LATTICE OF SUBALGEBRAS IN THE FINITELY GENERATED VARIETIES OF MV-ALGEBRAS

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Abstract. In this paper, we use the theory of natural duality to study subalgebra lattices in the finitely generated varieties of MV-algebras. With this tool, we obtain the dual atomicity of these lattices, and characterize the members of these varieties in which every subalgebra is an intersection of maximal subalgebras. Then, we determine the algebras that have a modular or distributive lattice of subalgebras.

1. Introduction

MV-algebras were introduced in 1958 by C.C. Chang (see [3] and [4]) as a many-valued counterpart of Boolean algebras. Their study in a logical and algebraic point of view led to many interesting results, as an algebraic proof of the completeness theorem of Łukasiewicz's infinite-valued propositional calculus (see [4]).

An MV-algebra can be viewed as an algebra $A = \langle A; \oplus, \odot, \neg, 0, 1\rangle$ of type (2,2,1,0,0) such that $\langle A; \oplus, 0\rangle$ is an Abelian monoid and satisfying the following identities: $\neg\neg x = x$, $x \oplus 1 = 1$, $\neg 0 = 1$, $x \odot y = \neg(\neg x \oplus \neg y)$, $(x \odot \neg y) \oplus y = (y \odot \neg x) \oplus x$ (we refer to [6] for a introduction to the theory of MV-algebras).

One of the most simple (and most important) example of MV-algebra is the real interval $[0, 1]$ endowed with the operations $x \oplus y = \min(x + y, 1)$, $x \odot y = \max(x + y - 1, 0)$ and $\neg x = 1 - x$.

Komori's classification of the subvarieties of the variety $\mathcal{MV}$ of MV-algebras (see [12]) underlines the importance of the subalgebras $Ł_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ (where n is a positive integer) of $[0, 1]$. Indeed, a finitely generated subvariety of $\mathcal{MV}$ is always a subvariety of $\mathbb{HSP}(Ł_n)$ for some positive integer n. The axiomatization of $\mathbb{HSP}(Ł_n)$ can be found in [9], and some results about its free or projective members can be found in [10] or [8].

In this paper, we study the subalgebra lattices of the members of the classes $\mathbb{HSP}(Ł_n)$ ($n \in \mathbb{N}$). Our main tool for this work is a categorical duality (in fact a strong natural duality) between these varieties and some topological quasi-varieties. This natural duality was first discovered by Cignoli in [5] and was developed in [13] by P. Niederkorn. This duality maps embeddings to surjective morphisms and conversely. It is so natural to try to study subalgebra lattices by studying quotient lattices in the dual category. The idea to study lattices of subalgebras by the way of a duality has already been applied in [11] using Priestley duality for example.

In the first part of this paper, we briefly recall the principles of the duality involved. We then characterize the set of *quotient structures* of a member of the dual category and show that it can be naturally endowed with a lattice structure which is the dual counterpart of subalgebra lattice structure in $\mathbb{HSP}(Ł_n)$. Then we prove, with the help of our duality, the dual atomicity of the lattice of subalgebras of the members of the variety. We also discuss the conditions we have to impose on the algebras to assure that each of their proper subalgebras is an intersection of maximal subalgebras. Finally, we study the modularity and the distributivity of these lattices.

2. The duality

As stated above, our main tool in the study of subalgebra lattices in the finitely generated varieties of MV-algebras is the theory of duality. Indeed, P. Niederkorn has developed in [13] a duality (in fact a strong natural duality) for the varieties $\mathbb{HSP}(Ł_n)$ that transforms onto morphisms into embeddings and conversely (this duality had already been discovered by R. Cignoli in [5]).

We first recall this duality. We use the standard notations of category theory and natural duality for which we refer to [7] or [13]. Hence, we denote our algebras by underlined Roman capital letters and our topological structures by "undertilded" Roman capital letters.

So, let us set a positive integer n for the sequel of the paper and denote by $\underline{Ł}_n$ the MV-subalgebra $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0,1]$. We define $\underset{\sim}{Ł}_n$ as the topological structure

$$\underset{\sim}{Ł}_n = \langle Ł_n; \{Ł_m \mid m \in \mathrm{div}(n)\}, \tau\rangle,$$

where τ is the discrete topology, $\mathrm{div}(n)$ is the set of positive divisors of n and $Ł_m$ (with $m \in \mathrm{div}(n)$) is viewed as an unary relation on $Ł_n$. If we denote by $\mathcal{A}$ (*resp.* $\mathcal{X}$) the category whose objects are the members of the variety $\mathbb{HSP}(\underline{Ł}_n)$ (*resp.* the members of the topological quasi-variety $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{Ł}_n)$, i.e. the topological structures that are isomorphic to a closed subspace of a power of $\underset{\sim}{Ł}_n$) and whose morphisms are the MV-homomorphisms (*resp.* the continuous maps respecting the relational structure of the members of $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{Ł}_n)$), the results concerning natural duality on finitely generated varieties of MV-algebras can be briefly summarized by the following proposition (see theorem 2.1 in [13]).

Proposition 2.1. *The category $\mathcal{A}$ and $\mathcal{X}$ are dually equivalent by the functors*

$$\mathsf{D} : \mathcal{A} \to \mathcal{X} : \begin{cases} \underline{A} \in \mathcal{A} \mapsto \mathsf{D}(\underline{A}) = \mathcal{A}(\underline{A}, \underline{Ł}_n) \\ f \in \mathcal{A}(\underline{A}, \underline{B}) \mapsto \mathsf{D}(f) \in \mathcal{X}(\mathsf{D}(\underline{B}), \mathsf{D}(\underline{A})), \end{cases}$$

where $\mathsf{D}(f)(u) = u \circ f$ for all $u \in \mathsf{D}(\underline{B})$, and

$$\mathsf{E} : \mathcal{X} \to \mathcal{A} : \begin{cases} \underset{\sim}{X} \in \mathcal{X} \mapsto \mathsf{E}(\underset{\sim}{X}) = \mathcal{X}(\underset{\sim}{X}, \underset{\sim}{Ł}_n) \\ \psi \in \mathcal{X}(\underset{\sim}{X}, \underset{\sim}{Y}) \mapsto \mathsf{E}(\psi) \in \mathcal{A}(\mathsf{E}(\underset{\sim}{Y}), \mathsf{E}(\underset{\sim}{X})), \end{cases}$$

where $\mathsf{E}(\psi)(\alpha) = \alpha \circ \psi$ for all $\alpha \in \mathsf{E}(\underset{\sim}{Y})$.

Moreover, these two functors map embeddings onto surjective morphisms and conversely.

Komori's classification of the subvarieties of the variety of MV-algebras stresses (Theorem 4.11 in [12]) that a class K of MV-algebras is a finitely generated subvariety of $\mathcal{MV}$ if and only if there are positive integers $n_1, \ldots, n_r$ such that $K = \mathbb{HSP}(Ł_{n_1}, \ldots, Ł_{n_r})$.

But, since $Ł_{n_1}, \ldots, Ł_{n_r}$ can be embedded in $Ł_{\mathrm{lcm}(n_1,\ldots,n_r)}$ (because the subalgebras of $Ł_n$ are exactly the algebras $Ł_m$ where m is a divisor of n), we have $\mathbb{HSP}(Ł_{n_1}, \ldots, Ł_{n_r}) \subseteq \mathbb{HSP}(Ł_{\mathrm{lcm}(n_1,\ldots,n_r)})$. We can thus use the dualities for $\mathbb{HSP}(Ł_n)$ to obtain a representation of every algebra of every finitely generated variety of MV-algebras, and can also use them to study subalgebra lattices in the latter varieties.

In the sequel, we denote by X the underlying topological space of the member $\underset{\sim}{X}$ of $\mathcal{X}$. Let us recall the characterization of the objects of $\mathcal{X}$ (see theorem 2.1 in [13]): the objects of $\mathcal{X}$ are exactly the topological structures

$$\underset{\sim}{X} = \langle X; \{r_m^{\underset{\sim}{X}} \mid m \in \mathrm{div}(n)\}, \tau\rangle,$$

where

(X1) the topology τ is Boolean;
(X2) $r_m^{\underset{\sim}{X}}$ is a closed subspace of X for every $m \in \mathrm{div}(n)$;
(X3) we have $r_n^{\underset{\sim}{X}} = X$ and $r_m^{\underset{\sim}{X}} \cap r_{m'}^{\underset{\sim}{X}} = r_{\gcd(m,m')}^{\underset{\sim}{X}}$ for all m and m' in $\mathrm{div}(n)$.

