A Full Discontinuous Galerkin Formulation Of Euler Bernoulli Beams In Linear Elasticity With Fractured Mechanic Applications

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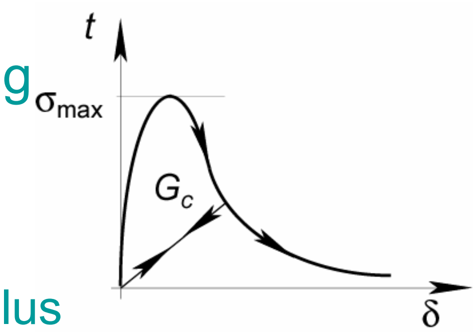
WCCM/APCOM – July 2010

Topics

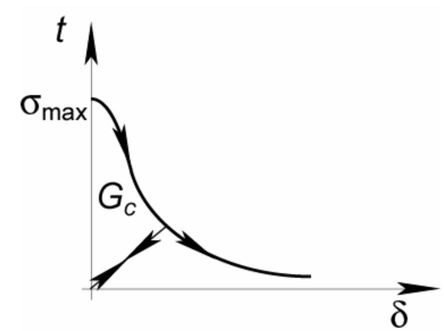
- Dynamic Fracture by Cohesive Approach
- Key principles of DG methods
- C0/DG formulation of thin structures
- Fracture of thin structures
  - Full DG formulation of beams
  - DG/Extrinsic cohesive law combination
  - Numerical example
- Conclusions & Perspectives

Dynamic Fracture by Cohesive Approach

- Two methods
  - Intrinsic Law
    - Cohesive elements inserted from the beginning
    - Drawbacks:
      - Efficient if a priori knowledge of the crack path
      - Mesh dependency [Xu & Needelman, 1994]
      - Initial slope modifies the effective elastic modulus
      - This slope should tend to infinity [Klein et al. 2001]:
        - » Alteration of a wave propagation
        - » Critical time step is reduced



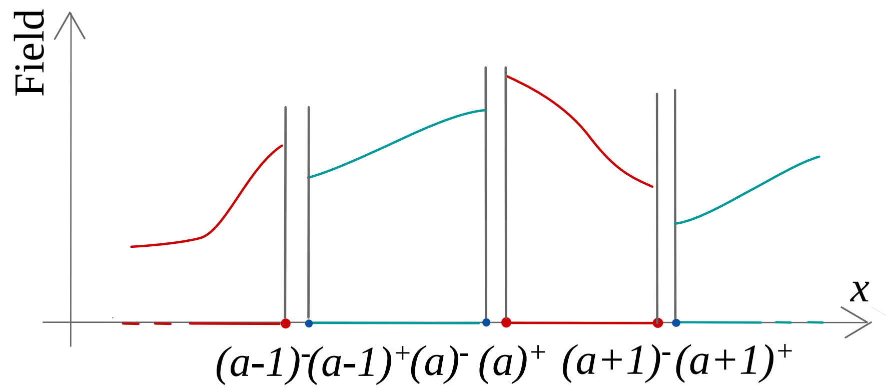
  - Extrinsic Law
    - Cohesive elements inserted on the fly when failure criterion is verified [Ortiz & Pandolfi 1999]
    - Drawback
      - Complex implementation in 3D (parallelization)



- New DG/extrinsic method [Seagraves, Jerusalem, Radovitzky, Noels]
  - Interface elements inserted from the beginning
  - Consistent and scalable approach

Key principles of DG methods

- Main idea
  - Finite-element discretization
  - Same **discontinuous** polynomial approximations for the
    - **Test** functions φ_h and
    - **Trial** functions $\delta\varphi$



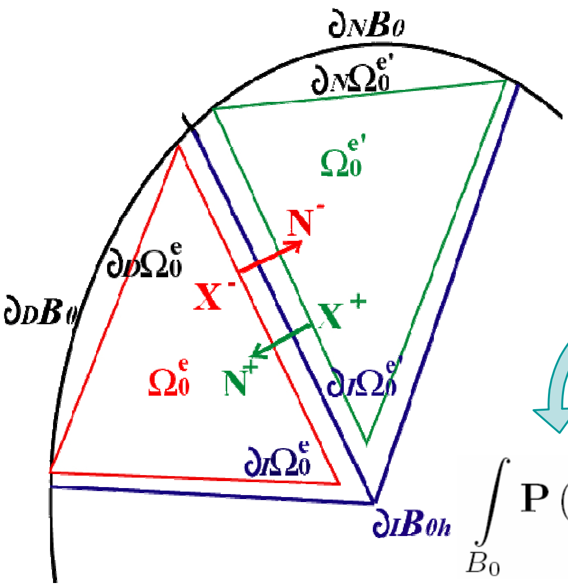
  - Definition of operators on the interface trace:
    - **Jump** operator: $[\![\bullet]\!] = \bullet^+ - \bullet^-$
    - **Mean** operator: $\langle\bullet\rangle = \frac{\bullet^+ + \bullet^-}{2}$
  - Continuity is weakly enforced, such that the method
    - Is consistent
    - Is stable
    - Has the optimal convergence rate

Key principles of DG methods

- Application to non-linear mechanics
  - Formulation in terms of the first Piola stress tensor $\mathbf{P}$

$$\nabla_0 \cdot \mathbf{P}^T = 0 \text{ in } \Omega \quad \& \quad \begin{cases} \mathbf{P} \cdot \boldsymbol{N} = \bar{\boldsymbol{T}} \text{ on } \partial_N \Omega \\ \varphi_h = \bar{\varphi}_h \text{ on } \partial_D B \end{cases}$$

  - New weak formulation obtained by integration by parts **on each element** Ω^e



$$\sum_e \int_{\Omega_0^e} \nabla_0 \cdot \mathbf{P}^T(\varphi_h) \cdot \delta\varphi \, dB = 0$$

$$\sum_e \int_{\Omega_0^e} -\mathbf{P}(\varphi_h) : \nabla_0 \delta\varphi \, dB + \sum_e \int_{\partial\Omega_0^e} \delta\varphi \cdot \mathbf{P}(\varphi_h) \cdot \boldsymbol{N} \, d\partial B = 0$$

$$\int_{B_0} \mathbf{P}(\varphi_h) : \nabla_0 \delta\varphi \, dB + \int_{\partial_I B_0} [\![\delta\varphi \cdot \mathbf{P}(\varphi_h)]\!] \cdot \boldsymbol{N}^- \, d\partial B = \int_{\partial_N B_0} \bar{\boldsymbol{T}} \cdot \delta\varphi \, d\partial B$$

?

