. Then for any polynomial P with degree less than $[\alpha]$,

$$|\Delta_h^{[\alpha]+1} f(x)| = |\Delta_h^{[\alpha]+1}(f(x) - P(x))| \leqslant \sum_{i=0}^{[\alpha]+1} |f(x + ih) - P(x + ih)|,$$

which implies

$$\sup_{|h|\leqslant 2^{-j}} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0,2^{-j}))} \leqslant ([\alpha]+2)\|f - P\|_{L^\infty(B(x_0,2^{-j}))}.$$

Taking the infimum over all the polynomials of degree less than $[\alpha]$ in the right-hand side of the last inequality leads to the desired result.

3.3. Wavelet criteria for pointwise irregularity

Concerning the pointwise irregularity, there is no result analogous to theorem 3. However, some stronger properties can be characterized. Let us recall that the wavelets are assumed to belong to $C^\gamma(\boldsymbol{R}^d)$, with $\gamma > \alpha$.

Theorem 4. *Let* $\alpha > 0$ *and* $f \in L^\infty_{loc}(\boldsymbol{R}^d)$. *If there exists* $C > 0$ *such that*

$$d_j(x_0) \geqslant C2^{-j\alpha}, \quad \forall j \geqslant 0, \tag{7}$$

then $f \in I^\alpha(x_0)$.

Proof. Let ℓ_0 such that for any i, $\operatorname{supp}(\psi^{(i)}) \subset B(0, 2^{\ell_0+1})$, assume that (7) is satisfied for some $C > 0$ and suppose that $f \in C_w^\alpha(x_0)$. By proposition 1, there exist a strictly increasing sequence of integers (j_n) and a sequence (P_n) of polynomials with degree less than α such that for any n,

$$\|f - P_n\|_{L^\infty(B(x_0,2^{-j_n}))} \leqslant \frac{C}{2\|\psi\|_{L^1(\boldsymbol{R}^d)}} 2^{-d(\ell_0+1)} 2^{-j_n\alpha}. \tag{8}$$

Now, let us fix $\lambda' \subset \lambda_{j_n+\ell_0+1}(x_0)$ and bound the wavelet coefficient $c_{\lambda'}$. Since ψ has sufficiently many vanishing moments,

$$\begin{aligned} c_{\lambda'} &= 2^{\ell' d} \int_{\boldsymbol{R}^d} f(x)\psi(2^{\ell'}x - k')\,\mathrm{d}x \\ &= 2^{\ell' d} \int_{B(k'/2^{\ell'}, 2^{-(\ell'-\ell_0-1)})} (f(x) - P_n(x))\psi(2^{\ell'}x - k')\,\mathrm{d}x. \end{aligned} \tag{9}$$

Remark that the assumption $\lambda' \subset \lambda_{j_n+\ell_0+1}(x_0)$ implies $\ell' \geqslant j_n + \ell_0 + 1$ and $B(k'/2^{\ell'}, 2^{-(\ell'-\ell_0-1)}) \subset B(x_0, 2^{-j_n})$. Equality (9) implies

$$|c_{\lambda'}| \leqslant 2^{d(\ell_0+1)} \|f - P_n\|_{L^\infty(B(x_0,2^{-j_n}))} \|\psi\|_{L^1(\boldsymbol{R}^d)},$$

which thanks to inequality (8) yield a contradiction with inequality (7).

Note that we do not have a wavelet characterization of the property $\overline{h}_f(x_0) = \alpha$, as it is the case for $\underline{h}_f(x_0)$ (see corollary 1). It is shown in section 3.4 that it cannot be so. Nevertheless, one can characterize the stronger property $f \in C_s^\alpha(x_0)$ using wavelets.

Proposition 2. *Let $f \in C^\alpha(x_0)$.*

(1) If there exist two constants $C_1, C_2 > 0$ depending only on x_0 such that

$$C_1 2^{-j\alpha} \leqslant d_j(x_0) \leqslant C_2 2^{-j\alpha} \qquad \forall j,$$

then $f \in I^\alpha(x_0)$.

