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# A FAMILY OF MMA APPROXIMATIONS FOR STRUCTURAL OPTIMIZATION

**Michael Bruyneel, Pierre Duysinx, and Claude Fleury** *

**Abstract** This paper proposes a new first order approximation scheme used for solving structural optimization problems. It is based on approximations of the MMA family (MMA and GCMMA), but it utilizes the gradients and/or the function values at two successive design points to improve the quality of the approximation. In addition, this scheme can consider simultaneously monotonous and non-monotonous structural behaviors. According to the characteristics of the treated problem, one of the approximations or a mix of them is automatically selected. Based on this approach, the accuracy of the approximated sub-problems is improved and the solution process can be sped up. Numerical results compare the effectiveness of the method with previously derived approximations of the MMA family for shape optimization of trusses and for composite design problems. The benefit of using mixed approximations is also discussed.

**Key words** Structural approximations, method of moving asymptotes

## 1 Introduction

The general statement of an optimization problem consists in minimizing an objective function $g_0(\mathbf{X})$ subject to behavior constraints $g_j(\mathbf{X})$ insuring the feasibility of the structural design.

$$\begin{array}{ll} \min\limits_{\mathbf{X}} & g_0(\mathbf{X}) \\ \text{s.t.:} & g_j(\mathbf{X}) \leq g_j^{max} \quad j = 1 \ldots m \\ & \underline{X}_i \leq X_i \leq \overline{X}_i \quad i = 1 \ldots n \end{array} \tag{1}$$

The $m+1$ functions $g_j(\mathbf{X})$ $(j = 0 \ldots m)$ are structural responses (e.g. mass, stresses, displacements) while the $n$ design variables $X_i (i = 1 \ldots n)$ can be the thickness of some structural members, geometric parameters, or fibres orientations for composite structures. Their range of variation is defined by lower ($\underline{X}_i$) and upper bounds ($\overline{X}_i$) that usually reflect technological and/or numerical considerations.

In most of the optimization problems, those structural responses are non linear (sometimes highly non linear) and their nature results in a monotonous or non-monotonous behavior with respect to a given design variable change. It comes that the direct solution of problem (1) with mathematical programming methods is too heavy from a computational point of view.

To solve structural problems such as (1), the approach that is used here is the approximation concept approach defined by Schmit and Fleury (1980) and that proved its efficiency and its general character. In this approach, the primary implicit optimization problem (1) is

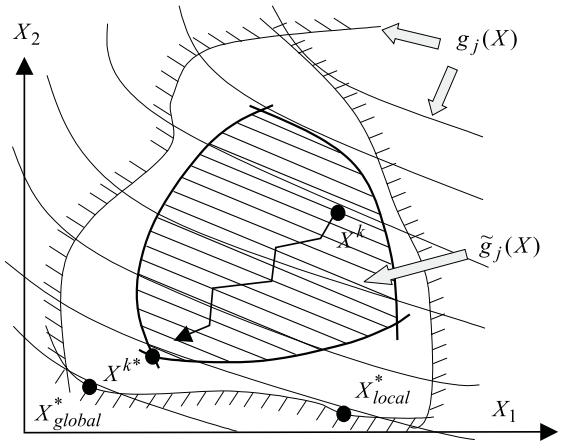


**Fig. 1** Original optimization problem (1) and its approximated form (2) around the current design point

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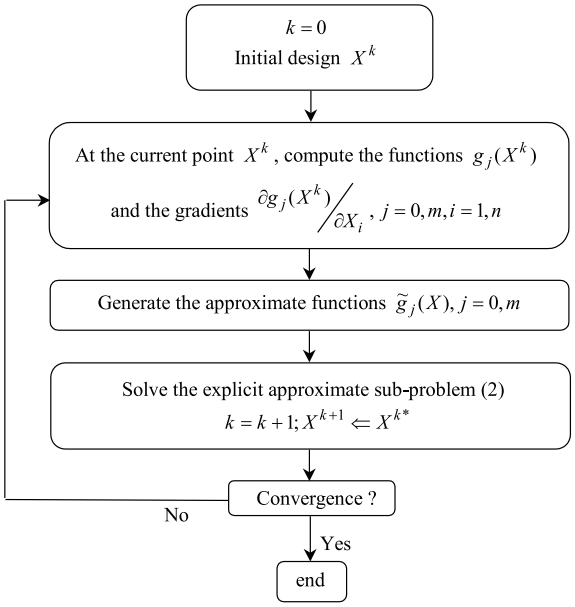



**Fig. 2** Iterative scheme of the optimization using the approximation concepts approach

replaced by the solution of a sequence of explicit approximated sub-problems (2)

$$\begin{array}{ll} \min\limits_{\mathbf{X}} & \tilde{g}_0(\mathbf{X}) \\ \text{s.t.:} & \tilde{g}_j(\mathbf{X}) \leq g_j^{max} \qquad j = 1 \ldots m \\ & \underline{X}_i \leq X_i \leq \overline{X}_i \qquad i = 1 \ldots n \end{array} \tag{2}$$

which are generated through first or second order Taylor series expansion of the structural functions in terms of specific intermediate linearization variables, for instance direct or reciprocal variables. Then, because the structural approximations $\tilde{g}_j(\mathbf{X})$ in (2) are convex and separable, a dual formulation (Fleury (1993)) can be efficiently used for solving each explicit sub-problem. The approximation procedure is sketched in Fig. 1 and the iterative optimization procedure using the approximation concepts approach is described in Fig. 2.

The number of structural analyses needed to reach the solution of (1) can be reduced when appropriate approximations are used. Their accuracy increases with the number of parameters they contain and/or the way they are computed. A lot of approximation schemes can be found in the literature (see Fleury (1989) for second order approximations, Fadel et al. (1990); Chickermane and Gea (1996) for two-point based schemes, and Xu and Grandhi (2000) for multipoint approximations). In this paper, we performed modifications of the approximations of the Method of Moving Asymptotes family proposed by Svanberg (1987, 1995a) in order to improve the quality of these approximations for highly non linear structural responses, and to make them reliable when dealing with design problems presenting a mix of monotonous and non-monotonous structural behaviors. The range of application of the modified approximations presented here is similar to the one of MMA, that is structural and, possibly, multidisciplinary optimization.

## 2 Monotonous approximations

Conlin scheme developed by Fleury and Braibant (1986) is a convex approximation based on the first order Taylor series expansion in terms of direct and reciprocal design variables. The approximation of a design function $g_j(\mathbf{X})$ is computed based on the function value and on the first derivatives at the current design point $\mathbf{X}^{(k)}$.

$$\begin{aligned} \tilde{g}_j(\mathbf{X}) = g_j(\mathbf{X}^{(k)}) &+ \sum_{+,i} \frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i}(X_i - X_i^{(k)}) \\ &- \sum_{-,i} (X_i^{(k)})^2 \frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i}(\frac{1}{X_i} - \frac{1}{X_i^{(k)}}) \end{aligned} \tag{3}$$

The symbols $\sum_{+,i}$ and $\sum_{-,i}$ denote the summations over terms having positive and negative first order derivatives.

