Two-Higgs-doublet model from the group-theoretic perspective

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Abstract

In the two-Higgs-doublet model, different Higgs doublets can be viewed as components of a generic "hyperspinor". We decompose the Higgs potential of this model into irreducible representations of the $SU(2)$ group of transformations of this hyperspinor. We discuss invariant combinations of Higgs potential parameters λ_i that arise in this decomposition and provide simple and concise sets of conditions for the hidden Z_2-symmetry, Peccei-Quinn symmetry, and explicit CP-conservation in 2HDM. We show that some results obtained previously by brute-force calculations are reduced to simple linear algebraic statements in our approach.

1   Introduction

The Electroweak Symmetry Breaking in the Standard Model is described usually with the Higgs mechanism. Its simplest realization is based on a single weak isodoublet of scalar fields, which couple to the gauge and matter fields and self-interact via the quartic potential, for review see [1]. Extended versions of the Higgs mechanisms are based on more elaborate Higgs sectors. The simplest extension is known as the two-Higgs-doublet model (2HDM), which makes use of two Higgs weak isodoublets of scalar fields ϕ_1 and ϕ_2. This model has been extensively studied in literature from various points of view, see [1, 2, 3, 4] and references therein.

The Higgs potential $V_H = V_2 + V_4$ contains quadratic and quartic parts, which in the most general case of the 2HDM are conventionally parametrized as

$$V_2 = -\frac{1}{2}\left[m_{11}^2(\phi_1^\dagger\phi_1) + m_{22}^2(\phi_2^\dagger\phi_2) + m_{12}^2(\phi_1^\dagger\phi_2) + m_{12}^{2\,*}(\phi_2^\dagger\phi_1)\right]; \tag{1}$$

$$V_4 = \frac{\lambda_1}{2}(\phi_1^\dagger\phi_1)^2 + \frac{\lambda_2}{2}(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) \tag{2}$$

$$+ \frac{1}{2}\left[\lambda_5(\phi_1^\dagger\phi_2)^2 + \lambda_5^*(\phi_2^\dagger\phi_1)^2\right] + \left\{\left[\lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)\right](\phi_1^\dagger\phi_2) + \text{h.c.}\right\}$$

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with 14 free parameters: real $m_{11}^2, m_{22}^2, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and complex $m_{12}^2, \lambda_5, \lambda_6, \lambda_7$. Note that in the case $m_{12}^2 = \lambda_6 = \lambda_7 = 0$ the potential remains invariant under transformations from the group Z_2:

$$\phi_1 \to -\phi_1\,, \phi_2 \to \phi_2\,, \quad \text{or} \quad \phi_1 \to \phi_1\,, \phi_2 \to -\phi_2\,. \tag{3}$$

The full Higgs lagrangian includes also kinetic terms, which can be off-diagonal in a general case, see discussion in [4]. Having written lagrangian, one usually proceeds with calculating physical observables in terms of this particular set of parameters.

Since the two Higgs doublets ϕ_1 and ϕ_2 have the same quantum numbers, they can be viewed as two components of a generic "hyperspinor". Unitary rotations that mix these two doublets leave the structure of the 2HDM potential and — more importantly — the physical observables unchanged, modifying only the parameters of the lagrangian. This freedom of choosing the basis for the two scalar doublets, known as the Higgs basis or reparametrization invariance, was exploited in discussion of symmetries and invariants of the 2HDM [4, 5, 6], CP-violation in the Higgs sector [4, 7, 8], perturbative unitarity conditions [9].

The importance of representing the lagrangian in a reparametrization-invariant way was appreciated as early as in 1974, [10], in the context of supersymmetric theories. In non-minimal SUSY, the two Higgs doublets were in fact members of a single Fayet-Sohnius hypermultiplet related with each other by the "flavor-$SU(2)$" symmetry, [11].