Key principles of DG methods

- Interface term rewritten as the sum of 3 terms
  - Introduction of the numerical flux $\boldsymbol{h}$

$$\int_{\partial_I B_0} [\![\delta\boldsymbol{\varphi} \cdot \mathbf{P}(\boldsymbol{\varphi}_h)]\!] \cdot \boldsymbol{N}^- \, d\partial B \rightarrow \int_{\partial_I B_0} [\![\delta\boldsymbol{\varphi}]\!] \cdot \boldsymbol{h}\left(\mathbf{P}^+, \mathbf{P}^-, \boldsymbol{N}^-\right) \, d\partial B$$

    - Has to be consistent: $\begin{cases} \boldsymbol{h}\left(\mathbf{P}^+, \mathbf{P}^-, \boldsymbol{N}^-\right) = -\boldsymbol{h}\left(\mathbf{P}^-, \mathbf{P}^+, \boldsymbol{N}^+\right) \\ \boldsymbol{h}\left(\mathbf{P}_{\text{exact}}, \mathbf{P}_{\text{exact}}, \boldsymbol{N}^-\right) = \mathbf{P}_{\text{exact}} \cdot \boldsymbol{N}^- \end{cases}$
    - One possible choice: $\boldsymbol{h}\left(\mathbf{P}^+, \mathbf{P}^-, \boldsymbol{N}^-\right) = \langle \mathbf{P} \rangle \cdot \boldsymbol{N}^-$
  - Weak enforcement of the compatibility

$$\int_{\partial_I B_0} [\![\boldsymbol{\varphi}_h]\!] \cdot \left\langle \frac{\partial \mathbf{P}}{\partial \mathbf{F}} : \boldsymbol{\nabla}_0 \delta\boldsymbol{\varphi} \right\rangle \cdot \boldsymbol{N}^- \, d\partial B$$

  - Stabilization controlled by parameter β, for all mesh sizes h^s

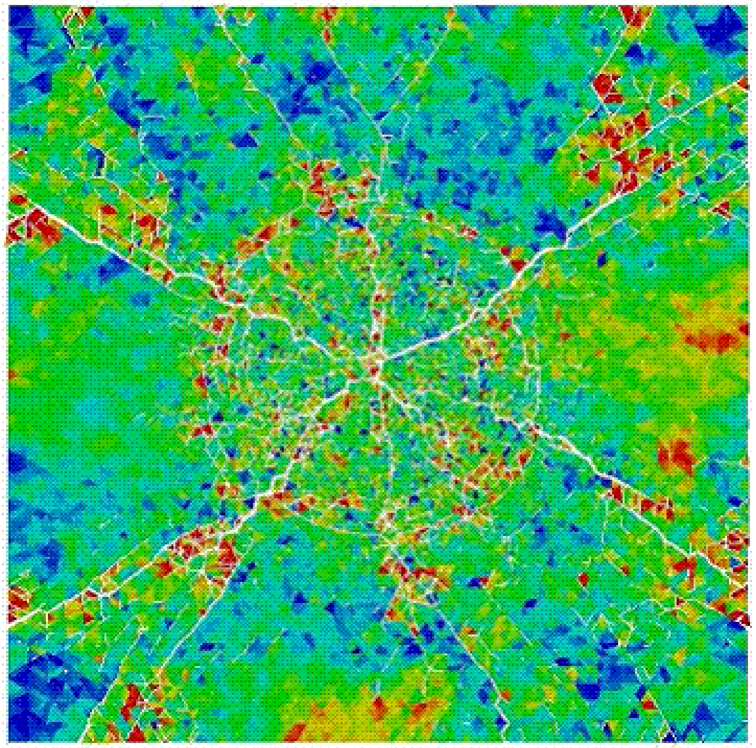
$$\int_{\partial_I B_0} [\![\boldsymbol{\varphi}_h]\!] \otimes \boldsymbol{N}^- : \left\langle \frac{\beta}{h^s} \frac{\partial \mathbf{P}}{\partial \mathbf{F}} \right\rangle : [\![\delta\boldsymbol{\varphi}]\!] \otimes \boldsymbol{N}^- \, d\partial B$$

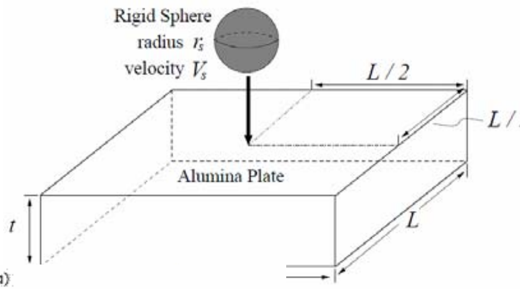
Noels & Radovitzky, IJNME 2006 & JAM 2006

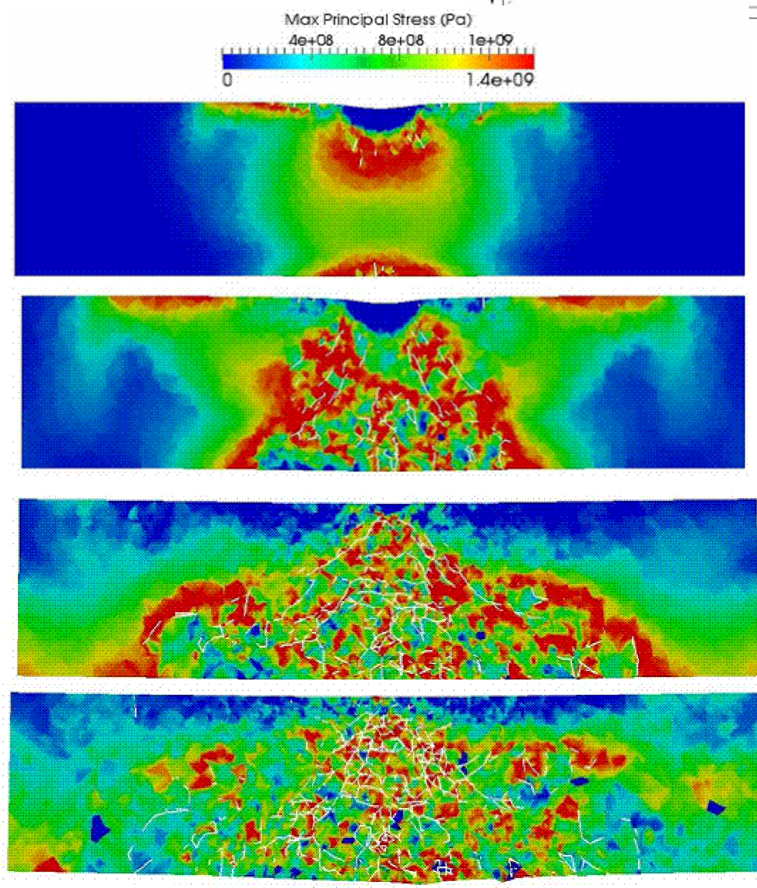
  - Those terms can also be explicitly derived from a variational formulation (Hu-Washizu-de Veubeke functional)