Let us also note that we can use the evaluation map to obtain a Boolean representation of any algebra $\underline{A}$ of $\mathcal{A}$. Indeed,

$$e_{\underline{A}} : \underline{A} \to \prod_{u \in \mathsf{D}(\underline{A})} u(\underline{A}) : a \mapsto (u(a))_{u \in \mathsf{D}(\underline{A})}$$

is a Boolean representation of $\underline{A}$ by its simple quotients (see [13] for details).

Finally, recall that we can obtain, as a direct consequence of the duality, the description of the dual of the finite members of $\mathcal{A}$ and $\mathcal{X}$.

Proposition 2.2 (proposition 2.2 in [13])**.** *Each finite member of $\mathcal{A}$ is isomorphic to a direct product of (finitely many) subalgebras of $\underline{Ł}_n$. The dual of such an algebra $\underline{A}$ is a finite discrete topological space containing one point for each factor in this product. The point corresponding to the factor $\underline{Ł}_m$ belongs to $r_{m'}^{\mathsf{D}(\underline{A})}$ if and only if m divides m'.*

Conversely, each finite (thus discrete) member $\underset{\sim}{X}$ of $\mathcal{X}$ gives rise to a dual algebra $\mathsf{E}(\underset{\sim}{X})$ isomorphic to a direct product of (finitely many) subalgebras of $\underline{Ł}_n$: one factor for each point in X. The factor corresponding to a point x of X is $\underline{Ł}_m$ where m is the greatest common divisor of the integers k such that x belongs to $r_k^{\underset{\sim}{X}}$.

Since lots of the structures and the algebras that we consider in the sequel are finite, we use the preceding proposition intensively throughout the paper. Note that we can recover from this proposition that every finite member of $\mathbb{HSP}(\underline{Ł}_n)$ is isomorphic to a direct product of $\underline{Ł}_m$ with $m \in \operatorname{div}(n)$.

3. The lattice structure of $\mathsf{Quot}(X)$

3.1. **Quotient structures in $\mathcal{X}$.** Because of the relational nature of the structures of $\mathcal{X}$, if we consider a surjective morphism $\pi : \underset{\sim}{X} \to \underset{\sim}{Y}$ between two members $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ of $\mathcal{X}$, we cannot recover the structure on $\underset{\sim}{Y}$ directly from $\ker(\pi)$. Indeed, we can in general define several structures on the topological space $X/\ker(\pi)$ which make π an $\mathcal{X}$-morphism.

This remark leads us to the following (natural) definition of a quotient structure of a member $\underset{\sim}{X}$ of $\mathcal{X}$. First recall that *a Boolean equivalence* R on a Boolean space X is an equivalence relation on X such that for all $(x, y) \notin R$ there exists a R-saturated clopen ω of X (i.e. a clopen ω which contains the R-class of z for each $z \in \omega$) containing x but not y. These equivalences are exactly the ones which make the topological space X/R a Boolean space (see [2] for the definition and the characterization of such equivalences). Note that if R is an equivalence on the set X and if x is an element of X, we denote by x^R the class of x for the equivalence R.

Definition 3.1. A *quotient structure* (or simply a *quotient*) of a member $\underset{\sim}{X}$ of $\mathcal{X}$ is a pair (R, Γ) where R is a Boolean equivalence on X and Γ is a set $\{r_m^{\langle X/R,\Gamma\rangle} \mid m \in \operatorname{div}(n)\}$ of closed subspaces of X/R such that

- the structure $\langle X/R; \Gamma, \tau\rangle$ (where τ is the quotient topology) is a member of $\mathcal{X}$;
- the quotient map $\pi_R : X \to X/R$ is an $\mathcal{X}$-morphism from $\underset{\sim}{X}$ to $\langle X/R; \Gamma, \tau\rangle$.

Such a quotient is denoted by $\langle X/R, \Gamma\rangle$ and we note $\mathsf{Quot}(\underset{\sim}{X})$ the set of the quotient structures of the element $\underset{\sim}{X}$ of $\mathcal{X}$.

In the sequel, $\underset{\sim}{X}$ always denotes a member of $\mathcal{X}$ and $\langle X/R, \Gamma\rangle$ an element of $\mathsf{Quot}(\underset{\sim}{X})$. But, we also use the notation $\langle X/R, \Gamma\rangle$ for the topological structure of $\mathcal{X}$ defined on X/R by Γ.

3.2. **The lattice structure of $\mathsf{Quot}(X)$.** We consider the natural order that exists on $\mathsf{Quot}(\underset{\sim}{X})$. Before this, let us recall that if X and Y are two topological spaces, if R is an equivalence relation on X and $\phi : X \to Y$ is a continuous map, then ϕ factors through the topological quotient X/R into a continuous map $\widetilde{\phi} : X/R \to Y$ if and only if $R \leq \ker(\phi)$. The map $\widetilde{\phi}$ is called *the factorization of ϕ through X/R.*

Definition 3.2. Let us consider two quotients $\langle X/R, \Gamma\rangle$ and $\langle X/S, \Delta\rangle$ of the structure $\underset{\sim}{X}$. We say that $\langle X/R, \Gamma\rangle \leq \langle X/S, \Delta\rangle$ if the following two conditions are fulfilled

(1) $R \leq S$;
(2) the factorization $\widetilde{\pi}_S$ of π_S through X/R is an $\mathcal{X}$-morphism from $\langle X/R, \Gamma\rangle$ to $\langle X/S, \Delta\rangle$.

It is obvious that the relation $\leq$ is a partial order on $\mathsf{Quot}(\underset{\sim}{X})$. Moreover, it defines on $\mathsf{Quot}(\underset{\sim}{X})$ a lattice structure which is described in the following lemma.

Lemma 3.3. *The partial order $\leq$ defines a lattice structure on $\mathsf{Quot}(\underset{\sim}{X})$. Moreover, if $\langle X/R, \Gamma\rangle$ and $\langle X/S, \Delta\rangle$ are two quotients of $\underset{\sim}{X}$, then*

- $\langle X/R,\Gamma\rangle \wedge \langle X/S,\Delta\rangle = \langle X/(R\wedge S),\Lambda\rangle$ *where for every divisor* m *of* n *we set*
$$r_m^{\langle X/(R\wedge S),\Lambda\rangle} = \widetilde{\pi}_R^{-1}(r_m^{\langle X/R,\Gamma\rangle}) \cap \widetilde{\pi}_S^{-1}(r_m^{\langle X/S,\Delta\rangle})$$
where $\widetilde{\pi}_R$ *(resp.* $\widetilde{\pi}_S$*) denotes the factorization of* π_R *(resp.* π_S*) through* $X/(R\wedge S)$*;*
- $\langle X/R,\Gamma\rangle \vee \langle X/S,\Delta\rangle = \langle X/(R\vee S),\Upsilon\rangle$ *where* $x \in r_m^{\langle X/(R\vee S),\Upsilon\rangle}$ *if* m *is multiple of*
$$\gcd(\{l \in \mathrm{div}(n) \mid \widetilde{\pi}_{R\vee S,R}^{-1}(x) \cap r_l^{\langle X/R,\Gamma\rangle} \neq \emptyset \textit{ or } \widetilde{\pi}_{R\vee S,S}^{-1}(x) \cap r_l^{\langle X/S,\Delta\rangle} \neq \emptyset\}),$$
where $\widetilde{\pi}_{R\vee S,R}$ *(resp.* $\widetilde{\pi}_{R\vee S,S}$*) denotes the factorization of* $\pi_{R\vee S}$ *through* X/R *(resp. through* X/S*).*

Proof. We first show that $\langle X/(R\wedge S),\Lambda\rangle$ is a quotient of $\underset{\sim}{X}$. Since the maps $\widetilde{\pi}_R$ and $\widetilde{\pi}_S$ are continuous, it is clear that the elements of Λ are closed subspaces. Moreover, the condition (X3) is clearly satisfied. We check directly that by construction the maps $\widetilde{\pi}_R$ and $\widetilde{\pi}_S$ are $\mathcal{X}$-morphisms.