In the Method of Moving Asymptotes or MMA, Svanberg (1987) generalizes Conlin by introducing two sets of new parameters, the lower and upper asymptotes, $L_i^{(k)}$ and $U_i^{(k)}$, in order to adjust the convexity of the approximation in accordance with the problem under consideration.

$$\begin{aligned} \tilde{g}_j(\mathbf{X}) = g_j(\mathbf{X}^{(k)}) &+ \sum_{+,i}^{n} p_{ij}^{(k)} \, (\frac{1}{U_i^{(k)} - X_i} - \frac{1}{U_i^{(k)} - X_i^{(k)}}) \\ &+ \sum_{-,i}^{n} q_{ij}^{(k)} \, (\frac{1}{X_i - L_i^{(k)}} - \frac{1}{X_i^{(k)} - L_i^{(k)}}) \end{aligned} \tag{4}$$

In (4) only one of the two coefficients $p_{ij}^{(k)}$ or $q_{ij}^{(k)}$ is different from zero at the same time for one design variable:

$$\begin{aligned} p_{ij}^{(k)} &= \max\{0, (U_i^{(k)} - X_i^{(k)})^2 \frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i}\} \\ q_{ij}^{(k)} &= \max\{0, -(X_i^{(k)} - L_i^{(k)})^2 \frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i}\} \end{aligned} \tag{5}$$

So for each design variable $X_i$, only one asymptote, either $L_i^{(k)}$ or $U_i^{(k)}$, is used in the approximation according to the sign of the first order derivative $\partial g_j(\mathbf{X}^{(k)})/\partial X_i$. Therefore, the approximation is monotonous, what ever can be the real behavior of the response function. MMA approximation is illustrated in Fig. 3 with the example of the strain energy density of a one-ply laminate in an optimal orientation problem.

From one iteration to another, the $n$ asymptotes $L_i^{(k)}$ and $U_i^{(k)}$ are updated according to heuristic rules (6)

$$\begin{aligned} L_i^{(k)} &= X_i^{(k)} - s_i \, (X_i^{(k-1)} - L_i^{(k-1)}) \\ U_i^{(k)} &= X_i^{(k)} + s_i \, (U_i^{(k-1)} - X_i^{(k-1)}) \end{aligned} \tag{6}$$

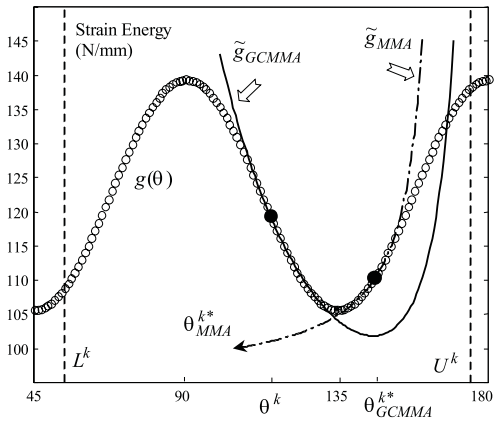



**Fig. 3** Approximations of the strain energy for optimal orientation in a one ply laminate

proposed by Svanberg (1987), where the parameter $s_i$ is computed based on the variation of the corresponding design variable values $X_i$ within 3 iteration steps.

The MMA approximation uses the same asymptotes for all the $m+1$ design functions $g_j(\mathbf{X})$ involved in the optimization problem. This definition does not give the flexibility of adjusting the approximation of each structural response in accordance with its own characteristics. Using (6), the asymptotes can be tightened for non linear functions, while they can not be relaxed at the same time for approximating in a reliable way a linear function. This led some researchers to introduce the Generalized Method of Moving Asymptotes (GMMA) by attaching a proper moving asymptote to each design variable in each design function, that is $L_{ij}^{(k)}$ or $U_{ij}^{(k)}$ are now used in place of $L_i^{(k)}$ or $U_i^{(k)}$ . The resulting GMMA approximation can be obtained from the MMA approximation (7), in which the dependency on both indices $i$ and $j$ has been introduced.

$$\tilde{g}_j(\mathbf{X}) = g_j(\mathbf{X}^{(k)}) + \sum_{+,i}^{n} p_{ij}^{(k)} \left(\frac{1}{U_{ij}^{(k)} - X_i} - \frac{1}{U_{ij}^{(k)} - X_i^{(k)}}\right) + \sum_{-,i}^{n} q_{ij}^{(k)} \left(\frac{1}{X_i - L_{ij}^{(k)}} - \frac{1}{X_i^{(k)} - L_{ij}^{(k)}}\right) \tag{7}$$

The asymptotes in (7) are computed based on the value of the non-mixed second order derivatives (Smaoui et al. (1988)) or based on an estimation of them (see Duysinx et al. (2000) for example). Alternatively as in Zhang and Fleury (1997), they can also be defined by the value of the considered structural response at the previous iteration, $g_j(\mathbf{X}^{(k-1)})$. In this case, the asymptotes for each constraint are computed by

$$\begin{aligned} L_{ij}^{(k)} &= X_i^{(k)} - s_j \left(X_i^{(k-1)} - L_{ij}^{(k-1)}\right) \\ U_{ij}^{(k)} &= X_i^{(k)} + s_j \left(U_{ij}^{(k-1)} - X_i^{(k-1)}\right) \end{aligned} \tag{8}$$

where the $s_j$ factor is adjusted to fit the approximation to the value of the function at the previous design point. This needs solving a one-dimensional line search (9) with a Newton-Raphson procedure:

$$\begin{aligned} &\tilde{g}_j(\mathbf{X}^{(k-1)}) = g_j(\mathbf{X}^{(k-1)}) \\ \Leftrightarrow\ &\tilde{g}_j(s_j) - g_j(\mathbf{X}^{(k-1)}) = 0 \end{aligned} \tag{9}$$

Based on (7), one can resort to the original MMA by forcing the asymptotes to take the same value for each design function. Furthermore, if $L_i^{(k)} = 0$ and $U_i^{(k)} \to \infty$, MMA is reduced to ConLin and to the linear expansion if further $L_i^{(k)} \to -\infty$.

## 3 Non-monotonous approximations

In Svanberg (1995a), the author derived a Globally Convergent version of the Method of Moving Asymptotes (GCMMA):

$$\tilde{g}_j(\mathbf{X}) = g_j(\mathbf{X}^{(k)}) + \sum_{i=1}^{n} p_{ij}^{(k)} \left(\frac{1}{U_i^{(k)} - X_i} - \frac{1}{U_i^{(k)} - X_i^{(k)}}\right) + \sum_{i=1}^{n} q_{ij}^{(k)} \left(\frac{1}{X_i - L_i^{(k)}} - \frac{1}{X_i^{(k)} - L_i^{(k)}}\right) \tag{10}$$

This is an extension of the MMA scheme (4), where both $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$ are simultaneously non-zero, which means that now both $L_i^{(k)}$ and $U_i^{(k)}$ are used *at the same time* to generate the approximation (10). This leads to the non-monotonous character of the approximation as illustrated in Fig. 3.

At the stage $k$ of the optimization process, GCMMA is defined by the function value $g_j(\mathbf{X}^{(k)})$, the parameters $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$, which are computed based on the first order derivatives, on a non-monotonic parameter $\rho_j^{(k)}$, and by the $n$ pairs of asymptotes $L_i^{(k)}$ and $U_i^{(k)}$. The non-monotonic parameter and the asymptotes are updated according to rules given in Svanberg (1987, 1995a) that insures the global convergence property of the approximation scheme. When $\rho_j^{(k)}$ is equal to zero, one resorts to the monotonous MMA approximation.

As suggested in Svanberg (1995b), the curvature of the GCMMA approximation can be improved by considering non-mixed second order derivatives in place of the non-monotonic parameter:

$$p_{ij}^{(k)} = \frac{(U_i^{(k)} - X_i^{(k)})^3}{2(U_i^{(k)} - L_i^{(k)})} \times \left(2\frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i} + (X_i^{(k)} - L_i^{(k)})\frac{\partial^2 g_j(\mathbf{X}^{(k)})}{\partial X_i^2}\right) \tag{11}$$

$$q_{ij}^{(k)} = \frac{(X_i^{(k)} - L_i^{(k)})^3}{2(U_i^{(k)} - L_i^{(k)})} \times \left(-2\frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i} + (U_i^{(k)} - X_i^{(k)})\frac{\partial^2 g_j(\mathbf{X}^{(k)})}{\partial X_i^2}\right) \tag{12}$$

In the later, this approximation is called GCMMA2.