Recently, a general reformulation the entire theory of the most general 2HDM in an explicitly $SU(2)$-covariant form was presented in [6]. The Higgs potential was written, following [3], in a compact form as

$$V_H = Y_{ab}(\phi_a^\dagger \phi_b) + Z_{abcd}(\phi_a^\dagger \phi_b)(\phi_c^\dagger \phi_d)\,, \quad \text{with} \quad a, b = 1, 2\,, \tag{4}$$

and tensors of various ranks were constructed from Y_{ab} and Z_{abcd}. By contracting all the subscripts in different ways, a number of quantities invariant under $SU(2)$ transformations were derived. The physical meaning of these invariants, however, did not appear to be particularly clear. Some specific choices of the Higgs potential parameters were found to be important for the overall structure of the 2HDM [6] as well as for its CP-properties [7, 8], however their origin and interpretation were lacking.

In this note we present a simple yet revealing group-theoretic study of the basis invariance in 2HDM. We identify the quartic Higgs potential as a $C(2,2)$-tensor and decompose it in the sum of irreducible representations (irreps) of $SU(2)$. Several invariant combinations of λ_i with transparent origin follow immediately from this decomposition. Not intending to give an exhaustive list of all possible conclusions, we simply demonstrate that the group-theoretic point of view proves useful in understanding properties of 2HDM. We discuss in particular the conditions for the hidden Z_2-symmetry and for the CP-conservation in 2HDM. We show that many previous results of the brute-force calculations acquire an elegant interpretation in our approach.

2   Decomposing quartic potential into irreps

We consider the two scalar doublets ϕ_1 and ϕ_2 as two components of a "hyperspinor" $\psi_\alpha = (\phi_1,\ \phi_2)^T$, abstracting from the "internal" structure of these doublets. The most general

rotations of this spinor are operators from $SU(2) \times U(1)$

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = e^{i\rho_0} \begin{pmatrix} e^{i\rho_1} \cos\theta & e^{i\rho_2} \sin\theta \\ -e^{-i\rho_2} \sin\theta & e^{-i\rho_1} \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} , \tag{5}$$

which is parametrized by $3 + 1$ independent parameters θ, ρ_1, ρ_2 and ρ_0. A particular case of this transformation $\rho_1 = \rho_0$, $\theta = 0$ was introduced in [12] as "rephasing transformation" and was used to remove the imaginary part of the parameter λ_5. In [9] it was exploited in derivation of perturbative unitarity conditions for CP-violating case from those of the CP-conserving one.

The corresponding "antispinor" $\tilde{\psi}^\beta = (\phi_1^\dagger, \phi_2^\dagger)$ transforms under the antifundamental representation of $SU(2)$. With the aid of the antisymmetric tensor $\epsilon_{\gamma\beta}$, it can be mapped to usual contravariant spinor $\tilde{\psi}_\gamma = \epsilon_{\gamma\beta}\tilde{\psi}^\beta = (\phi_2, -\phi_1)^T$, which transforms under the fundamental representation.

Let us focus on the quartic Higgs potential (2). Being a quantity constructed from two ϕ_i and two $\phi_i^\dagger$, it corresponds to a vector in the 16-dimensional complex space $C(2, 2)$ spanned on the basis tensors $\Psi_{\alpha_1\alpha_2}^{\beta_1\beta_2}$ with $\alpha_i, \beta_i = 1, 2$, constructed from various combinations of ϕ_i and $\phi_i^\dagger$, or, alternatively, to a vector in the isomorphic space $C(4, 0)$ spanned on the basis vectorss

$$\Psi_{\gamma_1\gamma_2\alpha_1\alpha_2} = \epsilon_{\gamma_1\beta_1}\epsilon_{\gamma_2\beta_2}\Psi_{\alpha_1\alpha_2}^{\beta_1\beta_2} . \tag{6}$$

Since terms like $(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_1)$ and $(\phi_1^\dagger\phi_1)(\phi_1^\dagger\phi_2)$ represent the same pieces of the potential, we in fact work not in the full 16-dimensional space, but in its 10-dimentional subspace symmetric under $(\alpha_1, \beta_1) \leftrightarrow (\alpha_2, \beta_2)$. We will label this subspace with the subscript "sym".