Furthermore, if we denote by $\langle X/T,\Sigma\rangle$ a quotient of $\underset{\sim}{X}$ which is lower than $\langle X/R,\Gamma\rangle$ and than $\langle X/S,\Delta\rangle$, then T must be lower than $R\wedge S$. It remains to prove that Λ defines the greatest structure on $X/(R\wedge S)$ which makes $\widetilde{\pi}_R$ and $\widetilde{\pi}_S$ $\mathcal{X}$-morphisms, but this is clear.

Now, let us prove the existence and the description of the supremum. It follows from the definition of $\langle X/(R\vee S),\Upsilon\rangle$ that

$$r_m^{\langle X/(R\vee S),\Upsilon\rangle} = \bigcup_{r\in\omega} \bigcup_{\substack{(m_1,\ldots,m_r)\in\mathrm{div}(n)^r\\ \gcd(m_1,\ldots,m_r)\in\mathrm{div}(m)}} I_{(m_1,\ldots,m_r)}, \tag{3.1}$$

where $I_{(m_1,\ldots,m_r)}$ is defined by

$$\bigcup_{1\le i\le r} \widetilde{\pi}_{R\vee S,\,R}(r_{m_1}^{\langle X/R,\Gamma\rangle}) \cap \ldots \cap \widetilde{\pi}_{R\vee S,\,R}(r_{m_i}^{\langle X/R,\Gamma\rangle}) \cap \widetilde{\pi}_{R\vee S,\,S}(r_{m_{i+1}}^{\langle X/S,\Delta\rangle}) \cap \ldots \cap \widetilde{\pi}_{R\vee S,\,S}(r_{m_r}^{\langle X/S,\Delta\rangle}).$$

Indeed, on the one hand, if m is divisible by

$$\gcd(\{l \in \mathrm{div}(n) \mid \widetilde{\pi}_{R\vee S,R}^{-1}(x) \cap r_l^{\langle X/R,\Gamma\rangle} \neq \emptyset \text{ or } \widetilde{\pi}_{R\vee S,S}^{-1}(x) \cap r_l^{\langle X/S,\Delta\rangle} \neq \emptyset\}),$$

if

$$\{m_1,\ldots,m_i\} = \{l \in \mathrm{div}(n) \mid \widetilde{\pi}_{R\vee S,\,R}^{-1}(x) \cap r_l^{\langle X/R,\Gamma\rangle} \neq \emptyset\}$$

and if

$$\{m_{i+1},\ldots,m_r\} = \{l \in \mathrm{div}(n) \mid \widetilde{\pi}_{R\vee S,\,S}^{-1}(x) \cap r_l^{\langle X/S,\Delta\rangle} \neq \emptyset\},$$

then x belongs to

$$\widetilde{\pi}_{R\vee S,\,R}(r_{m_1}^{\langle X/R,\Gamma\rangle}) \cap \cdots \cap \widetilde{\pi}_{R\vee S,\,R}(r_{m_i}^{\langle X/R,\Gamma\rangle}) \cap \widetilde{\pi}_{R\vee S,\,S}(r_{m_{i+1}}^{\langle X/S,\Delta\rangle}) \cap \cdots \cap \widetilde{\pi}_{R\vee S,\,S}(r_{m_r}^{\langle X/S,\Delta\rangle}).$$

Conversely, if $(m_1,\ldots,m_r)$ is a r-uple of divisors of n which satisfies

$$\gcd(m_1,\ldots,m_r) \in \mathrm{div}(m)$$

and if x is an element of

$$\widetilde{\pi}_{R\vee S,\,R}(r_{m_1}^{\langle X/R,\Gamma\rangle}) \cap \cdots \cap \widetilde{\pi}_{R\vee S,\,R}(r_{m_i}^{\langle X/R,\Gamma\rangle}) \cap \widetilde{\pi}_{R\vee S,\,S}(r_{m_{i+1}}^{\langle X/S,\Delta\rangle}) \cap \cdots \cap \widetilde{\pi}_{R\vee S,\,S}(r_{m_r}^{\langle X/S,\Delta\rangle})$$

for some $i \in \{1,\ldots,r\}$, then

$$\{m_1,\ldots,m_r\} \subseteq \{l \in \mathrm{div}(n) \mid \widetilde{\pi}_{R\vee S,R}^{-1}(x) \cap r_l^{\langle X/R,\Gamma\rangle} \neq \emptyset \text{ or } \widetilde{\pi}_{R\vee S,S}^{-1}(x) \cap r_l^{\langle X/S,\Delta\rangle} \neq \emptyset\}.$$

Thus,

$$\gcd(\{l \in \mathrm{div}(n) \mid \widetilde{\pi}_{R\vee S,R}^{-1}(x) \cap r_l^{\langle X/R,\Gamma\rangle} \neq \emptyset \text{ or } \widetilde{\pi}_{R\vee S,S}^{-1}(x) \cap r_l^{\langle X/S,\Delta\rangle} \neq \emptyset\})$$

divides $\gcd(m_1,\ldots,m_r)$ which is a divisor of m.

It first follows from the identity (3.1) that the subspaces $r_m^{\langle X/(R\vee S),\Upsilon\rangle}$ of $X/(R\vee S)$ are closed.

Then, we prove that the proposed structure $\langle X/(R\vee S),\Upsilon\rangle$ fulfills the condition (X3) of the axiomatization of the structures of $\mathcal{X}$. Indeed, if x belongs to $X/(R\vee S)$ and if we denote by m_x the integer

$$\gcd(\{l \in \mathrm{div}(n) \mid \widetilde{\pi}_{R\vee S,R}^{-1}(x) \cap r_l^{\langle X/R,\Gamma\rangle} \neq \emptyset \text{ or } \widetilde{\pi}_{R\vee S,S}^{-1}(x) \cap r_l^{\langle X/S,\Delta\rangle} \neq \emptyset\}),$$

it follows successively that

$$\begin{aligned} x \in r_m^{\langle X/(R\vee S),\Upsilon\rangle} \cap r_{m'}^{\langle X/(R\vee S),\Upsilon\rangle} &\Leftrightarrow m_x \in \mathrm{div}(m) \text{ and } m_x \in \mathrm{div}(m') \\ &\Leftrightarrow m_x \in \mathrm{div}(\gcd(m,m')) \\ &\Leftrightarrow x \in r_{\gcd(m,m')}^{\langle X/(R\vee S),\Upsilon\rangle}. \end{aligned}$$

We now prove that the quotient $\langle X/(R\vee S),\Upsilon\rangle$ is greater than $\langle X/R,\Gamma\rangle$ and than $\langle X/S,\Delta\rangle$. According to our definition, we have to check that the maps $\widetilde{\pi}_{R\vee S,R}$ and $\widetilde{\pi}_{R\vee S,S}$ are $\mathcal{X}$-morphisms. We only provide the proof for $\widetilde{\pi}_{R\vee S,R}$ because the proof for $\widetilde{\pi}_{R\vee S,S}$ is completely similar. If z belongs to $r_m^{\langle X/R,\Gamma\rangle}$, then m is an element of $\{l \in \mathrm{div}(n) \mid \widetilde{\pi}_{R\vee S,R}^{-1}(\widetilde{\pi}_{R\vee S,R}(z)) \cap r_l^{\langle X/R,\Gamma\rangle} \neq \emptyset\}$. Thus, by definition of Υ, we obtain that $\widetilde{\pi}_{R\vee S,R}(z) \in r_m^{\langle X/(R\vee S),\Upsilon\rangle}$.