In order to avoid the use of the non-monotonic parameter $\rho_j^{(k)}$ in the definition of GCMMA or the computation of the second order derivatives for GCMMA2, the idea developed here consists in using the gradients at two successive iterations. As initiated in Bruyneel et al. (1999), this gives rise to a family of novel Gradient Based MMA approximations, or GBMMA schemes.

In the first approximation relying on the Gradient Based MMA idea, named GBMMA1, the asymptotes $L_i^{(k)}$ and $U_i^{(k)}$ are at first updated according to the classical rule proposed in (6). Then the parameters $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$ of the GCMMA approximation (10) are determined by matching the first derivatives of $g_j(\mathbf{X})$ in the previous and current design points $\mathbf{X}^{(k-1)}$ and $\mathbf{X}^{(k)}$. The values of $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$ can be analytically extracted from the following sets of equations:

$$\begin{cases} \frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i} &= \frac{p_{ij}^{(k)}}{(U_i^{(k)}-X_i^{(k)})^2} - \frac{q_{ij}^{(k)}}{(X_i^{(k)}-L_i^{(k)})^2} \\ \frac{\partial g_j(\mathbf{X}^{(k-1)})}{\partial X_i} &= \frac{p_{ij}^{(k)}}{(U_i^{(k)}-X_i^{(k-1)})^2} - \frac{q_{ij}^{(k)}}{(X_i^{(k-1)}-L_i^{(k)})^2} \end{cases} \tag{13}$$

The second novel approximation, GBMMA2, is based on the GCMMA2 scheme. But here the idea is to replace the computed non-mixed second order derivatives in (11) and (12) by the following estimation:

$$\frac{\partial^2 g_j(\mathbf{X}^{(k)})}{\partial X_i^2} \simeq \frac{\frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i} - \frac{\partial g_j(\mathbf{X}^{(k-1)})}{\partial X_i}}{X_i^{(k)} - X_i^{(k-1)}} \tag{14}$$

It is shown in Duysinx et al. (2000) that making this kind of finite differences, (14) is the best diagonal quasi Newton update that can be obtained from the second order derivatives when the diagonal assumption is made a priori. When the two successive design points are far apart, it's obvious that the finite difference kind approximation of equation (14) of second derivatives becomes less precise. However, our numerical experiments showed that this information is even more realistic than Svanberg's heuristic formula based on the monotonous parameter $\rho_j^{(k)}$. Finally when coming to an accumulation point, estimations (14) become really valuable for convergence speed because the design steps are small.

GBMMA1 and GBMMA2 can be further generalized by attaching a pair of asymptotes $L_{ij}^{(k)}$ and $U_{ij}^{(k)}$ to each design variable and each design function (that is $2n \times (m+1)$ asymptotes). This leads to the GBMMA3 and GBMMA4 approximation schemes described in (15), which are respectively related to GBMMA1 and GBMMA2, for the selection of parameters $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$.

$$\tilde{g}_j(\mathbf{X}) = g_j(\mathbf{X}^{(k)}) + \sum_{i=1}^{n} p_{ij}^{(k)} \left(\frac{1}{U_{ij}^{(k)} - X_i} - \frac{1}{U_{ij}^{(k)} - X_i^{(k)}}\right) + \sum_{i=1}^{n} q_{ij}^{(k)} \left(\frac{1}{X_i - L_{ij}^{(k)}} - \frac{1}{X_i^{(k)} - L_{ij}^{(k)}}\right) \tag{15}$$

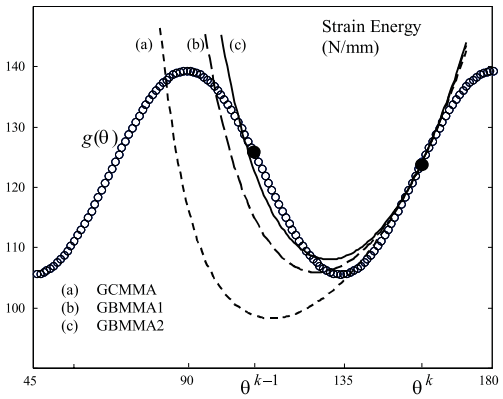



**Fig. 4** Approximations of the strain energy of a one ply laminate using GCMMA, GBMMA1 and GBMMA2

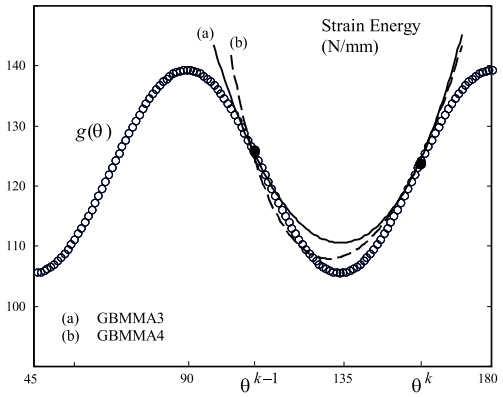



**Fig. 5** Approximations of the strain energy of a one ply laminate using GBMMA3 and GBMMA4

In the GBMMA3 and GBMMA4 schemes, the asymptotes are updated according to the relations (8-9) which are based on the fitting of the approximation to the function value at the previous iteration point.

The GCMMA, GBMMA1, and GBMMA2 approximation procedures are illustrated in Fig. 4, while the GBMMA3 and GBMMA4 approximations are given in Fig. 5.

Finally, one must also remark that a fifth approximation scheme can be imagined on the basis of (15) in order to generalize GCMMA. It consists in using the non-monotonic parameter $\rho_j^{(k)}$ while the asymptotes values are computed by the numerical fitting procedure (8-9). However, we could show from our numerical experimentations that, in the problem of Fig. 5, the solution of the fitting scheme (9) is not unique. There are two solutions $s_1^\star$ and $s_2^\star$ to the non-linear equation problem, leading to two possible approximations. Up to now, no rigorous rule has been found for selecting the best solution $s^\star$, which makes the procedure fragile for practical applications. Therefore, this approximation has been abandoned.

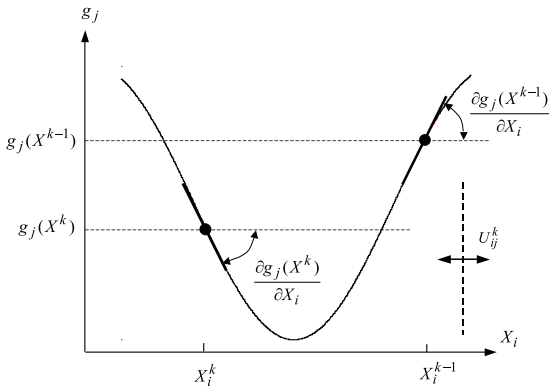



**Fig. 6** Fitting a convex approximation to the first derivatives (GBMMA1) and to both the function value and to its first derivatives (GBMMA3) at the previous iteration point are both possible

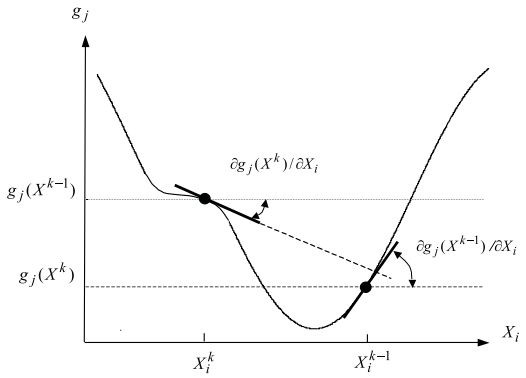



**Fig. 7** Fitting the first derivative is possible (GBMMA1) but it is not possible to match the function value at the previous iteration point (GBMMA3)

## 4
## Convexity check and selection

The key point in the derivation of approximation schemes is their convexity. This property allows using the theorem of Lagrangian duality for solving the sub-problems (2) and gives rise to the sufficient character to the Khun-Tucker optimality conditions.