(2) Assume that f is uniformly Hölderian. If f is anti-Hölderian of exponent α at x_0, then for any $\beta > 1$, there exist two constants $C_1, C_2 > 0$ depending only on x_0 such that

$$C_1 \frac{2^{-j\alpha}}{j^{\beta(\alpha+1)}} \leqslant d_j(x_0) \leqslant C_2 2^{-j\alpha} \qquad \forall j. \tag{10}$$

Proof. The first part of the proposition comes from theorem 4. Let us prove the second part of the proposition. We assume that for some $\varepsilon_0 > 0$, $f \in C^{\varepsilon_0}(\boldsymbol{R}^d)$. Since f belongs to $C^\alpha(x_0)$, we have, for some $C_2 > 0$,

$$d_j(x_0) \leqslant C_2 2^{-j\alpha} \qquad \forall j.$$

Suppose now that for any $C > 0$, there exists a strictly increasing sequence of integers $(j_n)_n$ such that,

$$d_{j_n}(x_0) \leqslant C \frac{2^{-j_n\alpha}}{j_n^{\beta(\alpha+1)}} \qquad \forall n.$$

We will show that this hypothesis leads to a contradiction. Define ℓ_0 such that $C_2 = 2^{\ell_0(M-\alpha)}C$ and define the sequence $(\ell_n)_{n\geqslant 1}$ recursively as follows:

$$\ell_1 = j_1 + \ell_0,\ \ell_n = \min\{\ell \geqslant \ell_{n-1} : \ell - \ell_0 - \beta \log_2 \ell \geqslant j_n\} \qquad \forall n \geqslant 2.$$

Now, let $|h| \leqslant 2^{-\ell_n}$ and x such that

$$[x, x + Mh] \subset B(x_0, 2^{-\ell_n}).$$

We may write

$$\Delta_h^M f(x) = \sum_k C_k \Delta_h^M \phi_k(x) + \sum_{i,j,k} c_{j,k}^{(i)} \Delta_h^M \psi_{j,k}^{(i)}(x) = \sum_{j\geqslant 0} \Delta_h^M f_j,$$

where $f_j(x) = \sum_k C_k \phi_k(x)$ if $j = 0$, $f_j(x) = \sum_{i,k} c_\lambda \psi_\lambda(x)$ otherwise.

Let $\alpha' > \alpha$ and define $L_n = [\alpha' \ell_n/\varepsilon_0] + 1$. We have, for n sufficiently large,

$$\left| \sum_{j\geqslant L_n} \Delta_h^M f_j(x) \right| \leqslant \sum_{j\geqslant L_n} \sum_{i,r,k} |c_\lambda|\, |\psi_\lambda(x + rh)|.$$

Since the wavelets have fast decay, for any $s > 0$ there exist some $M(s) > 0$ such that for any $y \in \boldsymbol{R}^d$,

$$|\psi^{(i)}(y)| \leqslant \frac{M(s)}{(1 + |y|)^s}.$$

Hence

$$\left|\sum_{j\geqslant L_n}\Delta_h^M f_j(x)\right| \leqslant M(s)\sum_{j\geqslant L_n}2^{-j\varepsilon_0}\sum_{i,r,k}\frac{1}{(1+|2^j(x+rh)-k|)^s}.$$

The usual inequality $\sup_{x\in\boldsymbol{R}^d}\sum_k \frac{1}{(1+|2^j x-k|)^s} < \infty$ leads to

$$\left|\sum_{j\geqslant L_n}\Delta_h^M f_j(x)\right| \leqslant C(s)2^{-L_n\varepsilon_0} \leqslant C(s)2^{-\ell_n\alpha'} \leqslant C(s)2^{-\ell_n\alpha}$$

for some $C(s) > 0$.

Let us now give an upper bound for $\sum_{j=\ell_n}^{L_n}\Delta_h^M f_j(x)$. Since the wavelets are compactly supported, if n is sufficiently large, we have, for any $y \in B(x_0, 2^{-\ell_n})$ and any $\lambda \not\subset 3\lambda_{j_n}(x_0)$, $\psi_\lambda(y) = 0$. Then

$$\begin{aligned}\left|\sum_{j=\ell_n}^{L_n}\Delta_h^M f_j(x)\right| &\leqslant M\sup_{y\in B(x_0,2^{-\ell_n})}\sum_{j=\ell_n}^{L_n}\sum_i\sum_{k,\lambda\subset 3\lambda_{j_n}(x_0)}|c_\lambda|\,|\psi_\lambda(y)|\\ &\leqslant M\sum_{j=\ell_n}^{L_n}C\frac{2^{-j_n\alpha}}{j_n^{\beta(\alpha+1)}}\sup_y\sum_k|\psi_\lambda(y)|\\ &\leqslant MCL_n\frac{2^{-j_n\alpha}}{j_n^{\beta(\alpha+1)}}.\end{aligned}$$