To conclude, we now prove that if $\langle X/T,\Sigma\rangle$ is a quotient of $\underset{\sim}{X}$ which is greater than $\langle X/R,\Gamma\rangle$ and than $\langle X/S,\Delta\rangle$, then it is greater than $\langle X/(R\vee S),\Upsilon\rangle$. We directly obtain by definition that $R\vee S \leq T$. Let us denote by $\widetilde{\pi}_{T,R\vee S} : X/(R\vee S) \to X/T$ the factorization of the quotient map $\pi_T : X \to X/T$ trough $X/(R\vee S)$. This map is continuous and we have to prove that it is an $\mathcal{X}$-morphism. Assume that x belongs to $r_m^{\langle X/(R\vee S),\Upsilon\rangle}$. It means that there is a positive integer r and some positive divisors $m_1,\ldots,m_r$ of n such that $\gcd(m_1,\ldots,m_r) \in \mathrm{div}(m)$ and $x \in I_{(m_1,\ldots,m_r)}$. We can then find a i in $\{1,\ldots,r\}$ such that x belongs to

$$\widetilde{\pi}_{R\vee S,R}(r_{m_1}^{\langle X/R,\Gamma\rangle}) \cap \ldots \cap \widetilde{\pi}_{R\vee S,R}(r_{m_i}^{\langle X/R,\Gamma\rangle}) \cap \widetilde{\pi}_{R\vee S,S}(r_{m_{i+1}}^{\langle X/S,\Delta\rangle}) \cap \ldots \cap \widetilde{\pi}_{R\vee S,S}(r_{m_r}^{\langle X/S,\Delta\rangle}).$$

Now, $\widetilde{\pi}_{T,R\vee S}(x)$ is equal to the image by $\widetilde{\pi}_{T,R}$ (which denotes the factorization or π_T trough X/R) of any element of $\widetilde{\pi}_{R\vee S,R}^{-1}(x)$ and is also equal to the image by $\widetilde{\pi}_{T,S}$ (which denotes the factorization of π_T trough X/S) of any element of $\widetilde{\pi}_{R\vee S,S}^{-1}(x)$. Thus, since $\widetilde{\pi}_{T,R}$ and $\widetilde{\pi}_{T,S}$ are both morphisms, we obtain that $\widetilde{\pi}_{T,R\vee S}(x)$ belongs to $r_{m_1}^{\langle X/T,\Sigma\rangle} \cap \ldots \cap r_{m_r}^{\langle X/T,\Sigma\rangle} = r_m^{\langle X/T,\Sigma\rangle}$. □

The reason for which we have introduced the lattice $\mathsf{Quot}(\underset{\sim}{X})$ is that it is our main tool in the study of the lattice of subalgebras of a member $\underline{A}$ of $\mathcal{A}$, which will be denoted by $\mathsf{Sub}(\underline{A})$ in the sequel.

Proposition 3.4. *If $\underline{A}$ is a member of $\mathcal{A}$ then the lattice $\mathsf{Sub}(\underline{A})$ is anti-isomorphic to $\mathsf{Quot}(\mathsf{D}(\underline{A}))$.*

Proof. This is a direct consequence of proposition 2.1. □

As it is our basic tool, we make use of proposition 3.4 throughout the paper without any reference.

4. Maximal elements in $\mathsf{Sub}(\underline{A})$

Definitions 4.1. A subalgebra $\underline{B}$ of $\underline{A}$ is *maximal* if it is maximal among the proper subalgebras of $\underline{A}$.

Similarly, a quotient structure $\langle X/R,\Gamma\rangle$ of $\underset{\sim}{X}$ is *minimal* if it is minimal among the non trivial quotients of $\underset{\sim}{X}$ (in other words, if it is an atom of $\mathsf{Quot}(\underset{\sim}{X})$).

As stated in the preceding proposition, finding maximal elements in $\mathsf{Sub}(\underline{A})$ is equivalent to finding minimal elements in $\mathsf{Quot}(\mathsf{D}(\underline{A}))$. Therefore, in this section, we characterize the minimal elements of $\mathsf{Quot}(\underset{\sim}{X})$ and show that this lattice is atomic for every $\underset{\sim}{X}$ in $\mathcal{X}$. Dually, it means that for every proper subalgebra $\underline{B}$ of $\underline{A}$, there is a maximal subalgebra $\underline{C}$ of $\underline{A}$ containing $\underline{B}$. We also examine the conditions under which every subalgebra of $\underline{A}$ is the intersection of maximal subalgebras.

4.1. Minimum quotient of X for a given Boolean equivalence.

Given a Boolean equivalence R on the underlying topological space of $\underset{\sim}{X}$, one can easily see that the set of the quotients of $\underset{\sim}{X}$ that are built on X/R is a convex sublattice of $\mathsf{Quot}(\underset{\sim}{X})$ that has a least element.

Lemma 4.2. *If R is a Boolean equivalence on X then the set containing all the structures of $\mathsf{Quot}(\underset{\sim}{X})$ that are built on X/R is a convex sublattice of $\mathsf{Quot}(\underset{\sim}{X})$ whose least element is the structure $\langle X/R, \Gamma_{\mathsf{Min}}^{X/R}\rangle$ where for every divisor m of n we set*

$$r_m^{\langle X/R,\Gamma_{\mathsf{Min}}^{X/R}\rangle} = \bigcup_{\substack{\{m_1,\dots,m_r\}\subseteq \operatorname{div}(n)\\ \gcd(m_1,\dots,m_r)\in\operatorname{div}(m)}} \pi_R(r_{m_1}^{\underset{\sim}{X}}) \cap \cdots \cap \pi_R(r_{m_r}^{\underset{\sim}{X}}).$$

Proof. One directly proves that this structure is a quotient of $\underset{\sim}{X}$. It is then obvious that all the quotients of $\underset{\sim}{X}$ that are constructed on X/R are greater than $\langle X/R, \Gamma_{\mathsf{Min}}^{X/R}\rangle$. □

In most cases, when the context is clear, we simply denote by Γ_{Min} the set $\Gamma_{\mathsf{Min}}^{X/R}$.

To get the algebraic interpretation of this construction, recall (proposition 3.1 in [13]) that the underlying topological space of $\mathsf{D}(\underline{A})$ is homeomorphic to the dual (under the Stone duality for Boolean algebras) of the set $\mathfrak{B}(\underline{A})$ of idempotent elements of $\underline{A}$ (an element x of $\underline{A}$ is *idempotent* if $x \oplus x = x$), which is the greatest subalgebra of $\underline{A}$ to be a Boolean algebra (see [3]).

Hence, Lemma 4.2 shows that for each subalgebra $\mathfrak{C}$ of $\mathfrak{B}(\underline{A})$ the set of elements of $\mathsf{Sub}(\underline{A})$ that have $\mathfrak{C}$ as set of idempotents is a sublattice of $\mathsf{Sub}(\underline{A})$ (that is obvious) and we have obtained a description of the dual of its greatest element.

Lemma 4.2 also gives us some candidates for atomic quotients, namely the structures $\langle X/R, \Gamma_{\mathsf{Min}}\rangle$ where R is a minimal Boolean equivalence on X(*i.e.* an equivalence R for which there exists x and y in X such that $X/R = \{\{z\} \mid z \in X, z \neq x, z \neq y\} \cup \{\{x, y\}\}$). As one should expect, these structures are not *always* atomic, since there could exist strictly between $\underset{\sim}{X}$ and $\langle X/R, \Gamma_{\mathsf{Min}}\rangle$ a structure constructed on the underlying topological space of $\underset{\sim}{X}$.

Lemma 4.3. *If R is a Boolean equivalence on X then*

(1) *the greatest structure $\langle X, \Delta_R\rangle$ constructed on the same underlying topological space as $\underset{\sim}{X}$ which makes $\pi : \langle X, \Delta_R\rangle \to \langle X/R, \Gamma_{\mathsf{Min}}\rangle$ an $\mathcal{X}$-morphism is defined by*
$$x \in r_m^{\langle X,\Delta_R\rangle} \quad\Leftrightarrow\quad \gcd(\{l \in \operatorname{div}(n) \mid x^R \cap r_l^{\underset{\sim}{X}} \neq \emptyset\}) \in \operatorname{div}(m);$$

(2) *if in addition R is a proper minimal Boolean equivalence, the structure $\langle X/R, \Gamma_{\mathsf{Min}}\rangle$ is an atom in $\mathsf{Quot}(\underset{\sim}{X})$ if and only if the sets $r_m^{\underset{\sim}{X}}$ are R-saturated (*i.e. *$r_m^{\underset{\sim}{X}}$ is a join of R-classes for all $m \in \operatorname{div}(n)$).*

Proof. The proof of (1) follows from the fact that
$$r_m^{\langle X,\Delta_R\rangle} = \pi_R^{-1}(r_m^{\langle X/R,\Gamma_{\mathsf{Min}}\rangle}).$$

The second result is a consequence of (1). Indeed, the saturation of the sets $r_m^{\underset{\sim}{X}}$ is equivalent to the equality of $\langle X, \Delta_R\rangle$ with $\underset{\sim}{X}$. □

The preceding Lemma gives us the atoms of $\mathsf{Quot}(\underset{\sim}{X})$ that are constructed on a proper quotient of X. To conclude our quest of atomic elements, we now have to find the minimal quotients that are built on the same topological space as $\underset{\sim}{X}$. Let us first introduce some notations.