Nonetheless, from the implementation point of view, it is important to note that for GMMA, GBMMA3 and GBMMA4, the primal-dual relations $\mathbf{X}(\lambda)$ can no longer be derived in closed-form, and a numerical one-dimensional scheme is then required to compute this relation.

For the approximations of the MMA family to be strictly convex, it is easy to show that $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$ have to be positive. For the MMA approximation, the following relation $L_i^{(k)} < X_i^{(k)} < U_i^{(k)}$ has to be verified. Besides, for GCMMA, the non-monotonic parameter $\rho_j^{(k)}$ has to be non negative too.

To be sure of the convexity of the GBMMAs approximations, tests have to be performed. GBMMA1 can always be used if the first order derivatives are of opposite signs at two successive iterations. For GBMMA2, the value of the approximated second order derivatives (14) have to be positive.

For GBMMA3 and GBMMA4 approximations, more complicated tests have to be derived. Without coming into too many details, it is possible to figure out which are these conditions on Figs. 6, 7 and 8.

In Fig. 6, all the GBMMAs approximations can be generated. Considering GBMMA3 or GBMMA4 schemes, it is possible to adjust the upper asymptote $U_{ij}^{(k)}$ with the fitting scheme (8-9) in such a way that the approximation will pass trough the preceding design point (while the lower asymptote $L_{ij}^{(k)}$ is updated according to Svanberg's classical formula (6)). But it is not always possible to realize this kind of fitting as it is suggested in the configuration sketched in Fig. 7. In this case, GBMMA1 (GBMMA2) can be generated based on the first derivative in $\mathbf{X}^{(k-1)}$, but the fitting procedure used to build GBMMA3 (GBMMA4) fails in finding a convex approximation.

When GBMMA3 (GBMMA4) can not be used for the contribution of one design variable in one design function, one can resort to its simplified form GBMMA1 (GBMMA2), and finally to GCMMA if necessary. The selection of the approximation then results from a kind of cascading process.

In addition, it was observed on numerical applications that it is interesting to use relation (14) for computing the approximation when the current design point is located in the neighboring of the optimum. Indeed, it makes sense that in the final convergence stages, second order information, even if estimated, improves convergence speeds. So a switch procedure from GBMMA1 (GBMMA3) to GBMMA2 (GBMMA4) is based on the following condition:

$$\frac{|X_i^k - X_i^{k-1}|}{\overline{X}_i - \underline{X}_i} < \text{SWITCH} \tag{16}$$

where the parameter SWITCH is selected by the user. A default value of $10^{-2}$ have given good results in our experiments.

Similar considerations can be made for monotonous approximations. A GMMA approximation based on the gradients and the function values at two successive design points can be derived (Duysinx et al. (2000)). Convexity tests should be able to point out their ability to be used, like in Fig. 8. Again, if GMMA can not be generated, MMA is then selected.

## 5
## Comparison of non-monotonous approximations

Let's now illustrate the different approximation schemes on a simple example. The following function is consid-

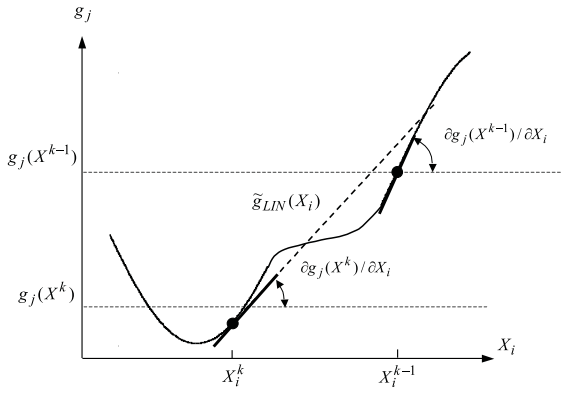



**Fig. 8** Fitting a convex approximation to the function value at the previous iteration point is not possible

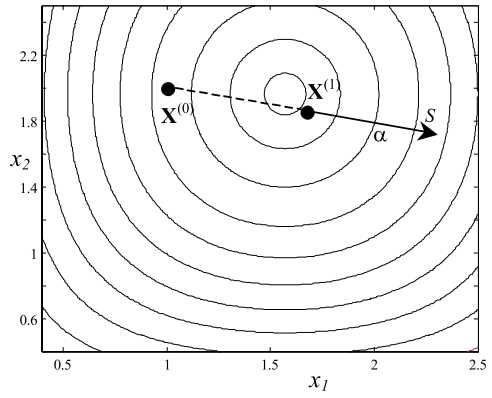



**Fig. 9** Illustrations of function (17). $\mathbf{X}^{(0)}$ and $\mathbf{X}^{(1)}$ are the two expansion points

ered:

$$g(x_1, x_2) \;=\; 2 \;-\; \sin(0.8\, x_2)\, \sin(x_1) \tag{17}$$

which is represented in Fig. 9.

Two design points, $\mathbf{X}^{(0)}$ and $\mathbf{X}^{(1)}$, are given. $\mathbf{X}^{(0)} = (1, 2)^T$ while $\mathbf{X}^{(1)}$ is the minimum found when the function $g(x_1, x_2)$ is approximated in $\mathbf{X}^{(0)}$ with GCMMA. The information available in these two successive iterations is used to generate different approximations around the point $\mathbf{X}^{(1)}$.

The test points, where the approximations are compared, are located along the vector $\mathbf{S}$ that is joining $\mathbf{X}^{(0)}$ to $\mathbf{X}^{(1)}$, i.e. $\mathbf{S} = \mathbf{X}^{(1)} - \mathbf{X}^{(0)}$.

$$\mathbf{X} \;=\; \mathbf{X}^{(1)} \;+\; \alpha\, \mathbf{S} \tag{18}$$

The parameter $\alpha$ is the step length along this $\mathbf{S}$ direction (see Fig. 9). When $\alpha = 0$, the corresponding point is $\mathbf{X}^{(1)}$, while $\alpha = -1$ characterizes $\mathbf{X}^{(0)}$.

The following side constraints on the design variables $\underline{x}_1 = \underline{x}_2 = 0.4$ and $\overline{x}_1 = \overline{x}_2 = 2.5$ are given. They are used in the definition of the two sets of moving asymptotes:

$$\begin{aligned} L_{x_1}^{(1)} &= x_1^{(1)} - 0.5 \times (\overline{x}_1 - \underline{x}_1) \\ U_{x_1}^{(1)} &= x_1^{(1)} + 0.5 \times (\overline{x}_1 - \underline{x}_1) \end{aligned} \tag{19}$$

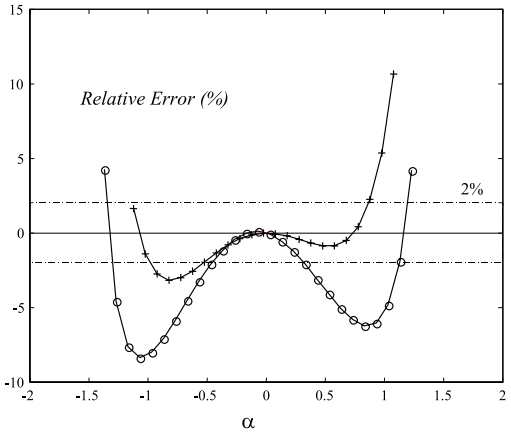



**Fig. 10** GCMMA and GCMMA2 approximations of function (17). Relative percentage error (O GCMMA, + GCMMA2)

and similar values for $L_{x_2}^{(1)}$ and $U_{x_2}^{(1)}$.