The coordinates of this vector either in $C(2, 2)_{sym}$ or in $C(4, 0)_{sym}$ are combinations of the parameters λ_i. Their transformation under rotations (5) of fields ϕ_i and, correspondingly, of basis vectors (6) defines a $C(2, 2)_{sym}$ or in $C(4, 0)_{sym}$ tensor (essentially the same tensor as Z_{abcd} in (4)). The space of all possible parametrizations of the quartic Higgs potential realizes, therefore, the $C(2, 2)_{sym}$ [or $C(4, 0)_{sym}$] representation of the $SU(2)$ group. One easily find its decomposition into irreps

$$(2 \otimes 2 \otimes 2 \otimes 2)_{sym} = 5 \oplus 3 \oplus 1 \oplus 1 . \tag{7}$$

For a transparent derivation of this result, one merges first $\alpha_1\alpha_2$: $2 \otimes 2 = 3 \oplus 1$, where 3 is symmetric and 1 is antisymmetric under $\alpha_1 \leftrightarrow \alpha_2$. The same holds for $\gamma_1\gamma_2$. Their product gives $((3 \oplus 1) \otimes (3 \oplus 1))_{sym} = 3 \otimes 3 \oplus 1 \otimes 1 = 5 \oplus 3 \oplus 1 \oplus 1$. In the conventional notation, the basis vectors in the space $C(2, 2)_{sym}$ that form these multiplets are

$$\begin{aligned}
|2, +2\rangle &= (\phi_2^\dagger\phi_1)(\phi_2^\dagger\phi_1) , \\
|2, +1\rangle &= (\phi_2^\dagger\phi_1)(\phi_2^\dagger\phi_2) - (\phi_2^\dagger\phi_1)(\phi_1^\dagger\phi_1) , \\
\text{5-plet} \quad |2, 0\rangle &= \frac{1}{\sqrt{6}}\left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2 - 2(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - 2(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1)\right] , \quad (8) \\
|2, -1\rangle &= (\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_1) - (\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_2) , \\
|2, -2\rangle &= (\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_2) ,
\end{aligned}$$

$$\begin{aligned}
|1, +1\rangle &= (\phi_2^\dagger\phi_1)\left[(\phi_1^\dagger\phi_1) + (\phi_2^\dagger\phi_2)\right] , \\
\text{3-plet} \quad |1, 0\rangle &= \frac{1}{\sqrt{2}}\left[(\phi_2^\dagger\phi_2) - (\phi_1^\dagger\phi_1)\right]\left[(\phi_1^\dagger\phi_1) + (\phi_2^\dagger\phi_2)\right] , \quad (9) \\
|1, -1\rangle &= -(\phi_1^\dagger\phi_2)\left[(\phi_1^\dagger\phi_1) + (\phi_2^\dagger\phi_2)\right] ,
\end{aligned}$$

$$\text{singlet}_1 \qquad |0_1, 0\rangle \;=\; \tfrac{1}{2}\left[(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2)\right]^2 , \tag{10}$$

$$\text{singlet}_2 \qquad |0_2, 0\rangle \;=\; \tfrac{1}{\sqrt{12}}\left[\left((\phi_1^\dagger \phi_1) - (\phi_2^\dagger \phi_2)\right)^2 + 4(\phi_2^\dagger \phi_1)(\phi_1^\dagger \phi_2)\right]. \tag{11}$$