Definition 4.4. If $\underset{\sim}{X}$ is an object of $\mathcal{X}$ and m is a divisor of n, we define the subset $s_m^{\underset{\sim}{X}}$ of X by
$$s_m^{\underset{\sim}{X}} = r_m^{\underset{\sim}{X}} \setminus \bigcup_{\substack{k\in\operatorname{div}(m)\\ k\neq m}} r_k^{\underset{\sim}{X}}.$$

Hence, if $\underline{A}$ is an algebra of $\mathcal{A}$ the set $s_m^{\mathsf{D}(\underline{A})}$ contains all the homomorphisms $u : \underline{A} \to \underline{\mathrm{L}}_n$ such that $u(\underline{A}) = \underline{\mathrm{L}}_m$. Moreover, the relation $r_m^{\mathsf{D}(\underline{A})}$ can be recovered by taking the union of the $s_l^{\mathsf{D}(\underline{A})}$ with l running through the divisors of m.

Lemma 4.5. *The structure $\langle X, \Gamma\rangle$ is minimal in $\mathsf{Quot}(\underset{\sim}{X})$ if and only if there is a divisor m of n, a prime divisor p of m and an element x of $s_m^{\underset{\sim}{X}}$ such that*
$$r_k^{\langle X,\Gamma\rangle} = \begin{cases} r_k^{\underset{\sim}{X}} & \text{if } \frac{m}{p} \notin \operatorname{div}(k)\\ r_k^{\underset{\sim}{X}} \cup \{x\} & \text{if } \frac{m}{p} \in \operatorname{div}(k),\end{cases}$$

for all $k \in \operatorname{div}(n)$.

Proof. This proof follows from the fact that $\{x\}$ is a closed subspace of X for all $x \in X$. □

Thus Lemma 4.3 and 4.5 give us the two ways to construct atomic elements in $\mathsf{Quot}(\underset{\sim}{X})$: either we consider the lowest structure $\langle X/R, \Gamma_{\mathsf{Min}}\rangle$ constructible on X/R where R is a suitable atomic Boolean equivalence on X or we shift an element from one of the $s_m^{\underset{\sim}{X}}$ to $s_{\frac{m}{p}}^{\langle X,\Gamma\rangle}$ where p is a prime factor of m.

As a consequence, we obtain the atomicity of $\mathsf{Quot}(\underset{\sim}{X})$. Just recall that a lattice L is *atomic* if for every x in L there exists an atom a under x.

Proposition 4.6. *If* $\underset{\sim}{X}$ *is a member of* $\mathcal{X}$, *then* $\mathsf{Quot}(\underset{\sim}{X})$ *is an atomic lattice. Hence, for any algebra* $\underline{A}$ *of* $\mathcal{A}$ *and any proper subalgebra* $\underline{B}$ *of* $\underline{A}$, *there exists a maximal subalgebra* $\underline{C}$ *of* $\underline{A}$ *containing* $\underline{B}$.

Proof. It is clear that if R is a non trivial Boolean relation on X for which there is an x in X and a divisor m of n such that $|x^R \cap s_m^{\underset{\sim}{X}}| \geq 2$ and if $\langle X/R, \Gamma\rangle$ is a quotient of $\underset{\sim}{X}$, then Lemma 4.3 provides us with an atom under $\langle X/R, \Gamma\rangle$ (namely the quotient $\langle X/S, \Gamma_{\mathsf{Min}}\rangle$ where S is any equivalence $S = \{(z, z) \mid z \in X\} \cup \{(x, y)\}$ where $y \neq x$ and $y \in x^R \cap s_m^{\underset{\sim}{X}}$).

Now, if $\langle X/R, \Gamma\rangle$ is a quotient of $\underset{\sim}{X}$ such that R is trivial or such that for all x in X and all divisor m of n we have $|x^R \cap s_m^{\underset{\sim}{X}}| < 2$, then by Lemma 4.5, we can find an atom under $\langle X/R, \Delta\rangle$. Indeed, on the one hand, if R is the identity relation, then, since the quotient $\langle X, \Delta\rangle$ is a proper quotient of $\underset{\sim}{X}$, there is an $m \in \operatorname{div}(n)$, a $q \in \operatorname{div}(m)$ and an x in X which has been shifted from $s_m^{\underset{\sim}{X}}$ to $s_{m/q}^{\langle X,\Gamma\rangle}$. Now, if p is any prime divisor of q, then the structure $\langle X, \Gamma'\rangle$ defined on X by shifting x from $s_m^{\underset{\sim}{X}}$ to $s_{m/p}^{\langle X,\Gamma\rangle}$ is an atom of $\mathsf{Quot}(\underset{\sim}{X})$ below $\langle X/R, \Gamma\rangle$. On the other hand, if R is not trivial and if for every x in X and every divisor m of n we have $|x^R \cap s_m^{\underset{\sim}{X}}| < 2$, then, if x^R is a non singleton class, there is two divisors m and m' of n and two elements y and z in x^R such that $y \in s_m^{\underset{\sim}{X}}$ and $z \in s_{m'}^{\underset{\sim}{X}}$. If m'' denotes the divisor of n such that $x^R \in s_{m''}^{\langle X/R,\Gamma\rangle}$ and if p is any prime divisor of m/m'' (resp. any prime divisor of m'/m'' if $m = m''$) then the structure $\langle X, \Delta\rangle$ defined on X by shifting y from $s_m^{\underset{\sim}{X}}$ to $s_{m/p}^{\langle X,\Delta\rangle}$ (resp. by shifting z from $s_{m'}^{\underset{\sim}{X}}$ to $s_{m'/p}^{\langle X,\Delta\rangle}$) is an atom of $\mathsf{Quot}(\underset{\sim}{X})$ under $\langle X/R, \Gamma\rangle$. □

4.2. **Minimal set of $\vee$-generators.** We are going to construct a minimal set of $\vee$-generators (we say that the subset G of $\mathsf{Quot}(\underset{\sim}{X})$ is a *set of* $\vee$*-generators* if every element of $\mathsf{Quot}(\underset{\sim}{X})$ is a supremum of elements of G) of $\mathsf{Quot}(\underset{\sim}{X})$ containing the atoms and determine when this set coincides with the set of the atoms of $\mathsf{Quot}(\underset{\sim}{X})$. Dually, we will be able to recognize the algebras $\underline{A}$ in which every proper subalgebra is the intersection of maximal subalgebras. Note that it is the case for every Boolean algebra (so any non trivial Boolean equivalence on a Boolean space X is the supremum of non trivial minimal Boolean relations on X). See [1] for the details.

In the sequel, we denote by $\mathfrak{B}(X)$ (resp. $\mathfrak{B}_{\mathsf{Min}}(X)$) the set of Boolean equivalences (resp. the set of non trivial minimal Boolean equivalences) on the Boolean space X and by $\mathrm{P}(m)$ the set of prime divisors of the integer m.

Definition 4.7. Suppose that $\underset{\sim}{X}$ is a member of $\mathcal{X}$ and that x is an element of $s_m^{\underset{\sim}{X}}$. If p is a prime divisor of m and if l is a positive integer such that $p^l \in \operatorname{div}(m)$, then we define the quotient

$$\langle X, \Gamma_{(x,\, m/p^l)}\rangle$$

of $\underset{\sim}{X}$ by

$$r_k^{\langle X,\Gamma_{(x,\, m/p^l)}\rangle} = \begin{cases} r_k^{\underset{\sim}{X}} & \text{if } \frac{m}{p^l} \notin \operatorname{div}(k) \\ r_k^{\underset{\sim}{X}} \cup \{x\} & \text{if } \frac{m}{p^l} \in \operatorname{div}(k) \end{cases}$$

for all $k \in \operatorname{div}(n)$.