Figures 10, 11, and 12 illustrate the behavior of the approximations along the direction $\mathbf{S}$. The relative error index is defined by:

$$\text{error} \;=\; \frac{\text{approximation} - \text{exact}}{\text{exact}} \tag{20}$$

It is an image of the conservative character of the approximation. A positive value indicates the tendency of the approximation to remain in the security region, while a negative value of this error index means that the real function is under-estimated. It also gives an indication about the accuracy of the approximation.

For GCMMA and GCMMA2, the relative error is small in the close neighboring of the expansion point $\mathbf{X}^{(1)}$ (Fig. 10). Their maximum relative errors in the domain $\alpha \in [-1.\,;\, 1.]$ is of -8.18% and 6.3%, respectively. GCMMA presents a lack of conservativity close to $\mathbf{X}^{(0)}$ ($\alpha = -1$). When gradients at $\mathbf{X}^{(0)}$ and $\mathbf{X}^{(1)}$ are used, the relative errors of GBMMA1 and GBMMA2 are close to zero for a large variation of the step length around $\mathbf{X}^{(1)}$ (Fig. 11). For GBMMA2, it is comprised between -2% and 2% for $\alpha \in [-0.8\,;\, 0.65]$. The weakest relative errors are obtained for GBMMA3 and GBMMA4. They are zero not only at the expansion point $\mathbf{X}^{(1)}$ but also in $\mathbf{X}^{(0)}$ as it is seen in Fig. 12. When GBMMA3 is considered, the relative error is never negative and is lower than 2% in the interval $\alpha \in [-1.5\,;\, 0.4]$. GBMMA4 has a relative error larger than -0.85% in the all domain.

If the function (17) is the objective function of a quasi unconstrained optimization problem, let's now compare the number of sub-problem generations that are needed to reach the solution. The number of analyses that are performed to find the minimum depends strongly on the quality of the approximation. From results given in Table 1, it is clear that all GBMMA schemes are quite superior to the GCMMA approximation.

Now let's consider $g(x_1, x_2)$ as a constraint of an optimization problem. The quality of its approximation will influence the feasibility of the intermediate solutions. For

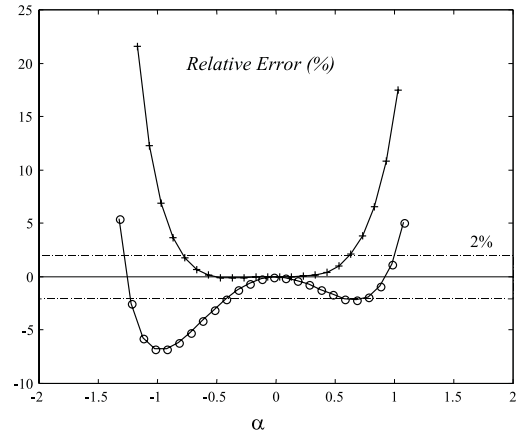



**Fig. 11** GBMMA1 and GBMMA2 approximations of function (17). Relative percentage error (O GBMMA1, + GBMMA2)

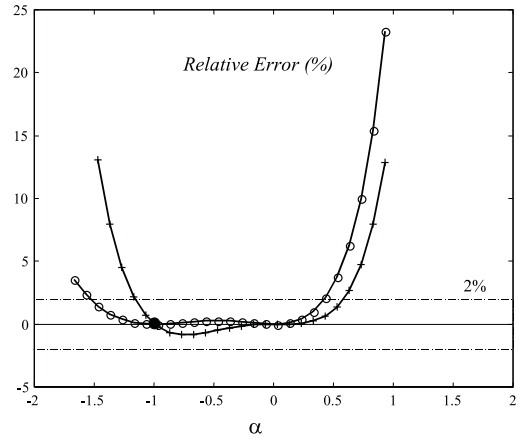



**Fig. 12** GBMMA3 and GBMMA4 approximations of function (17). Relative percentage error (O GBMMA3, + GBMMA4)

**Table 1** Number of iterations for the minimization of function (17). Starting point: $\mathbf{X}^{(0)} = (x_1, x_2) = (1, 2)$

| Approximation | $\|\nabla g\| < 10^{-2}$ | $\|\nabla g\| < 10^{-4}$ |
|---|---|---|
| GCMMA | 12 | 25 |
| GCMMA2 | 3 | 4 |
| GBMMA1 | 4 | 6 |
| GBMMA2 | 3 | 4 |
| GBMMA3 | 5 | 5 |
| GBMMA4 | 4 | 4 |

the following bound on the function (17),

$$g(x_1, x_2) \;=\; 2 \;-\; \sin(0.8\, x_2)\, \sin(x_1) \;\leq\; 1.3 \qquad (21)$$

the corresponding design domains are plotted in Fig. 13. Conservative character of the approximation leads to an approximated feasible domain that is everywhere inside of the real one. Again, it is clear that using the information from the previous design point in the GBMMA schemes allows to generate more accurate and conservative approximated feasible domains.

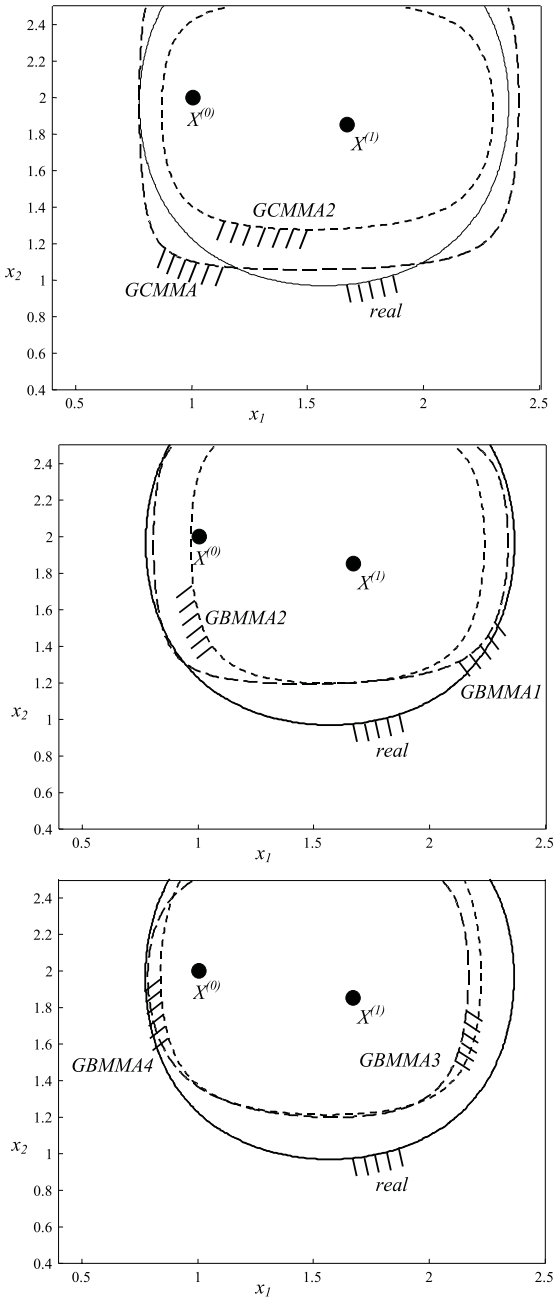



**Fig. 13** Illustration of the quality of the approximated feasible domain when (17) is the constraint of an optimization problem

## 6 Mixed approximations

When dealing with structural optimization problems including design variables of two different natures, for example in problems mixing sizing and shape variables or ply thickness and orientation variables in composite design, one is faced to a difficult task because of the simultaneous presence of monotonous and non-monotonous behaviors with respect to the design variables. In these conditions, most of usual approximation schemes have poor convergence properties or even fail to solve these

kinds of problems. This fact was noticed by Zhang et al. (1998) for truss configuration optimization. Those authors put forward the idea that a mixed approximation scheme was interesting in this case and they developed such an approximation in which *a priori* sizing variables are approximated by a GMMA scheme, whereas a diagonal quadratic approximation is used for shape variables.

