A Full Discontinuous Galerkin Formulation Of Euler Bernoulli Beams In Linear Elasticity With Fractured Mechanic Applications

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Topics

• Dynamic Fracture by Cohesive Approach

• Key principles of DG methods

• C0/DG formulation of thin structures

• Fracture of thin structures
  – Full DG formulation of beams
  – DG/Extrinsic cohesive law combination
  – Numerical example

• Conclusions & Perspectives
Dynamic Fracture by Cohesive Approach

• Two methods
  – Intrinsic Law
    • Cohesive elements inserted from the beginning
    • Drawbacks:
      – Efficient if a priori knowledge of the crack path
      – Mesh dependency [Xu & Needleman, 1994]
      – Initial slope modifies the effective elastic modulus
      – This slope should tend to infinity [Klein et al. 2001]:
        » Alteration of a wave propagation
        » Critical time step is reduced
  – Extrinsic Law
    • Cohesive elements inserted on the fly when failure criterion is verified [Ortiz & Pandolfi 1999]
    • Drawback
      – Complex implementation in 3D (parallelization)
  • New DG/extrinsic method [Seagraves, Jerusalem, Radovitzky, Noels]
    – Interface elements inserted from the beginning
    – Consistent and scalable approach
Key principles of DG methods

• **Main idea**
  – Finite-element discretization
  – Same **discontinuous** polynomial approximations for the

• **Test** functions $\varphi_h$ and
• **Trial** functions $\delta \varphi$

  – Definition of operators on the interface trace:
    • **Jump** operator: $[\bullet] = \bullet^+ - \bullet^-$
    • **Mean** operator: $\langle \bullet \rangle = \frac{\bullet^+ + \bullet^-}{2}$

  – Continuity is weakly enforced, such that the method
    • Is consistent
    • Is stable
    • Has the optimal convergence rate
Key principles of DG methods

- Application to non-linear mechanics
  - Formulation in terms of the first Piola stress tensor $P$
    \[
    \nabla_0 \cdot P^T = 0 \text{ in } \Omega \quad \& \quad \begin{cases} 
    P \cdot N = \bar{T} \text{ on } \partial_N \Omega \\
    \varphi_h = \bar{\varphi}_h \text{ on } \partial_D B 
    \end{cases}
    \]
  - New weak formulation obtained by integration by parts on each element $\Omega^e$
    \[
    \sum_{e} \int_{\Omega^e_0} \nabla_0 \cdot P^T (\varphi_h) \cdot \delta \varphi \, dB = 0 \\
    \sum_{e} \int_{\Omega^e_0} -P (\varphi_h) : \nabla_0 \delta \varphi \, dB + \sum_{e} \int_{\partial \Omega^e_0} \delta \varphi \cdot P (\varphi_h) \cdot N \, d\partial B = 0 \\
    \int_{\partial B_0} P (\varphi_h) : \nabla_0 \delta \varphi \, dB + \int_{\partial B_0} [\delta \varphi \cdot P (\varphi_h)] \cdot N^- \, d\partial B = \int_{\partial_N B_0} \bar{T} \cdot \delta \varphi \, d\partial B
    \]
Key principles of DG methods

• Interface term rewritten as the sum of 3 terms
  
  Introduction of the numerical flux $h$
  
  $\int_{\partial_1 B_0} \left[ \delta \varphi \cdot P(\varphi_h^+) \right] \cdot N^- \, \, d\partial B \rightarrow \int_{\partial_1 B_0} \left[ \delta \varphi \right] \cdot h(P^+, P^-, N^-) \, \, d\partial B$

  • Has to be consistent:
    
    $h(P^+, P^-, N^-) = -h(P^-, P^+, N^+)$
    
    $h(P_{\text{exact}}, P_{\text{exact}}, N^-) = P_{\text{exact}} \cdot N^-$

  • One possible choice:
    
    $h(P^+, P^-, N^-) = \langle P \rangle \cdot N^-$

  • Weak enforcement of the compatibility
    
    $\int_{\partial_1 B_0} \left[ \varphi_h^+ \right] \cdot \left\langle \frac{\partial P}{\partial F} : \nabla_0 \delta \varphi \right\rangle \cdot N^- \, \, d\partial B$