Key principles of DG methods

- Combination with extrinsic cohesive law
  - Scalable & Consistent



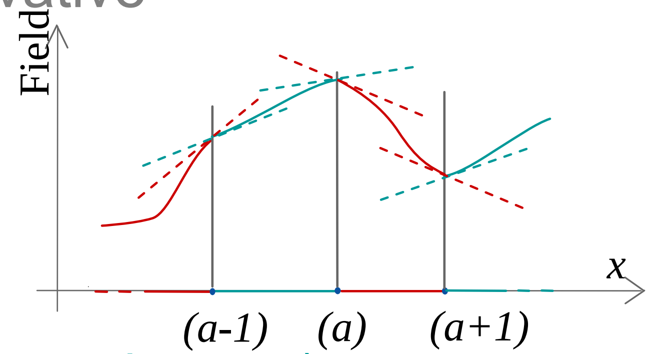




Radovitzky, Seagraves, Tupek, Noels CMAME Submitted

C0/DG formulation of thin structures

- Previous developments for thin bodies
  - Continuous field / discontinuous derivative
    - No new nodes
    - Weak enforcement of C^1 continuity
    - Displacement formulations of high-order differential equations



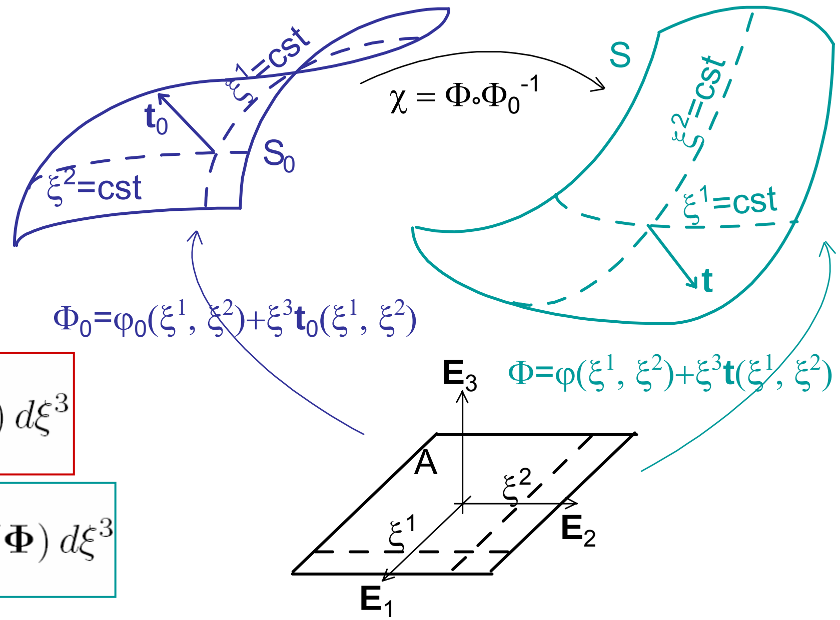
    - Usual shape functions in 3D (no new requirement)
    - Applications to
      - Beams, plates [Engel et al., CMAME 2002; Hansbo & Larson, CALCOLO 2002; Wells & Dung, CMAME 2007]
      - Linear & non-linear shells [Noels & Radovitzky, CMAME 2008; Noels IJNME 2009]
      - Damage & Strain Gradient [Wells et al., CMAME 2004; Molari, CMAME 2006; Bala-Chandran et al. 2008]

C0/DG formulation of thin structures

- Deformation mapping

  $\mathbf{F} = \nabla\Phi \circ [\nabla\Phi_0]^{-1}$ with

  $\nabla\Phi = \boldsymbol{g}_i \otimes \boldsymbol{E}^i$ & $\boldsymbol{g}_i = \nabla\Phi \boldsymbol{E}_i = \frac{\partial\Phi}{\partial\xi^i}$