In this work, we are continuing along this idea and we propose a mixed approximation based on monotonous and non-monotonous schemes from the MMA family we presented before. Using approximations of the same family is an advantage for the numerical implementation. We also provide a strategy to select the monotonous or the non-monotonous approximation schemes for each variable in each function. This insures the efficiency and the robustness of the procedure.

In order to build a very general mixed approximation of the MMA family, GBMMA and GMMA schemes are combined in (22)

$$\begin{aligned}\tilde{g}_j(\mathbf{X}) = g_j(\mathbf{X}^{(k)}) &+ \sum_{i\in A}^{n} p_{ij}^{(k)}\ (\frac{1}{U_{ij}^{(k)} - X_i} - \frac{1}{U_{ij}^{(k)} - X_i^{(k)}}) \\ &+ \sum_{i\in A}^{n} q_{ij}^{(k)}\ (\frac{1}{X_i - L_{ij}^{(k)}} - \frac{1}{X_i^{(k)} - L_{ij}^{(k)}}) \\ &+ \sum_{+,i\in B}^{n} p_{ij}^{(k)}\ (\frac{1}{U_{ij}^{(k)} - X_i} - \frac{1}{U_{ij}^{(k)} - X_i^{(k)}}) \\ &+ \sum_{-,i\in B}^{n} q_{ij}^{(k)}\ (\frac{1}{X_i - L_{ij}^{(k)}} - \frac{1}{X_i^{(k)} - L_{ij}^{(k)}}) \qquad (22)\end{aligned}$$

In this new GBMMA-GMMA approximation, the design variables are partitioned into two sets, namely A and B. For the design variables belonging to the set A the GBMMA scheme (15) is used, which introduces a non-monotonous contribution, whereas for variables from set B, a GMMA scheme (7) is considered, which gives rise to monotonous terms in the approximation.

From stage $k \geq 2$ of the iterative optimization process, an automatic strategy selects the partition of the design variables between the two sets A and B. The tests are based on the gradient values at two successive iterations, or more exactly on the variation of the sign of the first derivatives between the two design points. For each structural response $g_j(\mathbf{X})$ and each design variable $X_i$, one performs the following tests:

$$\frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i} \times \frac{\partial g_j(\mathbf{X}^{(k-1)})}{\partial X_i} > 0 \Rightarrow \text{GMMA} \qquad (23)$$

$$\frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i} \times \frac{\partial g_j(\mathbf{X}^{(k-1)})}{\partial X_i} < 0 \Rightarrow \text{GBMMA} \qquad (24)$$

$$\frac{\partial g_j(\mathbf{X}^{(k)})}{\partial X_i} - \frac{\partial g_j(\mathbf{X}^{(k-1)})}{\partial X_i} = 0 \Rightarrow \text{LINEAR} \qquad (25)$$

As suggested for the first time in Bruyneel and Fleury (2000), tests (23) to (25) are performed on a given number of iterations defined by the user parameter ICHECK

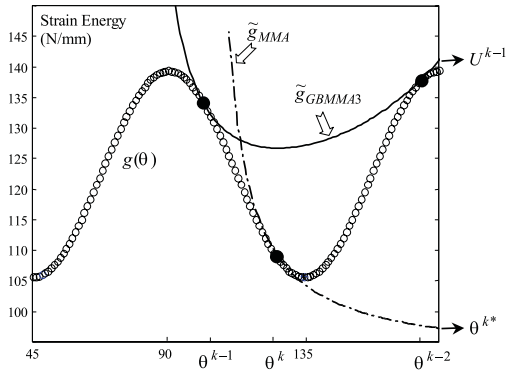



**Fig. 14** Selection of monotonous or non-monotonous approximations in a mixed scheme

to be sure to capture the true structural behaviors. During this checking phase, the use of monotonous approximations is forbidden for avoiding the risk of approximating a non-monotonous function with a monotonous one. This is illustrated in Fig. 14: between the iterations $k-2$ and $k-1$, the non-monotonous GBMMA3 scheme is selected based on test (24). If the detected non-monotonous behavior is not stored, a monotonous approximation could be built at the next step according to relation (23), because the gradients in $\theta^{(k-1)}$ and $\theta^{(k)}$ are now of the same sign. This would reject the new design point $\theta^{(k)\star}$ far from the current one and could slow down the overall optimization process.

If test (23) is verified during the ICHECK iterations, the behavior of the function $g_j(\mathbf{X})$ is considered to be monotonous with respect to $X_i$. This variable is then associated to the set B in (22) and its contribution in $g_j(\mathbf{X})$ is given by a monotonous approximation.

In practice, the choice of the value for the ICHECK parameter results from a compromise between security and speed: if this value is low, a non-monotonous structural behavior could be approximated using a monotonous approximation, and if its value is high, one then works mainly with non-monotonous (and perhaps too conservative) approximations. A typical value for ICHECK is 2.

A simpler scheme combining the GBMMA and the MMA approximations (called GBMMA-MMA) can be derived from (22). One needs just to define one set of lower asymptotes $L_i^{(k)}$ and one set of upper asymptotes $U_i^{(k)}$ for all the functions instead of $L_{ij}^{(k)}$ and $U_{ij}^{(k)}$. In this approximation scheme, the fitting procedure (8-9) is replaced by the simpler update procedure (6), so the function value at the preceding iteration is then not used anymore. This 'less expensive' scheme was used by Bruyneel and Fleury (2000) for the optimization of laminates over plies thickness and fibres orientations and showed good convergence properties.

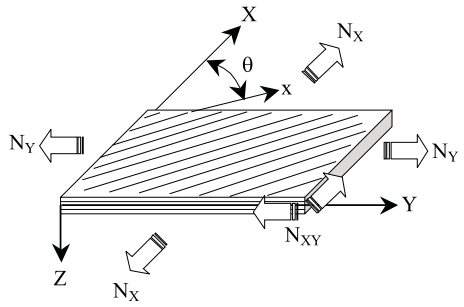



**Fig. 15** The considered laminate with the in plane loads, the structural and material axes

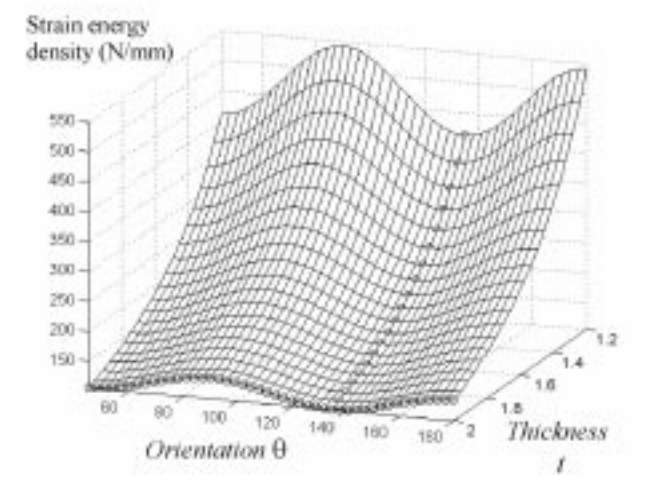


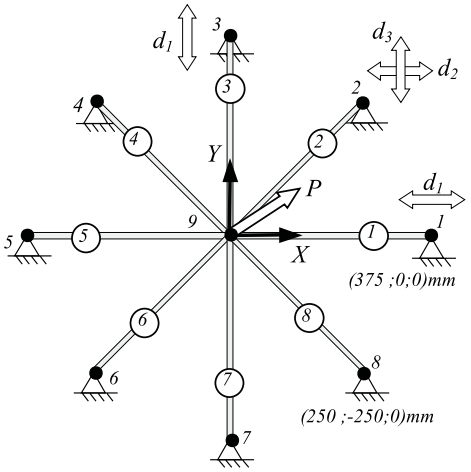

**Fig. 16** Variation of the strain energy density in an unidirectional composite subject to shear and torsion (Bruyneel and Fleury (2000))

## 7
## Numerical applications

The numerical tests are performed on design problems that exhibit high non linearity and a mix of monotonous and non-monotonous structural behaviors. Such characteristics can be found in problems dealing with configuration design of trusses or with fibers reinforced composite structures.