  • Stabilization controlled by parameter $\beta$, for all mesh sizes $h^s$
    
    $\int_{\partial_1 B_0} \left[ \varphi_h^+ \right] \otimes N^- \cdot \left\langle \frac{\beta}{h^s} \frac{\partial P}{\partial F} \right\rangle : \left[ \delta \varphi \right] \otimes N^- \, \, d\partial B$

  Noels & Radovitzky, IJNME 2006 & JAM 2006

• Those terms can also be explicitly derived from a variational formulation (Hu-Washizu-de Veubeke functional)
Key principles of DG methods

• Combination with extrinsic cohesive law
  – Scalable & Consistent
C0/DG formulation of thin structures

• Previous developments for thin bodies
  – Continuous field / discontinuous derivative
    • No new nodes
    • Weak enforcement of $C^1$ continuity
    • Displacement formulations of high-order differential equations
    • Usual shape functions in 3D (no new requirement)
  • Applications to
    – Beams, plates [Engel et al., CMAME 2002; Hansbo & Larson, CALCOLO 2002; Wells & Dung, CMAME 2007]
    – Linear & non-linear shells [Noels & Radovitzky, CMAME 2008; Noels IJNME 2009]
    – Damage & Strain Gradient [Wells et al., CMAME 2004; Molari, CMAME 2006; Balachandran et al. 2008]
C0/DG formulation of thin structures

- **Deformation mapping**
  \[
  F = \nabla \Phi \circ \left[ \nabla \Phi_0 \right]^{-1} \quad \text{with} \quad \nabla \Phi = g_i \otimes E_i \quad \& \quad g_i = \nabla \Phi E_i = \frac{\partial \Phi}{\partial \xi_i}
  \]

- **Resultant stress**
  - Tension
    \[
    n^\alpha = \frac{1}{\bar{j}} \int_{h_{\min}}^{h_{\max}} \sigma g^\alpha \det(\nabla \Phi) \, d\xi^3
    \]
  - Bending
    \[
    \tilde{m}^\alpha = \frac{1}{\bar{j}} \int_{h_{\min}}^{h_{\max}} \xi^3 \sigma g^\alpha \det(\nabla \Phi) \, d\xi^3
    \]

- **Shearing is neglected**
  - As \( t = \frac{\varphi_{,1} \wedge \varphi_{,2}}{||\varphi_{,1} \wedge \varphi_{,2}||} \) \( \Rightarrow \)
    \[
    \begin{align*}
    t^\alpha &= \chi^\mu_{,\alpha} \varphi_{,\mu} \\
    \bar{j} &= ||\varphi_{,1} \wedge \varphi_{,2}||
    \end{align*}
    \]

  - The formulation is displacement based only
  - Continuity on \( t \) is ensured weakly by DG method

\[\Phi_0 = \phi_0(\xi^1, \xi^2) + \xi^3 t_0(\xi^1, \xi^2)\]

\[\chi = \Phi_0 \Phi^{-1}\]
C0/DG formulation of thin structures

- Pinched open hemisphere
  - Properties:
    • 18-degree hole
    • Thickness 0.04 m; Radius 10 m
    • Young 68.25 MPa; Poisson 0.3
    • Quadratic, cubic & distorted el.
  - Comparison of the DG methods with literature

![Graph showing comparison of DG methods](image-url)
Fracture of Thin Structures

- Extension of DG/ECL combination to shells
  - We have to substitute the C0/DG formulation by a full DG
Fracture of Thin Structures

- Kinematics of linear beams
  - Beam’s equation are deduced from Kirchhoff-Love shell kinematics
    - So the DG formulations can be related to each other

- This time DG method is applied to
  - Shape functions
  - Derivative of shape functions
Fracture of Thin Structures