As in [14], since $d_j(x_0) \leqslant C_2 2^{-j\alpha}$ and the wavelets belong to $C^\gamma(\boldsymbol{R}^d)$,

$$\begin{aligned}\left|\sum_{j=0}^{j_n}\Delta_h^M f_j(x)\right| &\leqslant 2^{-\ell_n\gamma}C_2 j_n 2^{(\gamma-\alpha)j_n}\\ &= C2^{-\ell_n\gamma}2^{(\gamma-\alpha)(j_n+\ell_0)}j_n\end{aligned}$$

by definition of ℓ_0. Finally, one needs to give an upper bound for

$$\sum_{j=j_n}^{\ell_n}\Delta_h^M f_j = \left(\sum_{j=j_n}^{\ell_n}\sum\Delta_h^M f_j + \sum_{j=j_n}^{\ell_n}\sum_{\lambda\not\subset 3\lambda_{j_n}}\Delta_h^M f_j\right).$$

In the first sum of the right-hand side, one can use the upper bound $d_j(x_0) \leqslant C2^{-j_n\alpha}/j_n^{\beta(\alpha+1)}$ and in the second, $d_j(x_0) \leqslant C_2 2^{-j\alpha}$ to obtain an upper bound. Therefore $|\Delta_h^M f(x)| \leqslant C2^{-\ell_n\alpha}$ and $f \in C_w^\alpha(x_0)$.

In a similar way, proposition 3 gives a sufficient condition on wavelet coefficients for a function to be uniform anti-Hölderian of exponent α.

Proposition 3. *Let $\alpha > 0$. If there exist $C_1, C_2 > 0$ such that for any $x_0 \in \boldsymbol{R}^d$ and any dyadic cube λ of length side 2^{-j},*

$$C_1 2^{-j\alpha} \leqslant d_\lambda \leqslant C_2 2^{-j\alpha},$$

then f is both uniformly Hölderian and uniformly anti-Hölderian of exponent α.

Proof. If f is not uniformly anti-Hölderian, then, for any $C > 0$, there exist a strictly increasing sequence of integers $(j_n)_n$ and a sequence of real numbers $(x_n)_n$ such that

$$\sup_{|h|\leqslant 2^{-j_n}}\|\Delta_h^{[\alpha]+1}f\|_{L^\infty(B_h(x_n,2^{-j_n}))} \leqslant C2^{-j_n\alpha} \qquad \forall n.$$

Then using a proof similar to this of proposition 1, it follows that for any n, there exists a polynomial P_n of degree less than α such that

$$\|f(x) - P_n(x)\|_{L^\infty(B(x_n, 2^{-j_n}))} \leqslant C2^{-j_n\alpha}.$$

Using a similar approach to this of the proof of theorem 4, one deduces that, for any $C > 0$, there exist a strictly increasing sequence of integers $(j_n)_n$ and a sequence of real numbers $(x_n)_n$ such that

$$d_{j_n}(x_n) \leqslant C_0 C 2^{-j_n\alpha},$$

where C_0 only depends on the multi-resolution analysis; this leads to a contradiction.

3.4. An example showing that the reciprocal to theorem 4 is not always satisfied

We now study the pointwise irregularity at the origin of a family of wavelet series. These functions illustrate the difficulty to obtain an irregularity criterion relying on the wavelet leaders. Indeed, there is no result corresponding to theorem 3 for the irregularity.

We will use the Daubechies wavelet with two vanishing moments, ψ_2. Let $\alpha \in (0, 1)$, $\beta > 1$ and $f_{\alpha,\beta}$ defined as

$$f_{\alpha,\beta}(x) = -\sum_{n=0}^{\infty} 2^{-j_n\alpha} \sum_{j=j_n}^{j_{n+1}-1} \psi_2(2^j x - 1), \tag{11}$$

where $j_n = [\beta^n]$ can be modified so that $(j_n)_n$ is a strictly increasing sequence. The aim is to prove the following proposition.