Decomposition of the quartic potential (2) into these multiplets

$$V_4 = \sum_m a_{2,m} \cdot |2, m\rangle + \sum_n b_{1,n} \cdot |1, n\rangle + c|0_1, 0\rangle + d|0_2, 0\rangle \tag{12}$$

yields the following coefficients

$$a_{2,+2} = \frac{\lambda_5^*}{2}, \quad a_{2,+1} = -\frac{\lambda_6^* - \lambda_7^*}{2}, \quad a_{2,0} = \frac{\lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4}{\sqrt{24}},$$

$$a_{2,-2} = \frac{\lambda_5}{2}, \quad a_{2,-1} = \frac{\lambda_6 - \lambda_7}{2},$$

$$b_{1,+1} = \frac{\lambda_6^* + \lambda_7^*}{2}, \quad b_{1,0} = -\frac{\lambda_1 - \lambda_2}{2\sqrt{2}}, \quad b_{1,-1} = -\frac{\lambda_6 + \lambda_7}{2}$$

$$c = \frac{\lambda_1 + \lambda_2 + 2\lambda_3}{4} \quad d = \frac{\lambda_1 + \lambda_2 - 2\lambda_3 + 4\lambda_4}{\sqrt{48}} \tag{13}$$

It is also convenient to exploit the homomorphism from $SU(2)$ to $SO(3)$, switching from the triplet $b_{1,n}$ to a real vector $\vec{b} = (b_x, b_y, b_z)$ with

$$b_x = -\frac{1}{\sqrt{2}}\mathrm{Re}(\lambda_6 + \lambda_7), \quad b_y = -\frac{1}{\sqrt{2}}\mathrm{Im}(\lambda_6 + \lambda_7), \quad b_z = -\frac{1}{2\sqrt{2}}(\lambda_1 - \lambda_2). \tag{14}$$

Corresponding transformation of the 5-plet will turn it into a real traceless tensor a_{ij} $(i, j = x, y, z)$

$$a_{ij} = \frac{1}{2}\begin{pmatrix} \mathrm{Re}\lambda_5 - a & \mathrm{Im}\lambda_5 & \mathrm{Re}(\lambda_6 - \lambda_7) \\ \mathrm{Im}\lambda_5 & -\mathrm{Re}\lambda_5 - a & \mathrm{Im}(\lambda_6 - \lambda_7) \\ \mathrm{Re}(\lambda_6 - \lambda_7) & \mathrm{Im}(\lambda_6 - \lambda_7) & 2a \end{pmatrix}, \tag{15}$$

where $a \equiv (\lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4)/6$.

The quadratic part V_2 of the Higgs potential (1) can also be decomposed into irreps

$$V_2 = \sum_n y_{1,n} \cdot |1, n\rangle + f|0_3, 0\rangle \tag{16}$$

with

$$y_{1,+1} = -\frac{m_{12}^{2*}}{2}, \quad y_{1,0} = \frac{m_{11}^2 - m_{22}^2}{2\sqrt{2}}, \quad y_{1,-1} = \frac{m_{12}^2}{2}, \quad f = -\frac{m_{11}^2 + m_{22}^2}{2\sqrt{2}}, \tag{17}$$

or, analogously to (14),

$$y_x = \frac{1}{\sqrt{2}}\mathrm{Re}\,m_{12}^2, \quad y_y = \frac{1}{\sqrt{2}}\mathrm{Im}\,m_{12}^2, \quad y_z = \frac{1}{2\sqrt{2}}(m_{11}^2 - m_{22}^2). \tag{18}$$

3 Consequences

3.1 Invariants

Several combinations of λ_i remain invariant under an arbitrary $SU(2)$ rotation. Our decomposition immediately reveals four of them related to weights of each irrep in (12). Two of them are linear and the other two are quadratic combinations of λ_i:

$$
\begin{aligned}
\text{singlet}_1: & \quad \lambda_1 + \lambda_2 + 2\lambda_3 = \text{ const}, \\
\text{singlet}_2: & \quad \lambda_1 + \lambda_2 - 2\lambda_3 + 4\lambda_4 = \text{ const}, \\
\text{3-plet}: & \quad (\lambda_1 - \lambda_2)^2 + 4|\lambda_6 + \lambda_7|^2 = \text{ const}, \\
\text{5-plet}: & \quad |\lambda_5|^2 + |\lambda_6 - \lambda_7|^2 + \frac{1}{12}(\lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4)^2 = \text{ const}.
\end{aligned}
\tag{19}
$$