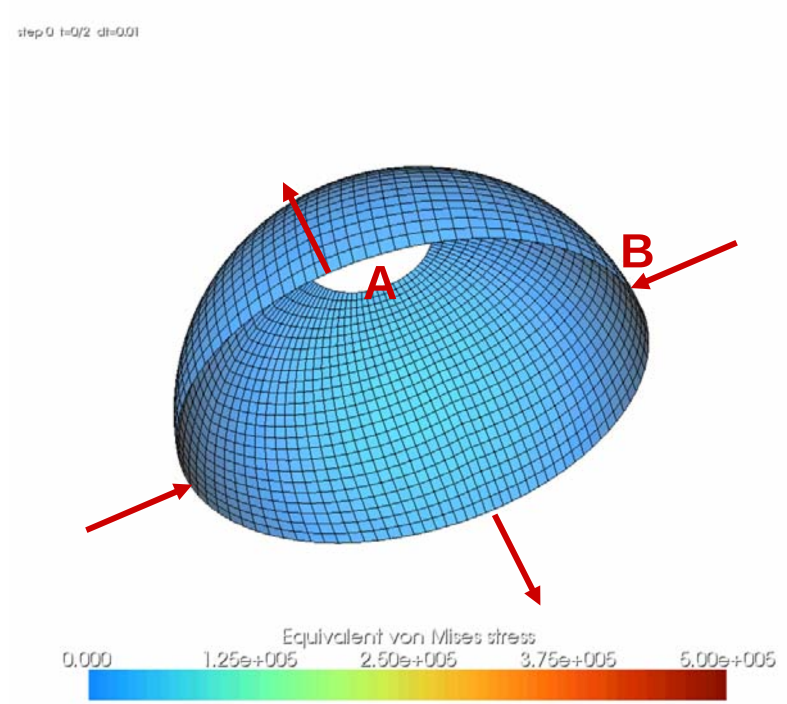


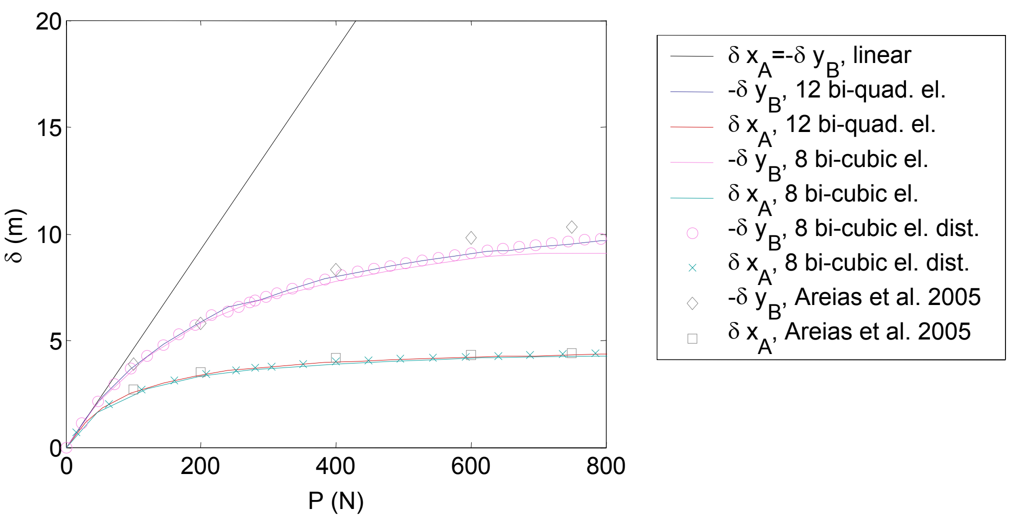
- Resultant stress
  - Tension $\boldsymbol{n}^\alpha = \frac{1}{\bar{j}} \int_{h_{\min 0}}^{h_{\max 0}} \boldsymbol{\sigma}\boldsymbol{g}^\alpha \det(\nabla\Phi)\, d\xi^3$
  - Bending $\tilde{\boldsymbol{m}}^\alpha = \frac{1}{\bar{j}} \int_{h_{\min 0}}^{h_{\max 0}} \xi^3 \boldsymbol{\sigma}\boldsymbol{g}^\alpha \det(\nabla\Phi)\, d\xi^3$

- Shearing is neglected
  - As $\boldsymbol{t} = \frac{\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}}{\|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\|}$ ⟹ $\begin{cases} \boldsymbol{t}_{,\alpha} = \lambda^\mu_\alpha \boldsymbol{\varphi}_{,\mu} \\ \bar{j} = \|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\| \end{cases}$
  - The formulation is displacement based only
  - Continuity on $\boldsymbol{t}$ is ensured weakly by DG method

C0/DG formulation of thin structures

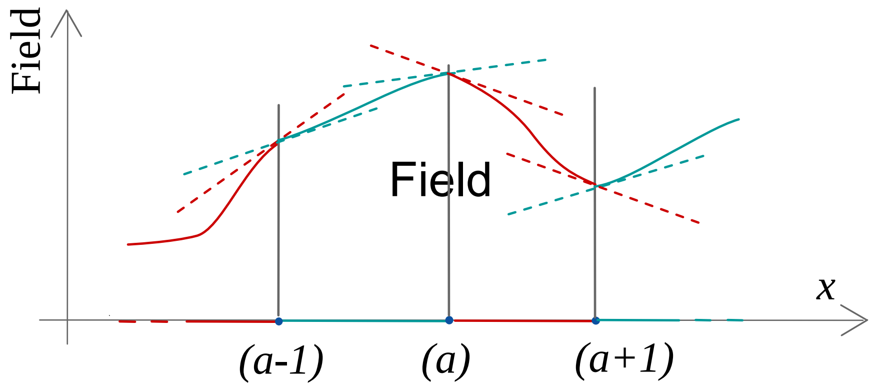
- Pinched open hemisphere
  - Properties:
    - 18-degree hole
    - Thickness 0.04 m; Radius 10 m
    - Young 68.25 MPa; Poisson 0.3
    - Quadratic, cubic & distorted el.
  - Comparison of the DG methods with literature