The first application concerns the geometric design of a 8 bars truss.

Then two composite examples dealing with symmetric laminates subject to in-plane loads are considered (see Fig. 15). The structural responses are computed based on the classical laminate theory (see e.g. Tsai and Hahn (1980)). The base material is the graphite-epoxy T300/5208. Both plies thickness and fibers orientations are the design variables. The objective function is the density of the strain energy of the laminate (see for example Fig. 16), which is computed based on the in-plane stiffness matrix $\mathbf{A}$ and the mid-plane strains $\varepsilon = (\varepsilon^0_{XX}, \varepsilon^0_{YY}, \gamma^0_{XY})^T$. Restrictions are imposed on the total thickness of the laminate and on the strength (Tsai-Wu failure criterion).



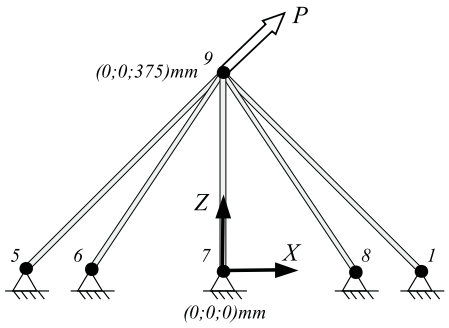




**Fig. 17** 8-bar truss geometry and design variables $d_1$ to $d_3$

For all the applications, the same stopping criteria are used. The solution is supposed to be reached when, for a feasible design, the relation (26) or (27) is verified. In the stopping criterion (26), the maximum relative difference is considered

$$\frac{|X_i^{(k)} - X_i^{(k-1)}|}{X_i^{(k-1)}} < TOL \tag{26}$$

$$\left| \frac{g_0(\mathbf{X}^{(k)}) - g_0(\mathbf{X}^{(k-1)})}{g_0(\mathbf{X}^{(k-1)})} \right| < TOL \tag{27}$$

### 7.1
### Example 1: eight bars truss geometric design

Let's consider the geometric design of the 8 bars truss presented in Fig. 17. The load $P_X = 60$ kN, $P_Y = 40$ kN, $P_Z = 40$ kN is applied at node 9. The material data are: $E = 210\ kN/mm^2$, $\rho = 7.81\ kg/dm^3$, $(\overline{\sigma} = -(\underline{\sigma}) = 150\ N/mm^2$. All the cross-section areas are constant and equal to $400\ mm^2$.

The structural weight is minimized subject to constraints imposed to axial stresses in the bars. The nodes 1 to 8 are allowed to move in the plane (X,Y) while position of node 9 is fixed. Due to symmetry conditions around (X,Z) and (Y,Z) planes, the coordinates of the 8

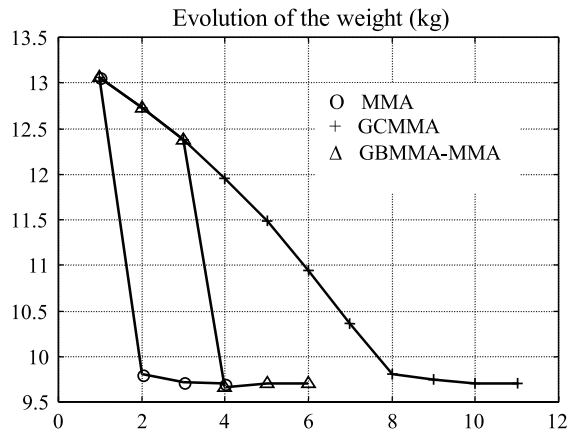



**Fig. 18** Geometric design. $TOL = 10^{-3}$ in (26)

first nodes can be ruled by 3 independent shape design variables as represented in Fig. 17.

Following the *a priori* guess suggested by Zhang et al. (1998), a non-monotonous approximation is used to solve this problem. As observed in Fig. 18, GCMMA can not reach the solution in less than 11 iterations. Indeed the structural responses are monotonous functions, which explains why the optimum is reached within 4 iterations when using MMA. Now with the help of the mixed GBMMA-MMA scheme and of its automatic selection strategy with ICHECK = 2, the optimization algorithm selects the less conservative MMA after 3 iterations and finds the optimum in 6 steps. The automatic scheme selection prevents the users from choosing non appropriate approximation schemes. This is a major advantage for large scale and difficult problems in which the choice is not obvious or misleading as here.

## 7.2
## Example 2: 4 plies symmetric laminate

This application puts into the light the benefit of using mixed approximations schemes for laminates designs. A 4 plies symmetric laminate is considered. The load case is the following: $N_X = 1000\ N/mm$, $N_Y = 1000\ N/mm$, $N_{XY} = 0\ N/mm$. The design variables are the thicknesses and fibres orientations of the 2 upper plies (since the laminate is symmetric).

$$\begin{array}{ll} \min\limits_{\theta,\mathbf{t}} & \frac{1}{2}\varepsilon^T \mathbf{A}\varepsilon \\ \text{s.t.:} & \mathrm{TW}(\theta_i, t_i) \ \leq \ 1 \qquad i = 1\ldots 2 \\ & \sum_{i=1}^{2} 2\,t_i \ \leq \ 15\text{mm} \\ & 0^\circ \ \leq \theta_i \ \leq \ 180^\circ \qquad i = 1\ldots 2 \\ & 0 \ < \ t_i \ \leq \ 10\,\text{mm} \quad i = 1\ldots 2 \end{array} \tag{28}$$

The starting point of the optimization process is: $\theta_1 = 100^\circ$, $\theta_2 = 50^\circ$, $t_1 = 0.5$ mm, $t_2 = 0.35$ mm.

In Fig. 19, when MMA is used, the convergence is controlled by the determination of the optimal orienta-

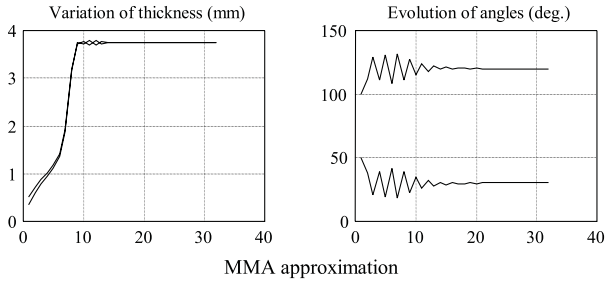



**Fig. 19** Evolution curves of ply thicknesses and orientations for MMA. $TOL = 10^{-3}$ in criterion (26)

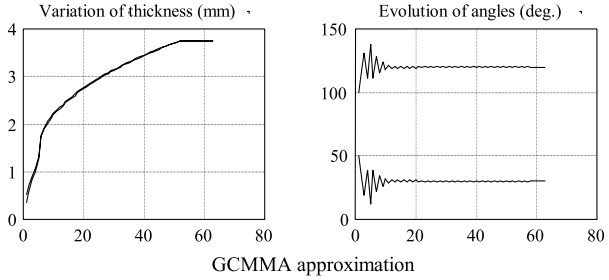



**Fig. 20** Evolution curves of ply thicknesses and orientations for GCMMA. $TOL = 10^{-3}$ in criterion (26)

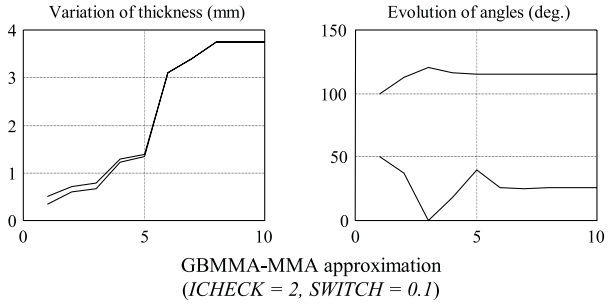



**Fig. 21** Evolution curves of ply thicknesses and orientations for GBMMA- MMA. $TOL = 10^{-3}$ in criterion (26)

tions. The solution is found thanks to a move limit strategy based on the asymptotes $L_i^{(k)}$ and $U_i^{(k)}$, as described by Svanberg (1987). In Fig. 20, the convergence process is slow because the structural responses in terms of the thickness are badly approximated by GCMMA which is too conservative in this case.