In total, there are 7 algebraically independent invariant combinations of λ_i only. This is not surprising, since we have 10 parameters of the Higgs potential and 3 degrees of freedom in reparametrization transformations. By denoting $b_i^{(k)} \equiv (a^k)_{ij} b_j$, one can select the following independent invariants

$$
c, \ d, \ \text{Tr}(a^2), \ \text{Tr}(a^3), \ \vec{b}^2, \ (\vec{b}\vec{b}^{(1)}), \ \epsilon_{ijk} b_i b_j^{(1)} b_k^{(2)},
\tag{20}
$$

where the last combination is just the scalar triple product of vectors $\vec{b}$, $\vec{b}^{(1)}$, and $\vec{b}^{(2)}$.

Any other algebraic function of λ_i invariant under a generic reparametrization transformation can be expressed as an algebraic function (and not always a polynomial) of invariants (20). The proof consists in straightforward application of linear algebra to our problem. Consider first invariant combinations constructed from the matrix a_{ij} only. They are built of scalars $\text{Tr}(a^k) = x_1^k + x_2^k + x_3^k$, where x_i are the eigenvalues of matrix a_{ij}. Since the characteristic polynomial for a_{ij} has coefficients proportional to $\text{Tr}(a^2)$ and $\text{Tr}(a^3)$, and since any symmetric polynomial of x_1, x_2, x_3 can be written as a polynomial of $x_1 + x_2 + x_3 = 0$, $x_1 x_2 + x_2 x_3 + x_3 x_1 = -\text{Tr}(a^2)/2$, and $x_1 x_2 x_3 = \text{Tr}(a^3)/3$, it follows that $\text{Tr}(a^k)$ for $k > 3$ is always a polynomial of $\text{Tr}(a^2)$ and $\text{Tr}(a^3)$.

Now consider invariants that contain a_{ij} as well vector $\vec{b}$. There are at most three linearly independent vectors, which can be chosen $\vec{b}$, $\vec{b}^{(1)}$, and their cross-product $[\vec{b}, \vec{b}^{(1)}]$. Any other $b_i^{(k)}$ is expressible as a linear combination of these three. On passing to scalar invariants, one recovers the last three expressions in (20).

The way the invariants (19) are derived offers a transparent interpretation of some results of direct calculations. For example, the authors of [6], after inspection of some reduced tensors, note that if $\lambda_1 = \lambda_2, \lambda_6 = -\lambda_7$ holds in one basis, it will also hold after an arbitrary $SU(2)$ rotation. In our approach, this follows immediately from the fact that absence of the triplet in decomposition (12) in a basis independent statement.

If one takes into account also the quadratic part (1) of the Higgs potential, a number of other invariants arises. They include (20) with $\vec{b} \to \vec{y}$ and additional mixed invariants that involve both $\vec{b}$ and $\vec{y}$. Writing down all of them would be a lengthy but rather straightforward excercise. It offers a much simpler procedure to listing all independent invariants than the number-crunching approach of [8].

3.2 Hidden Z_2-symmetry

The specific case $m_{12}^2 = \lambda_6 = \lambda_7 = 0$ is of particular interest in 2HDM, since in this case the Z_2-symmetry (3) of the Higgs potential is restored. In the case $\lambda_6 = \lambda_7 = 0$, but $m_{12}^2 \neq 0$ one speaks about soft Z_2-violation.

In the case of explicitly broken Z_2 symmetry by presence of m_{12}^2, λ_6 or λ_7, it might be still possible that there exists some $SU(2)$ rotation which sets these coefficients to zero (the hidden Z_2 symmetry). A question then arises: under what condition a general 2HDM potential possesses this hidden Z_2 symmetry. Of course, a brute-force calculation (start from (2), perform $SU(2)$ rotation, set resulting λ_6' and λ_7' to zero) can yield those conditions in an explicit algebraic form, for necessary formulae see [4, 6, 8]. However, the group theoretical analysis performed above offers an elegant formulation of these conditions.