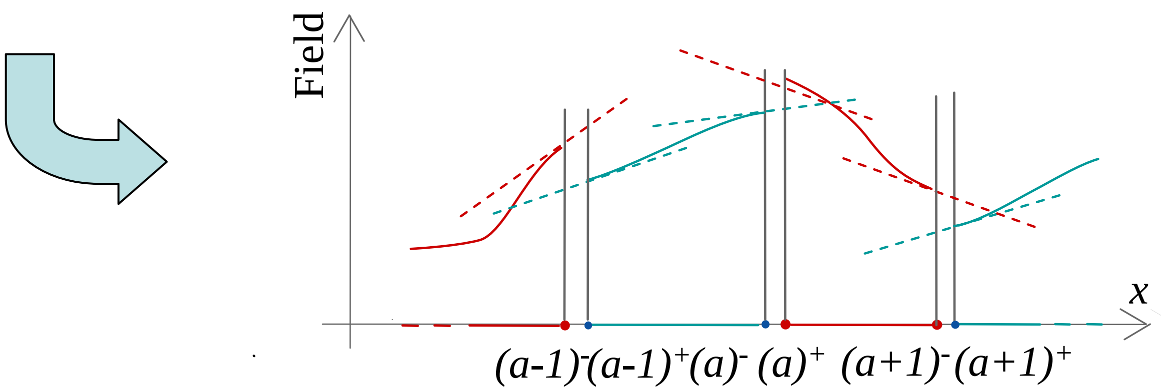




Fracture of Thin Structures

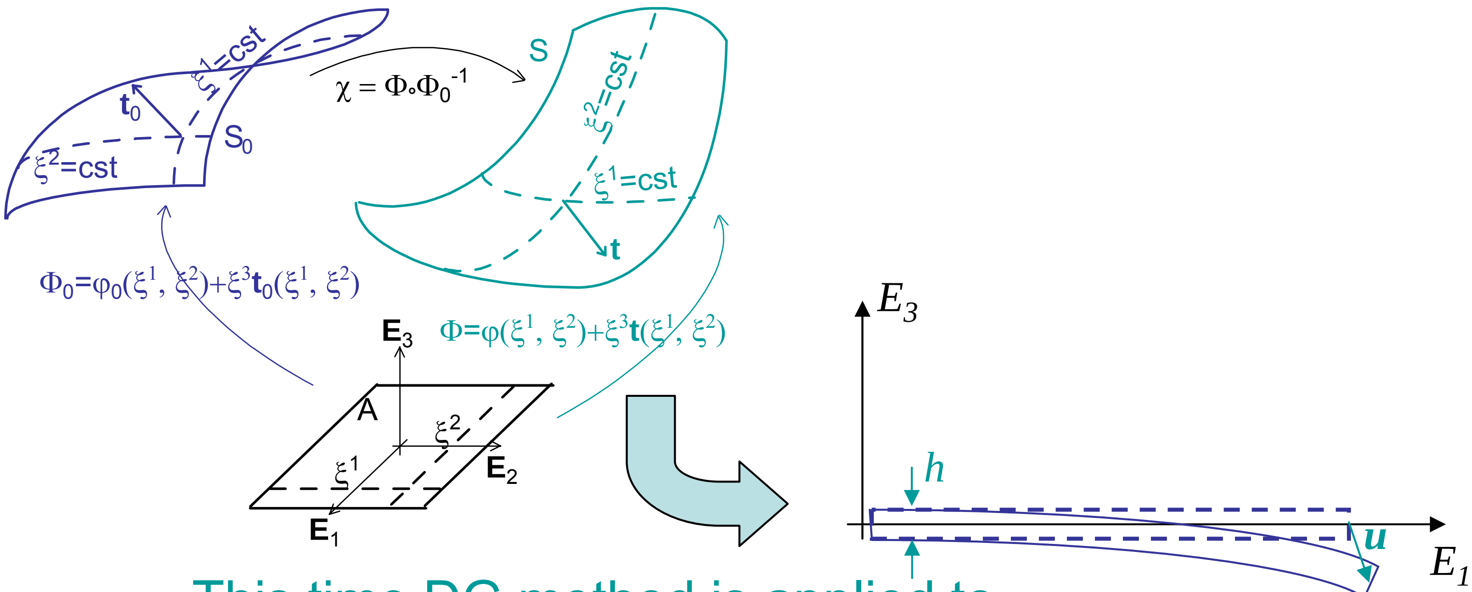
- Extension of DG/ECL combination to shells
  - We have to substitute the C0/DG formulation by a full DG





Fracture of Thin Structures

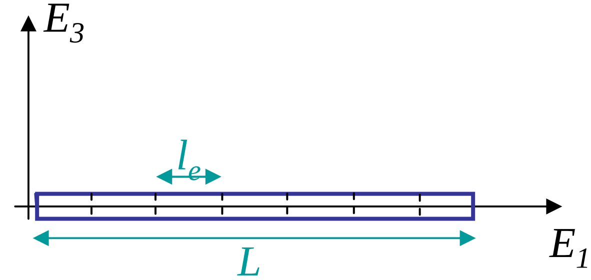
- Kinematics of linear beams
  - Beam's equation are deduced from Kirchhoff-Love shell kinematics
    - So the DG formulations can be related to each other



- This time DG method is applied to
  - Shape functions
  - Derivative of shape functions

Fracture of Thin Structures

- Full DG/ECL combination for Euler-Bernoulli beams
  - When rupture criterion is satisfied at an interface element
    - Shift from
      - DG terms ($\alpha_s = 0$) to
      - Cohesive terms ($\alpha_s = 1$)
      - $\gamma_s = 1$ until the end of fracture process $\gamma_s = 0$



$$
\begin{aligned}
&\sum_n \int_{l_e} \left[n^{11}\delta u_{1,1} + m^{11}\delta(-u_{3,11}) \right] dx \\
&+ \sum_s \Bigg\{ (1-\alpha_s) \Bigg(\langle n^{11} \rangle [\![\delta u_1]\!] + \langle Eh\delta u_{1,1} \rangle [\![u_1]\!] + [\![u_1]\!] \left\langle \frac{\beta_2 Eh}{h_s} \right\rangle [\![\delta u_1]\!] \\
&+ \langle m^{11} \rangle [\![\delta(-u_{3,1})]\!] + \left\langle \frac{Eh^3}{12}\delta(-u_{3,11}) \right\rangle [\![-u_{3,1}]\!] + [\![-u_{3,1}]\!] \left\langle \frac{\beta_1 Eh^3}{12 h_s} \right\rangle [\![-\delta u_{3,1}]\!] \Bigg) \\
&+ \gamma_s [\![u_3]\!] \left\langle \frac{\beta_3 Eh}{2(1+\nu)h_s} \right\rangle [\![\delta u_3]\!] \Bigg\} \\
&+ \sum_s \alpha_s \left(N(\Delta^*_{true})\delta [\![u_1]\!] + M(\Delta^*_{true})\delta [\![-u_{3,1}]\!] \right) = 0
\end{aligned}
$$

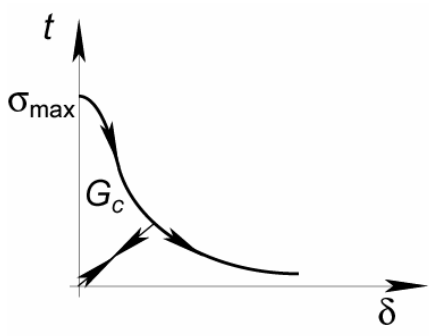
  - What remain to be defined are the cohesive terms

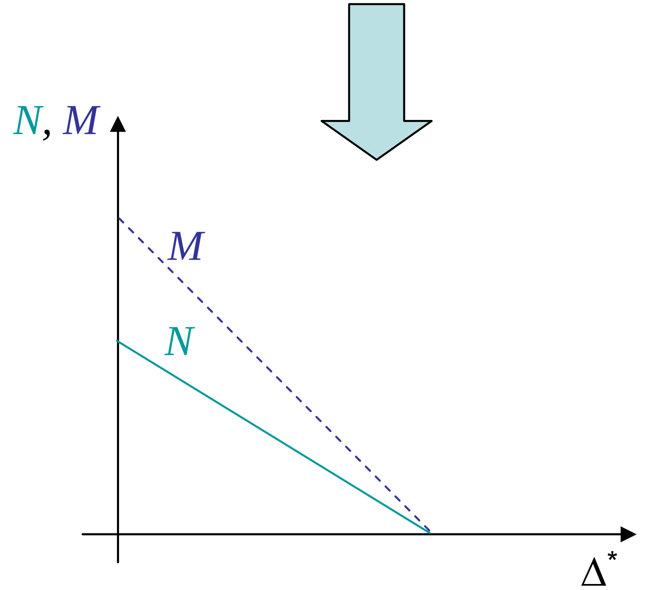
Fracture of Thin Structures

- New cohesive law for Euler-Bernoulli beams
  - Should take into account a through the thickness fracture
    - Problem : no element on the thickness
    - Very difficult to separate fractured and not fractured parts
  - Solution:
    - Application of cohesive law on
      - Resultant stress