Proposition 4. *Assume that $\alpha \in (0, 1/2)$ and $\beta > 1$. Then*

$$\overline{h}_{f_{\alpha,\beta}}(0) < \frac{\beta\alpha}{\beta + \alpha(\beta - 1)} < \limsup_{j\to\infty} \frac{-\log d_j(0)}{j \log 2} = \alpha.$$

We will use the following result.

Proposition 5. *The wavelet leaders of $f_{\alpha,\beta}$ satisfy the following relation,*

$$\limsup_{j\to\infty} \frac{-\log d_j(0)}{j \log 2} = \alpha.$$

Proof. The result is obvious, since $d_j(0) = 2^{-j_n\alpha}$ whenever $j \in \{j_n, \cdots, j_{n+1} - 1\}$.

We will also need the following lemma, which summarizes some useful properties of ψ_2.

Lemma 2. *Let $\psi = -\psi_2(\cdot - 1)$; the following properties are satisfied:*

- $\mathrm{supp}(\psi) \subset [0, 3]$,
- $\psi \in C^\gamma(\boldsymbol{R})$, *with* $\gamma = 1 - \log((1 + \sqrt{3})/2)/\log 2$,
- *if* $m \in \boldsymbol{N}$, $\psi(2^{-m}) = 2^{-m\gamma}(\sqrt{3} - 1)/2$,
- *both* $\psi(1)$ *and* $\psi(2)$ *are positive.*

Proof. The first assertion is proved in [7] whereas the second one is proved in [8] (theorem 3.1). Using the two scale difference equation satisfied by ϕ,

$$\phi(x) = \sum_p c_p \phi(2x - p),$$

where c_p are explicitly known real coefficients (see [7]), one has

$$\phi(2^{-m+1}) = 2\left(\frac{1+\sqrt{3}}{4}\right)^m,$$

for any $m \in \boldsymbol{N}$. The well-known relationship between ϕ and ψ_2 (see, e.g., [20]) leads to

$$\begin{aligned}\psi(2^{-m}) = & -\psi_2(2^{-m}-1) = \frac{\sqrt{3}-1}{4}\phi(2^{-m+1}) \\ &= \frac{\sqrt{3}-1}{2}\left(\frac{1+\sqrt{3}}{4}\right)^m = \frac{\sqrt{3}-1}{2}2^{-m\gamma}.\end{aligned}$$

Finally, the explicit computation of $\phi(1)$ and $\phi(2)$ ($\phi(1) = (1+\sqrt{3})/2$, $\phi(2) = (1-\sqrt{3})/2$) gives $\psi(1) > 0$ and $\psi(2) > 0$.

The upper Hölder exponent of $f_{\alpha,\beta}$ at the origin is given by the following proposition. Let us note that $\gamma = 0.550\,01 \pm 10^{-5} > 1/2$.

Proposition 6. *If $\alpha \in (0, 1/2)$ and $\beta > 1$, then*

$$\overline{h}_{f_{\alpha,\beta}}(0) = \frac{\beta\alpha\gamma}{\beta\gamma + \alpha(\beta-1)}.$$

Proof. We first give an upper bound for $\overline{h}_{f_{\alpha,\beta}}(0)$. Let $\ell \in \boldsymbol{N}$ and n_0 such that $j_{n_0} \leqslant \ell \leqslant j_{n_0+1} - 1$. Since $f(0) = 0$, we just have to give a lower bound for $|f(2^{-\ell})|$. Using the fact that $\operatorname{supp}(\psi_2) \subset [-1, 2]$, one has

$$f(2^{-\ell}) = -\sum_{n=0}^{n_0-1} 2^{-j_n\alpha} \sum_{j=j_n}^{j_{n+1}-1} \psi_2(2^j 2^{-\ell} - 1) = \sum_{n=0}^{n_0-1} 2^{-j_n\alpha} \sum_{j=j_n}^{j_{n+1}-1} \psi(2^j 2^{-\ell}).$$