Setting $\lambda_6 = \lambda_7 = 0$ requires removing *simultaneously* $\lambda_6 + \lambda_7$ from vector $\vec{b}$ (14) (by choosing z axis along $\vec{b}$) and $\lambda_6 - \lambda_7$ from tensor a_{ij} (15), which amounts to bringing tensor to its principal axes (recall that once $\lambda_6 = \lambda_7 = 0$, simple rephasing transformation removes $\mathrm{Im}\lambda_5$). Removing m_{12}^2 means that, in addition, $\vec{y} \| \vec{b}$. All this is possible if and only if the direction of vector $\vec{b} \| \vec{y}$ coincides with one of the principal axes of the tensor a_{ij}. Put in algebraic terms,

> hidden Z_2 symmetry holds if and only if vectors $\vec{b}$ and $\vec{y}$ are collinear and are eigenvectors of a_{ij}. $\qquad$ (21)

Degenerate cases like absence of vector or tensor in decomposition (12) are also included.

Some choices of parametrization considered in [6, 8] again have transparent meaning in our approach. In particular, the quartic potential was shown there to possess the hidden Z_2-symmetry, in particular, in two cases: when $\lambda_1 = \lambda_2, \lambda_6 = -\lambda_7$ or when $\lambda_1 + \lambda_2 = 2\lambda_3 + 2\lambda_4$, $\lambda_5 = 0$ and $\lambda_6 = \lambda_7$. In our language this conclusion immediately follows from the absence of triplet or 5-plet in (12), which automatically fulfils the above requirement.

3.3 Global Peccei-Quinn $U(1)$-symmetry

If the potential possesses the hidden Z_2 symmetry and if, in addition, λ_5 happens to be zero in the basis where $m_{12}^2 = \lambda_6 = \lambda_7 = 0$, the Higgs potential is said to possess the global Peccei-Quinn (PQ) $U(1)$-symmetry [14], although this situation was considered in [10] even before the work of Peccei and Quinn in the context of supersymmetric theories. A close inspection of the matrix a_{ij} reveals the reparametrization-invariant criterion of the existence of this symmetry:

> the PQ symmetry holds, if and only if two eigenvalues of matrix a_{ij} coincide *and* vectors $\vec{b}$ and $\vec{y}$ are both eigenvectors of a_{ij} corresponding to *the other*, third, egienvalue. $\qquad$ (22)

The latter condition in (22) removes possibilities $|\lambda_5| = \pm 3a$, in which case the spectrum of a_{ij} is also degenerate, but no PQ symmetry is realized.

3.4 Explicit CP-conservation

The two-Higgs doublet model enjoys so much attention because it provides room for CP-violation originating from the Higgs sector [15]. A neccessary and sufficient condition for the

Higgs potential to explicitly conserve CP is that all the coefficients in the Higgs potential be real, after an appropriate reparametrization transformation (Theorem 1 in [8]).

As can be seen directly from (14), (15), (18), condition $\operatorname{Im} m_{12}^2 = \operatorname{Im}\lambda_5 = \operatorname{Im}\lambda_6 = \operatorname{Im}\lambda_7 = 0$ means "decoupling" of the y-direction from the x and z-directions. Formulated in a reparametrization-invariant way, this means that

> the Higgs potential is explicitly CP-conserving if and only if there exists an eigenvector of a_{ij} orthogonal to both $\vec{b}$ and $\vec{y}$. $\hspace{2cm}$ (23)

In a non-degenerate case, when both $\vec{b}$ and $\vec{y}$ are non-zero, the above statement means that the cross-product of the two vectors must be an eigenvector of a_{ij}:

$$a_{ij}\epsilon_{jkl}b_k y_l \propto \epsilon_{ikl}b_k y_l . \tag{24}$$

Contracting (24) with b_i or y_i, one finds two scalar conditions for CP-conservation:

$$\epsilon_{jkl}a_{ij}b_i b_k y_l = 0 \quad \text{and} \quad \epsilon_{jkl}a_{ij}y_i b_k y_l = 0 . \tag{25}$$