        $n^{11} \Rightarrow N(\Delta^*)$
      - Resultant bending stress

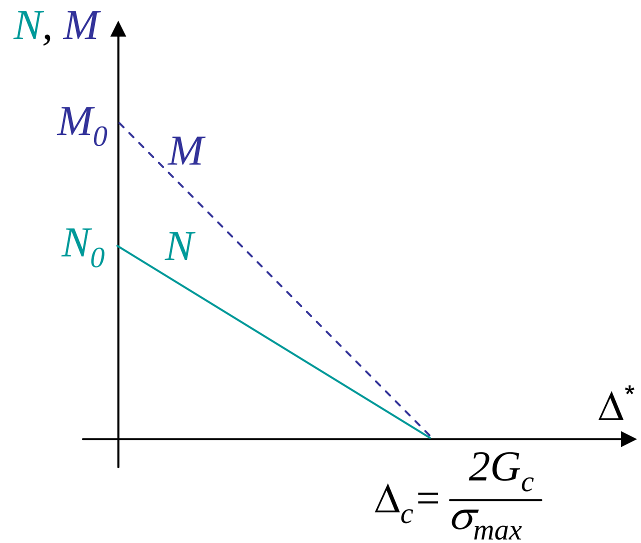
        $\tilde{m}^{11} \Rightarrow M(\Delta^*)$
    - In terms of a resultant opening Δ^*





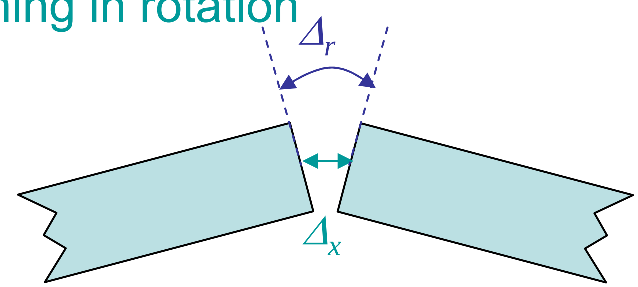
Fracture of Thin Structures

- Resultant opening Δ^* and cohesive laws $N(\Delta^*)$ & $M(\Delta^*)$
  - Defined such that
    - At fracture initiation
      - $N_0 = N(0)$ and $M_0 = M(0)$ satisfy $\sigma(\pm h/2) = \pm\,\sigma_{max}$
    - After fracture
      - Energy dissipated = $h\,G_C$



  - Solution
    - $\Delta^* = (1-\beta)\Delta_x + \beta\frac{h}{6}\Delta_r$
      - Δ_x = Opening is tension and Δ_r = Opening in rotation
      - Coupling parameter

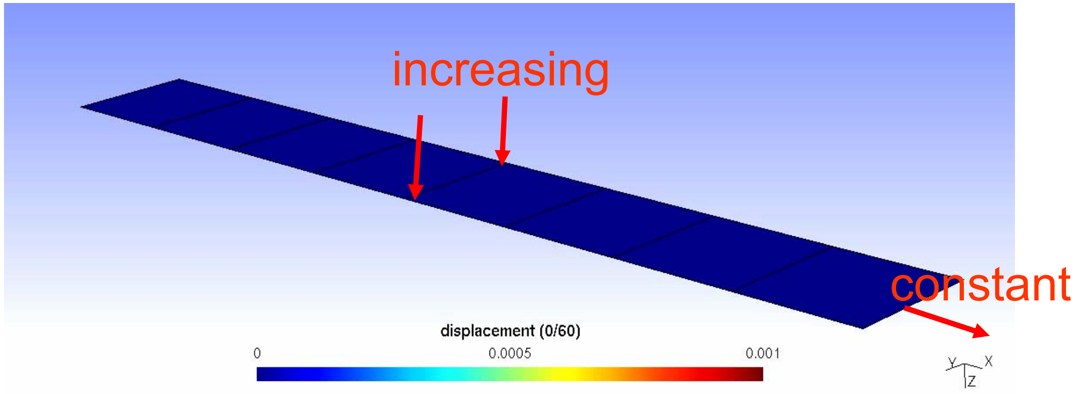
$$\beta = \frac{|6/hM_0|}{N_0 + |6/hM_0|}$$



    - Null resistance for $\Delta^* = \Delta_c = 2G_C/\sigma_{max}$

Fracture of Thin Structures

- Numerical example
  - DCB with pre-strain



- When the maximum stress is reached Beam should shift from a DCB configuration to 2 SCB configurations
- During the rupture process (2 cases)
  1. The variation of internal energy is larger than hG_C
     » rupture is achieved in 1 increment of displacement
  2. The variation of internal energy is smaller than hG_C
     » Complete rupture is achieved only if flexion is still increased
     » Whatever the pre-strain, after rupture, the energy variation should correspond to hG_C