Proposition 4.8. *The set*

$$G = \{\langle X/R, \Gamma_{\mathsf{Min}}\rangle \mid R \in \mathfrak{B}_{\mathsf{Min}}(X)\} \cup \bigcup_{m\in\mathrm{div}(n)} \bigcup_{x\in s_m^{\underset{\sim}{X}}} \{\langle X, \Gamma_{(x,\, m/p^l)}\rangle \mid p \in \mathrm{P}(m) \ \&\ p^l \in \mathrm{div}(m)\}$$

is a minimal set of $\vee$*-generators of* $\mathsf{Quot}(\underset{\sim}{X})$ *which contains the atoms of* $\mathsf{Quot}(\underset{\sim}{X})$*.*

Proof. It is clear that G contains the atoms of $\mathsf{Quot}(\underset{\sim}{X})$. Now, if $\langle X/S, \Delta\rangle$ is a quotient of $\underset{\sim}{X}$, the Boolean equivalence S is the supremum of a subset Φ of $\mathfrak{B}_{\mathsf{Min}}(X)$. We then construct the quotient $\langle X, \Delta'\rangle$ where Δ' is defined by

$$r_m^{\langle X,\Delta'\rangle} = \pi_S^{-1}(r_m^{\langle X/S,\Delta\rangle}).$$

This quotient is obviously the supremum of a subset G' of

$$\bigcup_{m\in\mathrm{div}(n)} \bigcup_{x\in s_m^{\underset{\sim}{X}}} \{\langle X, \Gamma_{(x,\, m/p^l)}\rangle \mid p \in \mathrm{P}(m) \ \&\ p^l \in \mathrm{div}(m)\}.$$

Hence, it follows that

$$\langle X/S, \Delta\rangle = \bigvee G' \vee \langle X/(\bigvee_{R\in\Phi} R), \Gamma_{\mathsf{Min}}\rangle.$$

Finally, one directly proves that the set G is minimal among the $\vee$-generating subsets of $\mathsf{Quot}(\underset{\sim}{X})$. ☐

By counting the elements of G (when X is a finite), we obtain the following corollary.

Corollary 4.9. *If* $\underset{\sim}{X}$ *is a finite* $\mathcal{X}$*-structure with more than two elements, then* $\mathsf{Quot}(\underset{\sim}{X})$ *has a* $\vee$*-generating set containing*

$$\binom{|X|}{2} + \sum_{m\in\mathrm{div}(n)} \sum_{p\in\mathrm{P}(m)} |\{l \in \omega \mid p^l \in \mathrm{div}(m)\}|.|s_m^{\underset{\sim}{X}}|$$

elements.

Hence, an $\mathcal{X}$-structure has a quotient lattice $\vee$-generated by its atoms if and only if the set G introduced in Proposition 4.8 reduces to the set of the atoms of $\mathsf{Quot}(\underset{\sim}{X})$. These structures are characterized in the following proposition.

Proposition 4.10. *An* $\mathcal{X}$*-structure* $\underset{\sim}{X}$ *has a quotient lattice* $\vee$*-generated by its atoms if and only if there is a square-free divisor* m *of* n *such that*

$$s_m^{\underset{\sim}{X}} = X.$$

Proof. Suppose that $\mathsf{Quot}(\underset{\sim}{X})$ is $\vee$-generated by its atoms, and that we can find two divisors m and m' of n and two elements x and y of X such that

$$x \in s_m^{\underset{\sim}{X}} \quad \text{and} \quad y \in s_{m'}^{\underset{\sim}{X}}.$$

Then, the quotient $\langle X/R, \Gamma_{\mathsf{Min}}\rangle$ where we set

$$X/R = \{\{u\} \mid u \in X \setminus \{x, y\}\} \cup \{\{x, y\}\}$$

is an element of G which is not an atom of $\mathsf{Quot}(\underset{\sim}{X})$, according to lemma 4.3.

Similarly, suppose that we can find a divisor m of n and a prime divisor p of m such that p^2 is still a divisor of m and

$$s_m^{\underset{\sim}{X}} = X.$$

Then, if x is an element of X, the quotient

$$\langle X, \Gamma_{(x,\, m/p^2)}\rangle$$

is a member of G but is not an atom of $\mathsf{Quot}(\underset{\sim}{X})$.

Let us now assume that there is a square-free divisor m of n such that $s_m^{\underset{\sim}{X}} = X$. Following our remark which precedes this proposition, it is sufficient to prove that the set G defined in

Proposition 4.8 is a subset of the set of the atoms of $\mathsf{Quot}(\underset{\sim}{X})$. First note that for any element x of X, we have

$$x \in r_k^{\underset{\sim}{X}} \Leftrightarrow m \in \mathrm{div}(k).$$

As a consequence, for every R in $\mathfrak{B}_{\mathsf{Min}}(X)$, the subspaces r_k ($k \in \mathrm{div}(n)$) are R-saturated and $\langle X/R, \Gamma_{\mathsf{Min}}\rangle$ is an atom, according to Lemma 4.3. We so have obtained that $\{\langle X/R, \Gamma_{\mathsf{Min}}\rangle \mid R \in \mathfrak{B}_{\mathsf{Min}}(X)\}$ is a subset of the set of atoms of $\mathsf{Quot}(\underset{\sim}{X})$. To conclude, note that since m is square-free,

$$\bigcup_{m\in\mathrm{div}(n)} \bigcup_{x\in s_m^{\underset{\sim}{X}}} \{\langle X, \Gamma_{(x,\, m/p^l)}\rangle \mid p \in \mathrm{P}(m) \ \&\ p^l \in \mathrm{div}(m)\} = \bigcup_{x\in s_m^{\underset{\sim}{X}}} \{\langle X, \Gamma_{(x,\, m/p)}\rangle \mid p \in \mathrm{P}(m)\}.$$

But Lemma 4.5 and the definition of $\langle X, \Gamma_{(x,\, m/p)}\rangle$ inform us that the quotients of the right-hand side of this identity are atoms, and we have eventually proved that every quotient of G is an atom. □

The dual algebraic counterpart of Proposition 4.10 is the following.

Proposition 4.11. *Suppose that $\underline{A}$ is a member of $\mathcal{A}$. Every proper subalgebra of $\underline{A}$ is an intersection of maximal subalgebras if and only if $\underline{A}$ is a Boolean power of the algebra $\underline{Ł}_m$ where m is a square-free divisor of n. If in addition $\underline{A}$ is finite, it is equivalent to say that $\underline{A}$ is isomorphic to a finite power of the algebra $\underline{Ł}_m$ for a square-free divisor m of n.*

Proof. Apply the duality to Proposition 4.10 with the help of our remark about Boolean representation in the section 2 for the first part of the statement and with the help of Proposition 2.2 for the second part. □

Since every finite MV-algebra is a member of the variety $\mathbb{HSP}(\underline{Ł}_n)$ for some positive integer n, the preceding proposition can also be viewed as a characterization of finite MV-algebras that have a dually atomic lattice of subalgebras.

5. Semimodularity of $\mathsf{Sub}(\underline{A})$

We use our duality to study the semimodularity of the lattice $\mathsf{Sub}(\underline{A})$. As usual, if $\leq$ is an order on L, we write $a \prec b$ (and say b *covers* a) if $a < b$ and there is no c such that $a < c < b$. Then a lattice L is *semimodular* if it satisfies for every a, b and c in L

$$a \prec b \Rightarrow (a \vee c = b \vee c \text{ or } a \vee c \prec b \vee c).$$

As our duality between $\mathsf{Sub}(\underline{A})$ and $\mathsf{Quot}(\mathsf{D}(\underline{A}))$ reverses the order, the lattice $\mathsf{Sub}(\underline{A})$ is semimodular if and only if $\mathsf{Quot}(\mathsf{D}(\underline{A}))$ is *dually semimodular*, i.e. it satisfies the following covering property

$$a \prec b \Rightarrow (a \wedge c = b \wedge c \text{ or } a \wedge c \prec b \wedge c),$$

for all a, b and c in $\mathsf{Quot}(\mathsf{D}(\underline{A}))$.