Therefore,

$$\begin{aligned} f(2^{-\ell}) &\geqslant C_1\left(\sum_{n=0}^{n_0-1} 2^{-j_n\alpha} \sum_{j=j_n+1}^{j_{n+1}-1} 2^{(j-\ell)\gamma} + 2^{-j_{n_0}\alpha} \sum_{j=j_{n_0}+1}^{\ell-1} 2^{(j-\ell)\gamma}\right) \\ &\geqslant C_1\left(2^{-\ell\gamma} \sum_{n=0}^{n_0-1} 2^{-j_n\alpha} 2^{j_{n+1}\gamma} + 2^{-\ell\gamma} 2^{-j_{n_0}\alpha} 2^{\ell\gamma}\right) \\ &\geqslant C_1\left(2^{-\ell\gamma} \sum_{n=0}^{n_0-1} 2^{j_n(\beta\gamma-\alpha)} + 2^{-j_{n_0}\alpha}\right) \\ &\geqslant C_1(2^{-\ell\gamma} 2^{j_{n_0-1}(\beta\gamma-\alpha)} + 2^{-j_{n_0}\alpha}). \end{aligned}$$

Let $t \in (1, \beta)$ be such that $j_{n_0} = \ell/t$. We have

$$f(2^{-\ell}) \geqslant C_1 2^{-\ell \min(\gamma - \frac{\gamma\beta-\alpha}{\beta t}, \frac{\alpha}{t})}.$$

Since we claim that $\alpha \leqslant 1/2 \leqslant \gamma\beta$,

$$\max_{t\in(1,\beta)}\left(\min\left(\gamma - \frac{\beta\gamma-\alpha}{\beta t}, \frac{\alpha}{t}\right)\right) = \frac{\beta\alpha\gamma}{\beta\gamma+\alpha(\beta-1)},$$

and thus, for any $\ell \geqslant j_0$,

$$f(2^{-\ell}) \geqslant C_1 2^{-\overline{h}\ell},$$

where $\overline{h} = \gamma\beta\alpha/(\gamma\beta + \alpha(\beta - 1))$. In other words, the following relation has been proved for any $\ell \geqslant j_0$:

$$\sup_{[x,x+h]\subset B(0,2^{-\ell})} |f(x+h) - f(x)| \geqslant C2^{-\overline{h}\ell},$$

which gives the required upper bound for $\overline{h}_{f_{\alpha,\beta}}(0)$.

Let us now check for a lower bound for $\overline{h}_{f_{\alpha,\beta}}(0)$. Since $\psi \in C^{\gamma}(\boldsymbol{R})$,

$$\forall |x| \leqslant 2^{-\ell}, \qquad |\psi(2^j x)| \leqslant C_2 2^{(j-\ell)\gamma},$$

for some $C_2 > 0$. If $n \in \boldsymbol{N}$, let us set $\ell_n = j_{n+1} - 1$. Since $\mathrm{supp}(\psi) \subset [0, 3]$, we have, for any given n_0 and any $|x| \leqslant 2^{-\ell_{n_0}}$,

$$|f_{\alpha,\beta}(x)| \leqslant C_2 \sum_{n=0}^{n_0} 2^{-j_n\alpha} \sum_{j=j_n}^{j_{n+1}-1} 2^{j-\ell_{n_0}} \leqslant C_2 2^{-\ell_{n_0}\gamma} 2^{\ell_{n_0}\frac{\gamma\beta-\alpha}{\beta}}.$$

The same arguments as above lead to the following inequality:

$$\sup_{|x|\leqslant 2^{-\ell_n}} |f_{\alpha,\beta}(x) - f_{\alpha,\beta}(0)| \leqslant C2^{-\overline{h}\ell_n},$$

which allows to conclude.

Since $\gamma < 1$,

$$\frac{\beta\alpha\gamma}{\beta\gamma + \alpha(\beta - 1)} < \frac{\beta\alpha}{\beta + \alpha(\beta - 1)}$$

and proposition 4 is then a direct consequence of proposition 5 and proposition 6.