In a degenerate case, when $\vec{b} = 0$ or $\vec{y} = 0$, we are left with a half of the requirement (23). For example, if $\vec{b} = 0$, we require that $\vec{y}$ be orthogonal to some of the eigenvectors of a_{ij}. This takes place if and only if the triple scalar product

$$[\vec{y}, \vec{y}^{(1)}, \vec{y}^{(2)}] = 0 , \quad \text{where} \quad y_i^{(1)} \equiv a_{ij}y_j , \quad y_i^{(2)} \equiv a_{ij}y_j^{(1)} , \tag{26}$$

In the other degenerate case, when $\vec{y} = 0$, the condition for CP-conservation reads:

$$[\vec{b}, \vec{b}^{(1)}, \vec{b}^{(2)}] = 0 . \tag{27}$$

Both (26) and (27) are obviously reparametrization-invariant conditions.

The condition (23) is arguably more compact and transparent than those published before. The reparametrization invariant conditions for the Higgs potential in 2HDM to be explicitly CP-conserving have been analyzed recently in [7] and [8]. The results of these two studies did not coincide: the authors of [7] found three sufficient and necessary conditions for the Higgs potential to be CP-conserving, while authors [8] list four independent conditions.

The two papers agree that for a non-degenerate case vanishing of two invariants, denoted I_1 and I_2 in [7] and I_{Y3Z} and I_{2Y2Z} in [8], are necessary and sufficient for CP-conservation. It is for degenerate cases that the conclusions of these two papers differ. Authors of [7] argued that for $m_1 = m_2$ the two invariants vanish, while the possibility for the CP-violation still remains within the quartic part of the Higgs potential only. In order to remove this possibility, they set to zero another invariant I_3 constructed from product of 6 tensors Z_{abcd}.

The authors of [8] provide their counterpart of this invariant, I_{6Z}, and proceed further to analyze another degenerate case $\lambda_1 = \lambda_2$, $\lambda_6 + \lambda_7 = 0$. They argue that all three previous invariants vanish in this case, yet the possibility for the CP-violation remains. To eliminate it, one must set to zero a fourth invariant, I_{3Y3Z}. They claim that setting to zero these four invariants is necessary and sufficient for explicit CP-conservation in the Higgs sector.

Our study confirms the results of [8]. We see that last degenerate case considered there means in our notation $\vec{b} = 0$, therefore their fourth invariant must represent the same condition as our (26). The other three invariants also have clear counterparts in our notation. A question,

however, remains about the *minimal* set of conditions that would grasp all cases. Clearly, (26) and (27) alone are not sufficient, since they do not include the requirement that both vectors be orthogonal to *the same* eigenvector of a_{ij}.

It appears that the flaw in the analysis of [7] was overlooking the possibility $c_1 = c_2$, $\theta_2 = \theta_1 + \pi$, which amounts to $\lambda_6 + \lambda_7 = 0$ in the usual notation. This leads to $I_1 = I_2 = 0$ without fixing the value of θ_1. Assuming, in addition, $a_1 = a_2$, one makes zero also the third invariant I_3, still without fixing θ_1. To remove this residual possibility for CP-violation, one must impose another, the fourth, condition.

In our analysis, we do not discuss *spontaneous CP*-violation, since it requires knowledge of the hyperspinor of the vacuum expectation values.

3.5 RG evolution of λ_i

The approach proposed here can also help better understand the generic properties of the renormalization-group (RG) evolution flows of quartic coupling constants λ_i in 2HDM. These equations were explicitly written in [13] in a general tensor-like form. In our approach, these are replaced by equations on matrix a_{ij}, on vector b_i and on scalars c and d.