The characterization of the members of $\mathcal{A}$ that have a semimodular lattice of subalgebras is obtained thanks to the two following lemmas. The first imposes restrictions on the size of the structure, and the second on the structure itself. Before going into details, note that if X is a Boolean space, if R is an equivalence on X whose classes are finite and for which only a finite number of these classes are not a singleton, then R is a Boolean equivalence on X. Indeed, if x and y are not equivalent, then there is for every z in x^R a clopen ω_z that contains z but no element of any other non singleton class. Then, $\bigcup_{z\in x^R} \omega_z$ is a R-saturated clopen of X that separates x and y.

Lemma 5.1. *If $\underset{\sim}{X}$ is an $\mathcal{X}$-structure such that $|\underset{\sim}{X}| \geq 4$ then $\mathsf{Quot}(\underset{\sim}{X})$ is not dually semimodular.*

Proof. Assume that x, y, z and t are four different elements in $\underset{\sim}{X}$. Let us consider the two equivalences R and S where R denotes the equivalence generated by $\{(x, y), (z, t)\}$ and where S is the equivalence generated by $\{(x, y), (y, z), (z, t)\}$. We then construct the two quotients $\langle X/R, \Gamma\rangle$ and $\langle X/S, \Delta\rangle$ defined by

$$r_m^{\langle X/R,\Gamma\rangle} = \pi_R(r_m^{\underset{\sim}{X}}) \cup \{\pi_R(x), \pi_R(z)\} \quad \forall m \in \mathrm{div}(n),$$

and

$$r_m^{\langle X/S,\Delta\rangle} = \pi_S(r_m^{\underset{\sim}{X}}) \cup \{\pi_S(x)\} \quad \forall m \in \mathrm{div}(n).$$

Hence, we have $\langle X/R,\Gamma\rangle \prec \langle X/S,\Delta\rangle$. Finally, consider the relation T on X where T is generated by $\{(x,z),(y,t)\}$ and define the structure $\langle X/T,\Upsilon\rangle$ by

$$r_m^{\langle X/T,\Upsilon\rangle} = \pi_T(r_m^{\underset{\sim}{X}}) \cup \{\pi_T(x),\pi_T(y)\} \quad \forall m \in \mathrm{div}(n).$$

We obtain that $\langle X/R,\Gamma\rangle \wedge \langle X/T,\Upsilon\rangle = \langle X,\Sigma\rangle$ where Σ is defined by

$$r_m^{\langle X,\Sigma\rangle} = r_m^{\underset{\sim}{X}} \cup \{x,y,z,t\} \quad \forall m \in \mathrm{div}(n),$$

and that $\langle X/S,\Delta\rangle \wedge \langle X/T,\Upsilon\rangle = \langle X/T,\Upsilon\rangle$. Therefore, $\langle X/R,\Gamma\rangle \wedge \langle X/T,\Upsilon\rangle \not\prec \langle X/S,\Delta\rangle \wedge \langle X/T,\Upsilon\rangle$ and $\langle X/R,\Gamma\rangle \wedge \langle X/T,\Upsilon\rangle \neq \langle X/S,\Delta\rangle \wedge \langle X/T,\Upsilon\rangle$. □

Lemma 5.2. *Assume that you can find in $\underset{\sim}{X}$ two elements x and y such that $x \in s_{m_x}^{\underset{\sim}{X}}$ and $y \in s_{m_y}^{\underset{\sim}{X}}$ with $\gcd(m_x,m_y) \neq 1$. Then $\mathsf{Quot}(\underset{\sim}{X})$ is not dually semimodular.*

Proof. Let us consider the structure $\langle X/R,\Gamma_{\mathsf{Min}}\rangle$ where R is the equivalence generated by (x,y). To construct a structure $\langle X/R,\Delta\rangle$ covering $\langle X/R,\Gamma_{\mathsf{Min}}\rangle$, consider a prime factor p of $m = \gcd(m_x,m_y)$ and define Δ by

$$r_k^{\langle X/R,\Delta\rangle} = \begin{cases} r_k^{\langle X/R,\Gamma_{\mathsf{Min}}\rangle} & \text{if } \frac{m}{p} \notin \mathrm{div}(k) \\ r_k^{\langle X/R,\Gamma_{\mathsf{Min}}\rangle} \cup \{\pi_R(x)\} & \text{if } \frac{m}{p} \in \mathrm{div}(k). \end{cases}$$

We finally consider the structure $\langle X,\Upsilon\rangle$ defined by

$$r_k^{\langle X,\Upsilon\rangle} = \begin{cases} r_k^{\underset{\sim}{X}} & \text{if } \frac{m}{p} \notin \mathrm{div}(k) \\ r_k^{\underset{\sim}{X}} \cup \{x,y\} & \text{if } \frac{m}{p} \in \mathrm{div}(k). \end{cases}$$

Then, the structure $\langle X/R,\Gamma_{\mathsf{Min}}\rangle \wedge \langle X,\Upsilon\rangle$ is obtained from $\underset{\sim}{X}$ by shifting x and y into

$$s_m^{\langle X/R,\Gamma_{\mathsf{Min}}\rangle \wedge \langle X,\Upsilon\rangle}$$

and $\langle X/R,\Delta\rangle \wedge \langle X,\Upsilon\rangle$ is obtained from $\underset{\sim}{X}$ by shifting x and y into $s_{\frac{m}{p}}^{\langle X/R,\Delta\rangle \wedge \langle X,\Upsilon\rangle}$. Hence, the two preceding structures are not covering each other. □

Proposition 5.3. *The lattice $\mathsf{Quot}(\underset{\sim}{X})$ is dually semimodular if and only if one of the following conditions is fulfilled*

1. $|\underset{\sim}{X}| = 1$;
2. $\underset{\sim}{X} = \{x,y\}$ *and* $x \in s_{m_x}^{\underset{\sim}{X}}$, $y \in s_{m_y}^{\underset{\sim}{X}}$ *with* $\gcd(m_x,m_y) = 1$;
3. $\underset{\sim}{X} = \{x,y,z\}$ *and* $x \in s_{m_x}^{\underset{\sim}{X}}$, $y \in s_{m_y}^{\underset{\sim}{X}}$, $z \in s_{m_z}^{\underset{\sim}{X}}$ *with* m_x, m_y *and* m_z *pairwise relatively prime.*

Proof. If $|\underset{\sim}{X}| = 1$ then $\mathsf{Quot}(\underset{\sim}{X})$ is isomorphic to the lattice $\mathsf{div}(m)$ for a divisor m of n (apply the machinery of Proposition 2.2 about the dualization of finite structures to this one point structure $\underset{\sim}{X}$), and hence is dually semimodular.

If $\underset{\sim}{X}$ satisfies the second condition, then the lattice $\mathsf{Quot}(\underset{\sim}{X})$ is isomorphic to $\mathsf{div}(m_x) \times \mathsf{div}(m_y) \cup \{1\}$ where 1 is defined as an element covering $(1,1)$.

If $|\underset{\sim}{X}| = 3$, the proposed structures are obtained according to the restrictions of the preceding lemmas. It is then a matter of computation to verify that these structures are dually semimodular. □

So we have the following exhaustive enumeration of the members of $\mathcal{A}$ that have a semimodular lattice of subalgebras. Of course, as for proposition 4.11, this proposition can be easily extended to the class of finite MV-algebras.

Proposition 5.4. *Assume that $\underline{A}$ is a member of $\mathbb{HSP}(\underline{L}_n)$. Then, the lattice $\mathsf{Sub}(\underline{A})$ is semimodular if and only if $\underline{A}$ is isomorphic to one of the following algebras*

- $\underline{L}_m$ *where m is a divisor of n,*
- $\underline{L}_m \times \underline{L}_{m'}$ *where m and m' are relatively prime divisors of n,*

- $\underline{L}_m \times \underline{L}_{m'} \times \underline{L}_{m''}$ *where* m, m' *and* m'' *are pairwise relatively prime divisors of* n.