4. Proof of the prevalence results

4.1. Proof of theorem 1

The proof of our two prevalence results relies on the stochastic process technique. Recall that random element X on a complete metric space E is a measurable mapping X defined on a probability space $(\Omega, \mathcal{A}, P)$ with values in E. For any random element on E, one can define a probability on E by the formula

$$P_X(A) = P\{X \in A\}.$$

If we consider as measure μ, $\mu = P_X$ in the definition of a Haar-null set given in section 1, we see that in order to prove that a set is Haar-null, it is sufficient to check that

$$\forall f \in E, \qquad P_X(A + f) = 0.$$

We now show that the spaces $SM^{\alpha}(\boldsymbol{R}^d)$ are prevalent subsets of $C^{\alpha}(\boldsymbol{R}^d)$. Theorem 1 directly follows from proposition 7:

Proposition 7. *For f in a prevalent subset of $C^{\alpha}(\boldsymbol{R}^d)$, there exist $C_0 > 0$ and j_0 such that*

$$\forall j \geqslant j_0, \quad \forall \lambda \text{ such that } |\lambda| = 2^{-j}, \qquad |d_\lambda| \geqslant C_0 2^{-j\alpha}.$$

Remark 1. Proposition 7 also holds if we replace the notion of prevalence with a quasi-sure property based on the Baire's category theorem; see [14] (proposition 5).

Proof. Let us recall that the wavelet basis $(\psi^{(i)}_{j,k})_{i,j,k}$ is assumed to be compactly supported. Let (n_λ) be independently identically distributed (i.i.d.) Bernoulli random variables and consider the random field defined as follows:

$$X(x) = \sum_{i=1}^{2^d-1} \sum_{j\geqslant 0} \sum_{|k|\leqslant 2^{jd}} (-1)^{n_\lambda} 2^{-\alpha j} \psi_\lambda(x).$$

The sample paths of $\{X(x)\}_{x\in \boldsymbol{R}^d}$ belong to $C^\alpha(\boldsymbol{R}^d)$ almost surely. It is then sufficient to show that, for any function f belonging to $C^\alpha(\boldsymbol{R}^d)$, there exist some integer j_0 such that

$$\forall j \geqslant j_0, \quad \forall \lambda \quad \text{such that} \quad |\lambda| = 2^{-j}, \qquad d_\lambda(f+X) \geqslant \frac{2^{-j\alpha}}{2} \text{ a.s.} \tag{12}$$

To prove property (12), we use an approach similar to [2]. By definition of the wavelet leaders

$$P\left(d_\lambda(f+X) \leqslant \frac{2^{-j\alpha}}{2}\right) = P\left(\bigcap_{\lambda'\subset\lambda} |c'_\lambda(f+X)| \leqslant \frac{2^{-j\alpha}}{2}\right).$$

We now use the independence of the wavelet coefficients of $X+f$ and deduce that

$$\begin{aligned} P\left(d_\lambda(f+X) \leqslant \frac{2^{-j\alpha}}{2}\right) &= \prod_{\lambda'\subset\lambda} P\left(|c_{\lambda'}(f+X)| \leqslant \frac{2^{-j\alpha}}{2}\right) \\ &= \prod_{\lambda'\subset\lambda} P\left(-\frac{2^{-j\alpha}}{2} \leqslant c_{\lambda'}(f+X) \leqslant \frac{2^{-j\alpha}}{2}\right) \\ &= \prod_{\lambda'\subset\lambda} P\left(-\frac{2^{-j\alpha}}{2} - c_{\lambda'}(f) \leqslant c_{\lambda'}(X) \leqslant \frac{2^{-j\alpha}}{2} - c_{\lambda'}(f)\right). \end{aligned}$$

Let $j' = j + [\log(4\|f\|_{C^\alpha})/\log 2^\alpha] + 1$; one has

$$\begin{aligned} P\left(d_\lambda(f+X) \leqslant \frac{2^{-j\alpha}}{2}\right) &\leqslant \prod_{\lambda'\subset\lambda,\, |\lambda'|=2^{-j'}} P\left(-\frac{2^{-j\alpha}}{2} - c_{\lambda'}(f) \leqslant c_{\lambda'}(X) \leqslant \frac{2^{-j\alpha}}{2} - c_{\lambda'}(f)\right) \\ &\leqslant \prod_{\lambda'\subset\lambda,\, |\lambda'|=2^{-j'}} P\left(-\frac{1}{2} - \frac{c_{\lambda'}(f)}{2^{-j\alpha}} \leqslant (-1)^{n_{\lambda'}} \leqslant \frac{1}{2} - \frac{c_{\lambda'}(f)}{2^{-j\alpha}}\right) \end{aligned}$$