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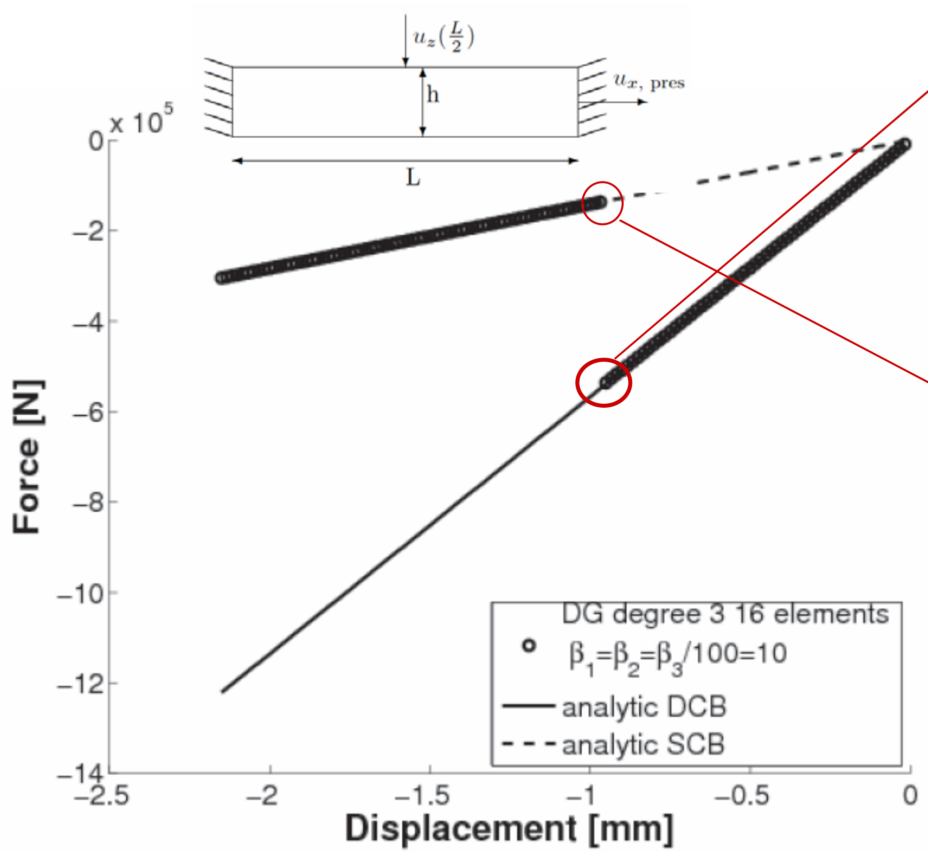
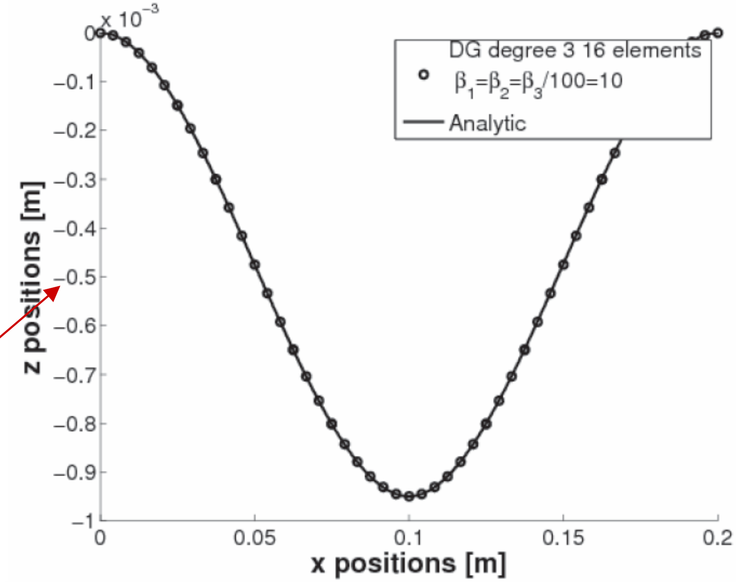
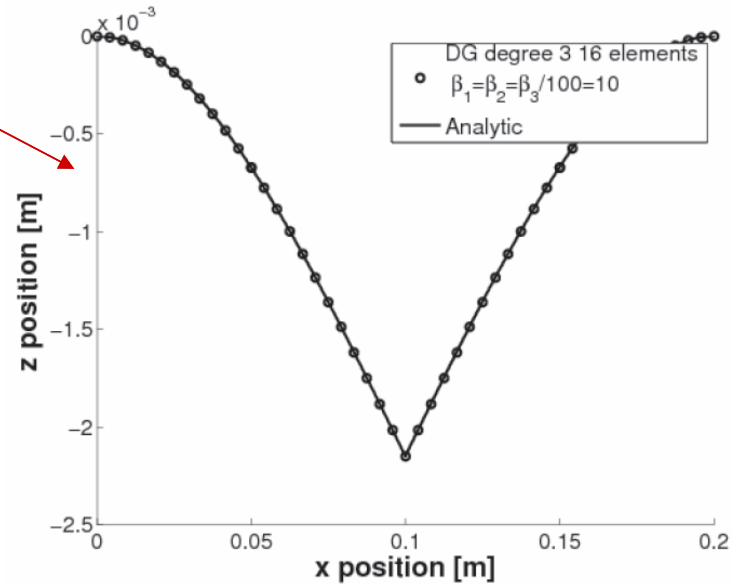
Fracture of Thin Structures