Restoration of symmetries under the RG flows is a well known phenomenon, for instance, in $O(n)$ models, see [16, 17]. One might expect similar phenomena to happen in 2HDM. In order to see this from equations without solving them, one can write these equations in a generic tensorial form as

$$\frac{da_{ij}}{dt} = \beta_a \left(a_{ik}a_{kj} - \frac{1}{3}\mathrm{Tr}(a^2)\delta_{ij} \right) + \beta_b \left(b_ib_j - \frac{1}{3}b^2\delta_{ij} \right) + trivial\,, \tag{28}$$

$$\frac{db_i}{dt} = \beta_{ab}\,a_{ij}b_j + trivial\,. \tag{29}$$

Here *trivial* indicates terms with trivial tensorial structure, *i.e.* $\propto a_{ij}$ for (28) and $\propto b_i$ for (29). We focus now on the relative orientation of the vector $\vec{b}$ and the principal axes of a_{ij}. One sees that there is a Z_2-symmetric fixed point of these equations. There are grounds to expect that this fixed point is *stable*. Indeed, in a suitable gauge theory with a single $SU(N)$ field [18] (instead of the electroweak $SU(2) \times U(1)$ structure), the asymptotically-free solution will have

$$a_{ij}(t) = \frac{\bar{a}_{ij}}{t}\,, \quad b_i(t) = \frac{\bar{b}_i}{t^k}\,,$$

for some $k \geq 1$ to be determined from the equations, which turns (28) and (29) into algebraic equations. In particular, (29) after this substitution simply states that vector $\vec{b}$ is an eigenvector of a_{ij}. We conclude that the Z_2 symmetry is restored in this particular theory in the scaling limit.

Although in the true electroweak theory such a simple scaling limit does not exist, the tendency towards restoration of Z_2 symmetry still might take place. Indeed, contraction of a real symmetric matrix with a real vector can be represented as a double vector product $[\vec{\omega}[\vec{\omega}\vec{b}]]$ modulo to diagonal terms. Thus, vector $\vec{b}$ is attracted towards one of the principal axes of a_{ij} under the RG evolution. A more detailed analysis is needed to establish the symmetry dynamics under the RG flow.

3.6 Extensions and generalizations

In the above study we did not provide expressions for the vacuum expectation values of the 2HDM. This analysis requires further study, since in this case the fundamental degree of freedom, the hyperspinor ψ_α splits into two daughter hyperspinors of the vacuum expectation values and residual fields. The equation for the hyperspinor of vacuum expectation values was written in [6] without explicit solution. On the other hand, the properties of vacuum in the 2HDM have received recently some attention, [19]. We expect that our group-theoretical analysis might be extended to embrace these issues as well. Algebraic structures that should arise along these lines require a closer look.

Finally, we note that our approach is applicable to more involved realizations of spontaneous breaking of the electroweak symmetry. For example, in a general n-Higgs doublet model (nHDM), the quartic potential has $n^2(n^2 + 1)/2$ terms (including mutually conjugate ones) giving rise to equally many real parameters λ_i. The number of irreps in $C(2,2)_{sym}$ is 5 for $n = 3$ and 6 for $n > 3$, including two scalars. Thus, for nHDM one can identify two invariants linear in λ_i, 3 (for $n = 3$) or 4 (for $n > 3$) invariants quadratic in λ_i, and many further invariants of higher order, which can be obtained by contracting various irreps. Although in this case the simplicity of the linear algebraic analysis of 2HDM case is lost, one might still expect to gain some insight from the group-theoretic approach.

4 Summary

We presented a simple and transparent group-theoretic interpretation of several phenomena in 2HDM. We treated the quartic potential as a vector in $C(2,2)$ space formed by the hyperspinor (ϕ_1, ϕ_2) and decomposed it into a sum of irreducible representations of the $SU(2)$ group: two scalars, one triplet and one 5-plet. This helped us reduce lengthy tensor analysis of 2HDM to simple linear algebra.

Within our formalism, we discussed the meaning of some invariants, which have been discovered previously with the aid of straightforward calculations. We presented simple, explicit and reparametrization-invariant conditions for Z_2-symmetry, Peccei-Quinn symmetry and CP-conservation of the Higgs potential in 2HDM. We argue that further development of this approach both to 2HDM and more involved Higgs sectors should prove very interesting.

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