Since for such cubes λ' (with side length $2^{-j'}$),

$$|c_{\lambda'}| \leqslant \|f\|_{C^\alpha} 2^{-j'\alpha} \leqslant \|f\|_{C^\alpha} 2^{-j\alpha} \frac{1}{4\|f\|_{C^\alpha}} \leqslant \frac{2^{-j\alpha}}{4},$$

if $c_{\lambda'}(f) \geqslant 0$, one has $1/2 - c_{\lambda'}(f)/2^{-j\alpha} < 1$, whereas, if $c_{\lambda'}(f) < 0$ then $-1/2 - c_{\lambda'}(f)/2^{-j\alpha} > -1$. Thus, since $P((-1)^{\eta_{\lambda'}} = -1) = P((-1)^{\eta_{\lambda'}} = 1) = 1/2$,

$$P\left(-\frac{1}{2} - \frac{c_{\lambda'}(f)}{2^{-j\alpha}} \leqslant (-1)^{n_{\lambda'}} \leqslant \frac{1}{2} - \frac{c_{\lambda'}(f)}{2^{-j\alpha}}\right) \leqslant \frac{1}{2}.$$

Therefore,

$$P(d_\lambda(f+X) \leqslant 2^{-j\alpha} j^{-4\alpha}) \leqslant \left(\frac{1}{2}\right)^{j^2} \leqslant \exp(-j^2)$$

and thus

$$\sum_\lambda P\left(d_\lambda(f+X) \leqslant \frac{2^{-j\alpha}}{2}\right) \leqslant \sum_{j\in\boldsymbol{N}} \exp(-j^2) < \infty.$$

The Borel–Cantelli lemma then implies inequality (12), which is the required conclusion.

Proof of theorem 1. Theorem 1 then directly follows from proposition 7 and from the wavelet criterion for strongly monoHölder functions stated in theorem 4.

4.2. Proof of theorem 2

We first briefly recall the definition of the Hausdorff dimension (see, e.g., [11] for more details). Let $\delta > 0$ and define the quantity

$$\mathcal{H}^{\delta}_{\varepsilon}(E) = \inf\left\{\sum_{i=1}^{\infty} |E_i|^{\delta} : E \subset \bigcup_{i=1}^{\infty} E_i, |E_i| \leqslant \varepsilon\right\}.$$

The Hausdorff measure is defined from $\mathcal{H}^{\delta}_{\varepsilon}$ as ε goes to 0.

Definition 7. *The outer measure $\mathcal{H}^{\delta}$ defined as*

$$\mathcal{H}^{\delta}(E) = \sup_{\varepsilon>0} \mathcal{H}^{\delta}_{\varepsilon}(E)$$

is a metric outer measure. Its restriction to the σ-algebra of the $\mathcal{H}^{\delta}$-measurable sets defines the Hausdorff measure of dimension δ.

Since the outer measure $\mathcal{H}^{\delta}$ is metric, the σ-algebra includes the Borelian sets.

The Hausdorff measure $\mathcal{H}^{\delta}$ is decreasing. Moreover, $\mathcal{H}^{\delta}(E) > 0$ implies $\mathcal{H}^{\delta'}(E) = \infty$ if $\delta' < \delta$. We are then lead to the following definition.

Definition 8. *The Hausdorff dimension* $\dim_{\mathcal{H}}(E)$ *of a set $E \subset \boldsymbol{R}^d$ is defined as follows:*

$$\dim_{\mathcal{H}}(E) = \sup\{\delta : \mathcal{H}^{\delta}(E) = \infty\}.$$

We now prove that the relation $\dim_{\mathcal{H}}\Gamma(f) = d + 1 - \alpha$, connecting the Hausdorff dimension of the graph of a function f and its uniform Hölder exponent, is satisfied for any function belonging to a prevalent set of $C^{\alpha}(\boldsymbol{R}^d)$.