• Full DG/ECL combination for Euler-Bernoulli beams
  – When rupture criterion is satisfied at an interface element
    • Shift from
      – DG terms ($\alpha_s = 0$)
      – Cohesive terms ($\alpha_s = 1$)
      – $\gamma_s = 1$ until the end of fracture process $\gamma_s = 0$

\[
\begin{align*}
\sum_n \int_{l_e} \left[ n^{11} \delta u_{1,1} + m^{11} \delta(-u_{3,11}) \right] dx \\
+ \sum_s \left\{ (1 - \alpha_s) \left( \langle n^{11} \rangle \[ \delta u_1 \] + \langle Eh \delta u_{1,1} \rangle \[ u_1 \] + \left[ u_1 \right] \left( \frac{\beta_2 Eh}{h_s} \right) \[ \delta u_1 \] \right) \\
+ \langle m^{11} \rangle \[ \delta(-u_{3,11}) \] + \left( \frac{Eh^3}{12} \delta(-u_{3,11}) \right) \left[ -u_{3,11} \right] + \left[ -u_{3,11} \right] \left( \frac{\beta_1 Eh^3}{12 h_s} \right) \left[ -\delta u_{3,11} \right] \right\} \\
+ \gamma_s \left[ u_3 \right] \left( \frac{\beta_3 Eh}{2(1 + \nu) h_s} \right) \[ \delta u_3 \] \\
+ \sum_s \alpha_s \left( N(\Delta_{true}^*) \delta \left[ u_1 \right] + M(\Delta_{true}^*) \delta \left[ -u_{3,11} \right] \right) = 0
\end{align*}
\]

– What remain to be defined are the cohesive terms
Fracture of Thin Structures

• New cohesive law for Euler-Bernoulli beams
  – Should take into account a through the thickness fracture
    • Problem: no element on the thickness
    • Very difficult to separate fractured and not fractured parts
  – Solution:
    • Application of cohesive law on
      – Resultant stress
        \( n^{11} \rightarrow N(\Delta^*) \)
      – Resultant bending stress
        \( \tilde{m}^{11} \rightarrow M(\Delta^*) \)
    • In terms of a resultant opening \( \Delta^* \)
Fracture of Thin Structures

- Resultant opening $\Delta^*$ and cohesive laws $N(\Delta^*)$ & $M(\Delta^*)$

  - Defined such that
    - At fracture initiation
      - $N_0 = N(0)$ and $M_0 = M(0)$
      - $\sigma(\pm h/2) = \pm \sigma_{\text{max}}$
    - After fracture
      - Energy dissipated = $h G_C$

  - Solution
    - $\Delta^* = (1 - \beta) \Delta_x + \beta \frac{h}{6} \Delta_r$
      - $\Delta_x$ = Opening is tension and $\Delta_r$ = Opening in rotation
      - Coupling parameter
        - $\beta = \frac{|6hM_0|}{N_0 + |6hM_0|}$
    - Null resistance for $\Delta^* = \Delta_c = 2G_C/\sigma_{\text{max}}$
Fracture of Thin Structures

• Numerical example
  – DCB with pre-strain

• When the maximum stress is reached Beam should shift from a DCB configuration to 2 SCB configurations
• During the rupture process (2 cases)
  1. The variation of internal energy is larger than $hG_C$
     » rupture is achieved in 1 increment of displacement
  2. The variation of internal energy is smaller than $hG_C$
     » Complete rupture is achieved only if flexion is still increased
     » Whatever the pre-strain, after rupture, the energy variation should correspond to $hG_C$
Fracture of Thin Structures

- Instable fracture
  - Geometry such that variation of internal energy > $hG_C$
Fracture of Thin Structures

- **Stable fracture**
  - Geometry such that variation of internal energy $< hG_C$
Fracture of Thin Structures

- Stable fracture
  - Effect of pre-strain
    - Dissipated energy always = $hG_C$
Conclusions & Perspectives

• Development of discontinuous Galerkin formulations
  – Formulation of high-order differential equations
    • Full DG formulation of beams
      – New degree of freedom
      – No rotation degree or freedom
      – As interface elements exist: cohesive law can be inserted

• Perspectives:
  – Extension to non-linear shells
  – Plasticity & ductile material