- Instable fracture
  - Geometry such that variation of internal energy > hG_C







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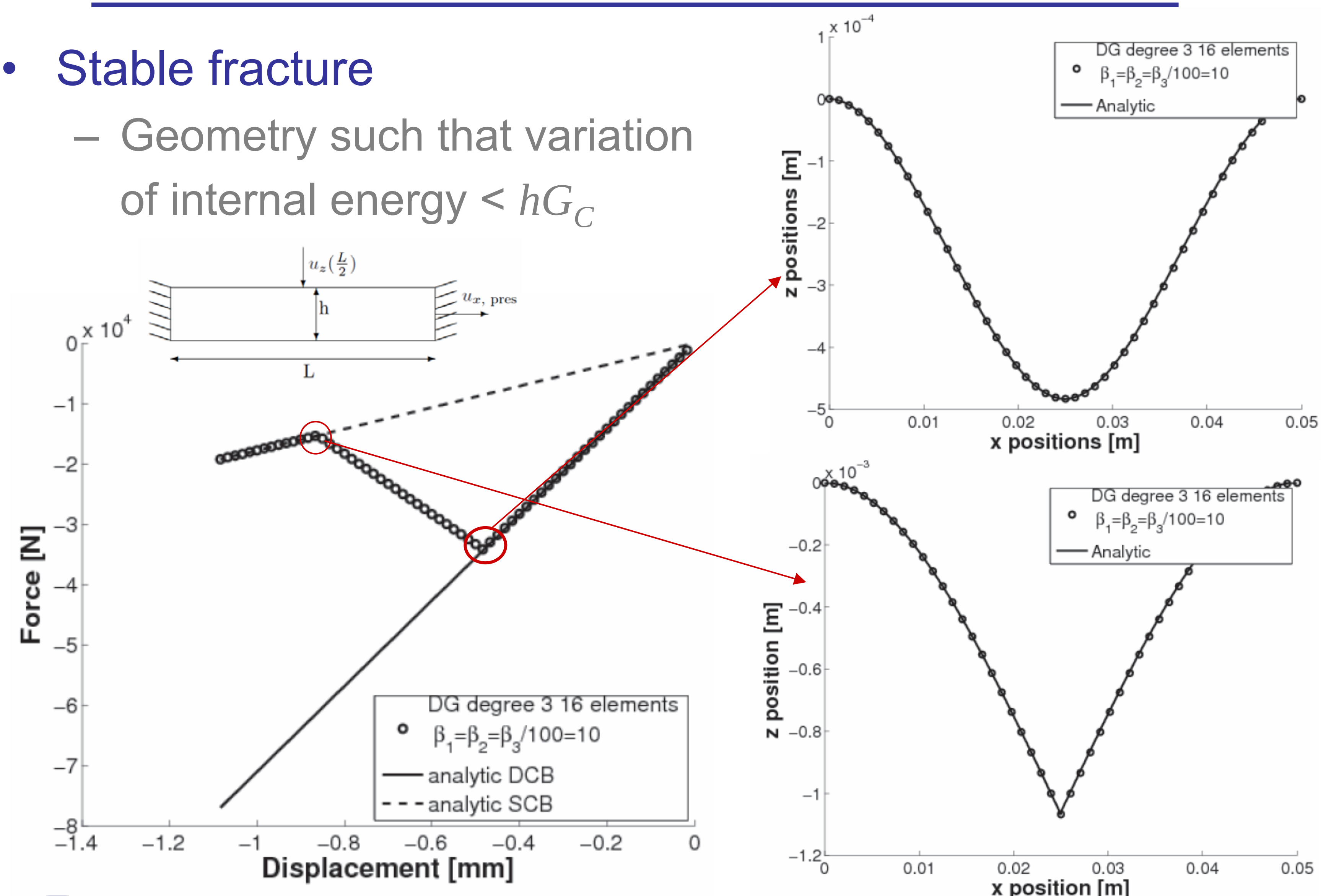
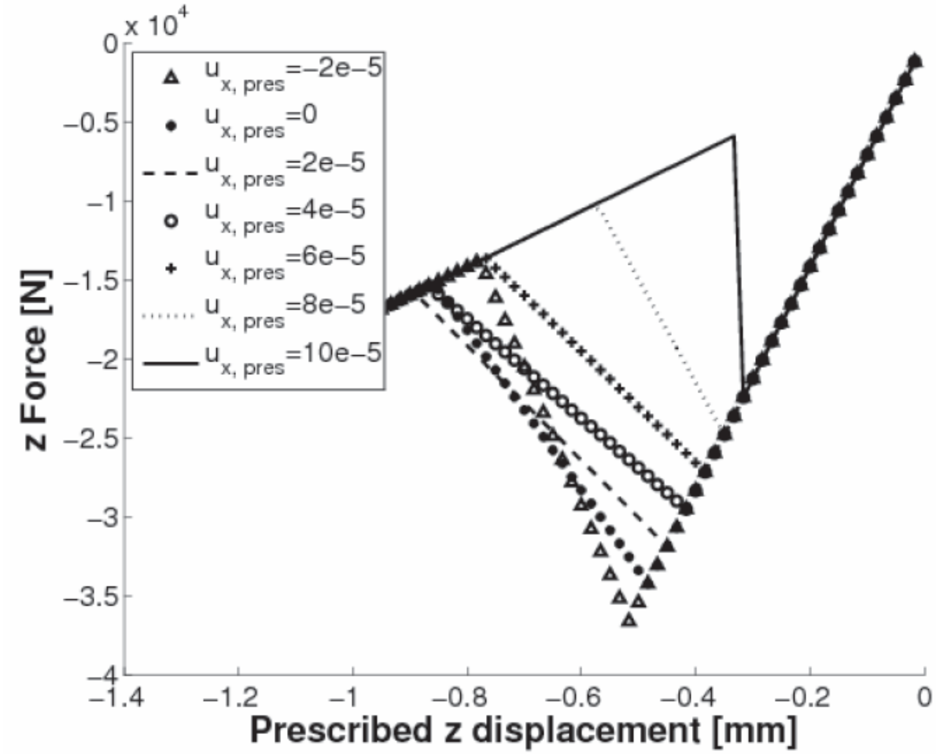
Fracture of Thin Structures

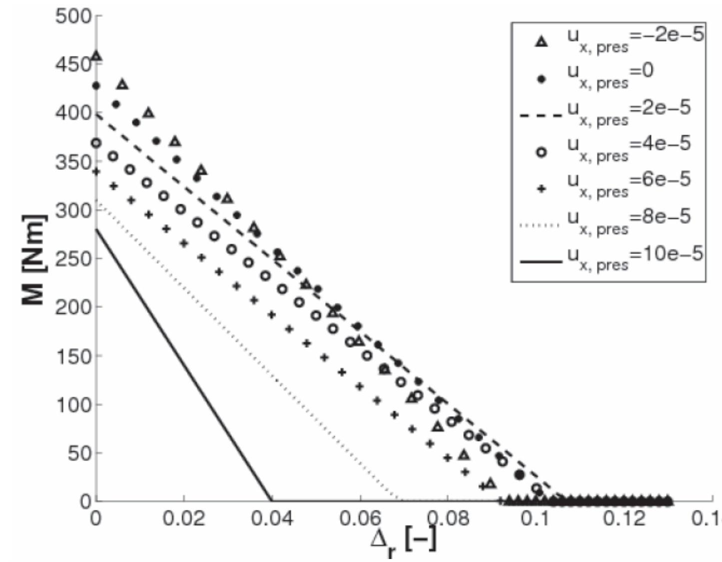
- Stable fracture
  - Geometry such that variation of internal energy < hG_C

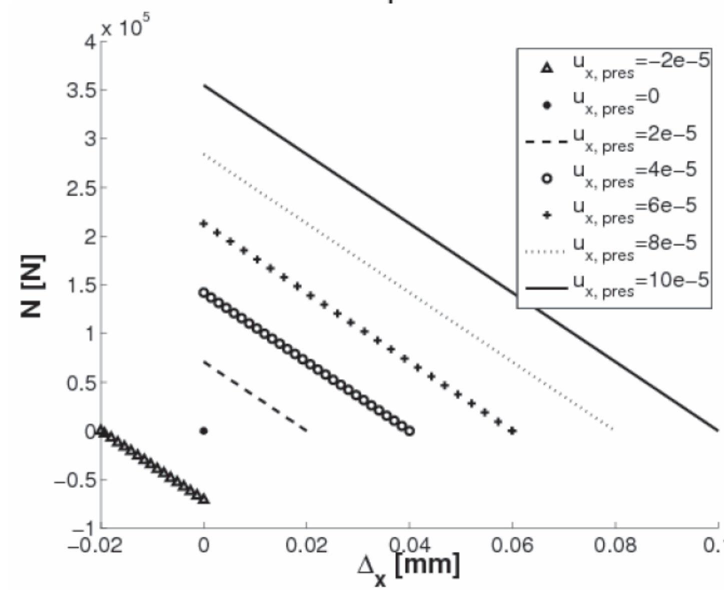


Fracture of Thin Structures

- Stable fracture
  - Effect of pre-strain
    - Dissipated energy always = hG_C







Conclusions & Perspectives

- Development of discontinuous Galerkin formulations
  - Formulation of high-order differential equations
    - Full DG formulation of beams
      - New degree of freedom
      - No rotation degree or freedom
      - As interface elements exist: cohesive law can be inserted

- Perspectives :
  - Extension to non-linear shells
  - Plasticity & ductile material