When the approximation scheme can consider separately monotonous and non-monotonous behaviors, the number of structural analyses required to reach the optimum is drastically decreased to 10 (Fig. 21). During the two first iterations (ICHECK = 2), the problem is approximated by a non monotonous MMA and the detection of the structural behaviors (monotonous or not) is performed. At the iteration 3, the contribution of the thickesses in the structural responses is approximated by the monotonous MMA, while gradient based MMA terms are used to generated a highly reliable non monotonous

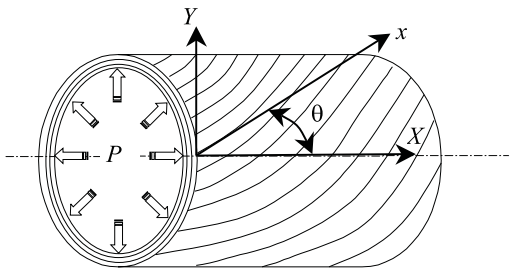



**Fig. 22** Composite cylinder problem

**Table 2** Starting point for the composite cylinder optimization problem

| Initial Orientations $(\theta_1, \theta_2, \theta_3, \theta_4)$ $(\theta_5, \theta_6, \theta_7, \theta_8)$ in degrees | Initial thicknesses $(t_1, t_2, t_3, t_4)$ $(t_5, t_6, t_7, t_8)$ in mm |
|---|---|
| $(0°, 135°, 45°, 90°)$ | $(0.1; 0.2; 0.1; 0.2)$ |
| $(90°, 45°, 135°, 0°)$ | $(0.1; 0.2; 0.1; 0.2)$ |

approximation of the structural responses for the fibers orientations.

### 7.3
### Example 3: Composite cylinder optimization

The design of a closed composite cylindrical container (see Fig. 22) subject to an internal pressure of $P = 10$ bars is considered (29).

$$\begin{array}{ll} \min\limits_{\theta,\,\mathbf{t}} & \frac{1}{2}\varepsilon^T \mathbf{A}\varepsilon \\ \text{s.t.:} & \mathrm{TW}(\theta_i, t_i) \;\leq\; 1 \qquad i = 1\ldots 8 \\ & \sum_{i=1}^{8} 2\,t_i \;\leq\; 10\text{mm} \\ & 0° \;\leq \theta_i \;\leq\; 180° \qquad i = 1\ldots 8 \\ & 0 \;<\; t_i \;\leq\; 10\,\text{mm} \quad i = 1\ldots 8 \end{array} \tag{29}$$

The radius of this thin-walled cylinder is 1 meter. The initial design is given in Table 2.

The number of structural analyses performed to reach a feasible local optimum are given in Table 3 when different types of approximations are used. Two differents values of the stopping precision parameter $TOL$ are used in criterion (27). In Figs. 23 and 24, convergence curves are provided for MMA and GBMMA-GMMA. In these figures, the value of the constraints is normalized so that they are violated when they take a value larger than unity. The maximum violation at each iteration is plotted.

When MMA is used, large oscillations are observed for the successive fibres orientations values (Fig. 23). This is due to the bad approximation of the structural responses in terms of those design variables. GCMMA2 is quite slow (Table 3) because it is degenerated to the first order approximation when second derivatives are negative.

**Table 3** Iterations versus approximation type for the optimization of the composite cylinder

| Approximation | Number of iterations $TOL = 0.05$ | $TOL = 0.01$ |
|---|---|---|
| MMA | 29 | 51 |
| GCMMA | 16 | 18 |
| GCMMA2 | 19 | 24 |
| GBMMA1 | 9 | 9 |
| GBMMA2 | 9 | 9 |
| GBMMA3 | 8 | 9 |
| GBMMA4 | 8 | 9 |
| GBMMA-MMA | 6 | 7 |
| GBMMA-GMMA | 6 | 7 |

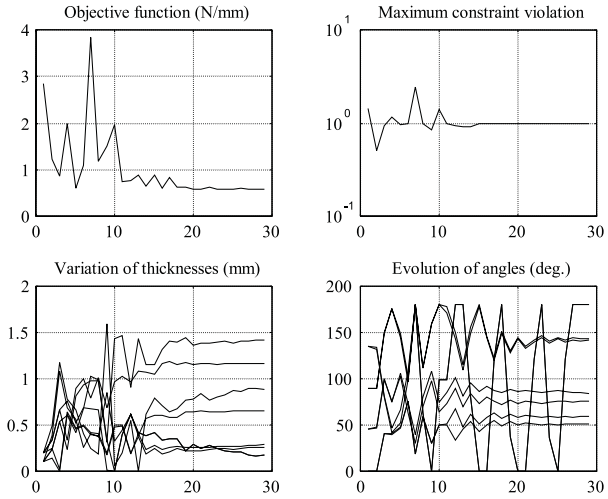



**Fig. 23** Iteration history for MMA

From Table 3, it is clear that resorting to Gradient Based MMA approximations and using information from previous design point can bring a major reduction of the number of iterations in this kind of composite design. Moreover, the mixed schemes can even further improve the convergence speed. From Fig. 24, one can see that the mixed GBMMA-GMMA scheme leads to monotonous convergence curves (in terms of the objective function as well as in terms of the design variables evolutions).

## 8
## Conclusions

This paper proposed a review of the approximations of the MMA family. New non-monotonous approximations were presented. It was shown how to mix monotonous and non-monotonous approximations in order to derive a general mixed approximation scheme GBMMA-GMMA, based on gradients and function values at two successive design points. The contribution of a given design variable in the approximation of a structural response

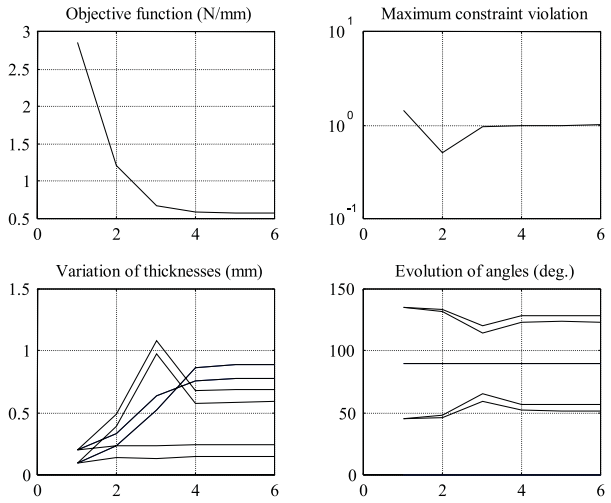



**Fig. 24** Iteration history for GBMMA-GMMA (ICHECK =2, SWITCH = 0.01)

results from its monotonous or non monotonous behavior detected during a checking phase. The computation of the parameters entering the selected approximation results from a cascading process. Numerical applications compared the effectiveness of the proposed schemes with previously derived approximations. The use of the information from previous design point allows to improve the accuracy of the approximation and tends to speed up the convergence process. The automatic selection of the approximation type prevents from using an approximation scheme that could slow down the overall optimization process. A comparison of the developed approximations to other well-known optimization techniques is in progress.

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