From now on, we will assume that the support of the wavelet ψ is a compact set not included in $[0, 1)^d$. The following result is directly obtained by considering proposition 1 and theorem 2 of [25].

Proposition 8. *Let X be the following random wavelet series:*

$$X(x) = \sum_{i=0}^{2^d-1} \sum_{j\geqslant 0} \sum_{k\in\boldsymbol{Z}^d} c_{\lambda}\psi_{\lambda}(x),$$

where c_{λ} are independent centred Gaussian random variables with standard deviation σ_{λ}. The following equality is satisfied almost surely:

$$\dim_{\mathcal{H}}\Gamma(X + f) \geqslant s,$$

where

$$s = \limsup_{J\to\infty} \liminf_{j\to\infty} (-j)^{-1} \log_2 \min_{j\leqslant\ell\leqslant j+J} \sum_{k} \min\left(1, \frac{2^{-\ell}}{\sqrt{2\pi}\sigma_{\lambda}}\right) 2^{-2\ell}.$$

We can now prove theorem 2. Let $(\xi_{\lambda(i,j,k)})_{i,j,k}$ be i.i.d. standard Gaussian random variables. We consider the following Gaussian field:

$$X(x) = \sum_{i=1}^{2^d-1} \sum_{j\geqslant 0} \sum_{|k|\leqslant 2^{jd}} \frac{\xi_{\lambda(i,j,k)}}{j^2\sqrt{\log j}} 2^{-\alpha j}\psi_{\lambda(i,j,k)}(x).$$

If f belongs to $C^{\alpha}(\boldsymbol{R}^d)$, $f + X$ belongs to $C^{\alpha'}(\boldsymbol{R}^d)$ for any $\alpha' < \alpha$ almost surely and thus

$$\dim_{\mathcal{H}} \Gamma(X + f) \leqslant d + 1 - \alpha.$$

Conversely, we have

$$\sigma_\lambda = \boldsymbol{E}\left| \frac{\xi_\lambda}{j^2\sqrt{\log j}} 2^{-\alpha j} \right|^2 = \frac{2^{-2\alpha j}}{j^4 \log j}$$

and thus

$$\begin{aligned}\sum_{|k| \leqslant 2^{\ell d}} \min\left(1, \frac{2^{-\ell}}{\sqrt{2\pi}\sigma_\lambda}\right) 2^{-2\ell d} &= \sum_{|k| \leqslant 2^{\ell d}} \min\left(1, \frac{2^{-\ell} 2^{\ell\alpha} \ell^2 \sqrt{\log(\ell)}}{\sqrt{2\pi}}\right) 2^{-2\ell d} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{|k| \leqslant 2^{\ell d}} 2^{\ell(\alpha - 1 - 2d)} \ell^2 \sqrt{\log(\ell)} \\ &= \frac{1}{\sqrt{2\pi}} 2^{\ell(\alpha-1-d)} \ell^2 \sqrt{\log \ell}.\end{aligned}$$

Finally, since $\alpha - 1 - d > 0$, we have, for any j and J,

$$\min_{j \leqslant \ell \leqslant j+J} \sum_k \min\left(1, \frac{2^{-\ell}}{\sqrt{2\pi}\sigma_\lambda}\right) 2^{-2\ell} = \frac{1}{\sqrt{2\pi}} 2^{j(\alpha-1-d)} j^2 \sqrt{\log j}.$$

This allows us to determine the index s of proposition 8:

$$\begin{aligned} s &= \limsup_{J\to\infty} \liminf_{j\to\infty} (-j)^{-1} \log_2 \left(\min_{j \leqslant \ell \leqslant j+J} 2^{\ell(\alpha-1-d)} \ell^2 \sqrt{\log \ell} \right) \\ &= \limsup_{J\to\infty} \liminf_{j\to\infty} (-j)^{-1} \log_2 \left(\frac{1}{\sqrt{2\pi}} 2^{j(\alpha-1-d)} j^2 \sqrt{\log j} \right) \\ &= d + 1 - \alpha. \end{aligned}$$

Therefore, we have, using proposition 8,

$$\dim_{\mathcal{H}} \Gamma(X + f) \geqslant d + 1 - \alpha$$

almost surely, which leads to the conclusion.

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