Proof. By Proposition 3.4, the lattice $\mathsf{Sub}(\underline{A})$ is anti-isomorphic to the lattice $\mathsf{Quot}(\mathsf{D}(\underline{A}))$. Hence, $\mathsf{Sub}(\underline{A})$ is semimodular if and only if $\mathsf{Quot}(\mathsf{D}(\underline{A}))$ is dually semimodular. Thus, Proposition 5.3 characterizes the dual of the algebras of $\underline{A}$ which have a semimodular lattice of subalgebras. Finally, an application of Proposition 2.2 to these structures gives the desired result. □

At this point of our development, one can wonder if the fact that the algebras whose subalgebra lattice is semimodular are finite is peculiar to the finitely generated varieties of MV-algebras, or if it is also true in some non-finitely generated varieties. We show that the algebra $\mathcal{C}$ of Chang (introduced by C.C. Chang in [3]) is an infinite algebra whose lattice of subalgebras is semimodular. Since this algebra can be found in every non finitely generated subvariety of $\mathcal{MV}$, we can then conclude that the finitely generated varieties of MV-algebras are the only ones whose algebras having a semimodular lattice of subalgebras are among the finite algebras.

First recall that the MV-algebra $\mathcal{C} = \langle C, \oplus, \odot, \neg, (0,0), (1,0)\rangle$ of Chang is defined on

$$C = \{(0,a) \mid a \in \mathbb{Z}^+\} \cup \{(1,b) \mid b \in \mathbb{Z}^-\},$$

by

$$(i,x) \oplus (j,y) = \begin{cases} (0, x+y) & \text{if } i+j=0 \\ (1, \min(0, x+y)) & \text{if } i+j=1 \\ (1,0) & \text{if } i+j=2 \end{cases}$$

and

$$\neg(i,x) = \begin{cases} (0,-x) & \text{if } i=1 \\ (1,-x) & \text{if } i=0. \end{cases}$$

One can convince oneself quite easily that every element in $\mathsf{Sub}(\mathcal{C})$ is isomorphic to $\mathcal{C}$ and that $\mathsf{Sub}(\mathcal{C})$ is an isomorphic copy of the lattice $\mathsf{div}(\mathbb{Z}^+ \setminus \{0\})$ of the positive integers ordered by divisibility and is so semimodular.

6. Modularity and distributivity of $\mathsf{Sub}(\underline{A})$

Our next job is to determine for which algebras $\underline{A}$ the lattice $\mathsf{Sub}(\underline{A})$ is dually semimodular. Since our duality reverses the order, this question is equivalent to find the structures $\underset{\sim}{X}$ in $\mathcal{X}$ such that $\mathsf{Quot}(\underset{\sim}{X})$ is semimodular.

Proposition 6.1. *If* $\underset{\sim}{X}$ *is a member of* $\mathcal{X}$, *then* $\mathsf{Quot}(\underset{\sim}{X})$ *is semimodular. Equivalently, if* $\underline{A}$ *is a member of* $\mathcal{A}$, *then* $\mathsf{Sub}(\underline{A})$ *is dually semimodular.*

Proof. If $\langle X/R, \Gamma\rangle$ belongs to $\mathsf{Quot}(\underset{\sim}{X})$, the set of quotients of $\underset{\sim}{X}$ that cover $\langle X/R, \Gamma\rangle$ is exactly the set of the atoms of the filter of $\mathsf{Quot}(\underset{\sim}{X})$ generated by $\langle X/R, \Gamma\rangle$. This filter is in turn isomorphic to $\mathsf{Quot}(\langle X/R, \Gamma\rangle)$. Thus, Lemma 4.3, Lemma 4.5 and the remark succeeding Lemma 4.5 give the two only possibilities that we should consider to construct a quotient of $\underset{\sim}{X}$ which covers $\langle X/R, \Gamma\rangle$.

Hence, first assume that $\langle X/R, \Gamma\rangle$ and $\langle X/R, \Gamma'\rangle$ are two quotients of $\underset{\sim}{X}$ such that Γ' is obtained by shifting an element $x \in s_m^{\langle X/R,\Gamma\rangle}$ into $s_{\frac{m}{p}}^{\langle X/R,\Gamma'\rangle}$ where p is a prime divisor of m:

$$r_k^{\langle X/R,\Gamma'\rangle} = \begin{cases} r_k^{\langle X/R,\Gamma\rangle} & \text{if } \frac{m}{p} \notin \mathrm{div}(k) \\ r_k^{\langle X/R,\Gamma\rangle} \cup \{x\} & \text{if } \frac{m}{p} \in \mathrm{div}(k), \end{cases}$$

so that $\langle X/R, \Gamma\rangle \prec \langle X/R, \Gamma'\rangle$.

Now, if $\langle X/S, \Delta\rangle$ is an element of $\mathsf{Quot}(\underset{\sim}{X})$, the structure $\langle X/R, \Gamma\rangle \vee \langle X/S, \Delta\rangle$ and $\langle X/R, \Gamma'\rangle \vee \langle X/S, \Delta\rangle$ are defined on $X/(R \vee S)$.

Moreover, if y is an element of $X/(R \vee S)$ and if $x \notin \widetilde{\pi}_{R\vee S,\,R}^{-1}(y)$, then $y \in s_m^{\langle X/R,\Gamma\rangle\vee\langle X/S,\Delta\rangle}$ if and only if $y \in s_m^{\langle X/R,\Gamma'\rangle\vee\langle X/S,\Delta\rangle}$. But if y is the element of $X/(R\vee S)$ such that $\widetilde{\pi}_{R\vee S,\,R}(x) = y$, then $y \in s_m^{\langle X/R,\Gamma\rangle\vee\langle X/S,\Delta\rangle}$ and $y \in s_m^{\langle X/R,\Gamma'\rangle\vee\langle X/S,\Delta\rangle}$ or $y \in s_{\frac{m}{p}}^{\langle X/R,\Gamma'\rangle\vee\langle X/S,\Delta\rangle}$. Therefore, $\langle X/R, \Gamma\rangle \vee \langle X/S, \Delta\rangle = \langle X/R, \Gamma'\rangle \vee \langle X/S, \Delta\rangle$ or $\langle X/R, \Gamma\rangle \vee \langle X/S, \Delta\rangle \prec \langle X/R, \Gamma'\rangle \vee \langle X/S, \Delta\rangle$.

Consider then two quotients $\langle X/R, \Gamma\rangle \prec \langle X/R', \Gamma'\rangle$ for which there are a divisor m of n and two elements $x, y \in s_m^{\langle X/R,\Gamma\rangle}$ such that X/R' is the partition of X defined by

$$X/R' = \{z^R \mid z \notin \pi_R^{-1}(x) \cup \pi_R^{-1}(y)\} \cup \{\pi_R^{-1}(x) \cup \pi_R^{-1}(y)\},$$

(the set Γ' being the one defined to ensure that the structure $\langle X/R, \Gamma'\rangle$ covers $\langle X/R, \Gamma\rangle$).

If $\langle X/S, \Delta\rangle$ is a third quotient of $\underset{\sim}{X}$, we first show that the Boolean equivalences $R' \vee S$ and $R \vee S$ are equal or that the first covers the second (*i.e.* we show that the lattice $\mathfrak{B}(X)$ is semimodular). Indeed, if we denote by x_1 an element of $\pi_R^{-1}(x)$ and by y_1 an element of $\pi_R^{-1}(y)$, it follows that $R \vee S = R' \vee S$ if and only if $(x_1, y_1) \in R \vee S$. Otherwise, if $R \vee S \neq R' \vee S$, it follows that

$$X/(R' \vee S) = \{z^{R\vee S} \mid z \notin \pi_R^{-1}(x) \cup \pi_R^{-1}(y)\} \cup \{x_1^{R\vee S} \cup y_1^{R\vee S}\},$$

and hence $R \vee S \prec R' \vee S$.

Now, if $R \vee S = R' \vee S$, then one easily shows that $\langle X/R, \Gamma\rangle \vee \langle X/S, \Delta\rangle = \langle X/R', \Gamma'\rangle \vee \langle X/S, \Delta\rangle$ and if $R \vee S \prec R' \vee S$ that $\langle X/R, \Gamma\rangle \vee \langle X/S, \Delta\rangle \prec \langle X/R', \Gamma'\rangle \vee \langle X/S, \Delta\rangle$. □

Recall that a lattice L is *modular* if it satisfies

$$(x \geq z) \Rightarrow (x \wedge y) \vee z = x \wedge (y \vee z).$$

Since the modularity of a lattice implies the semimodularity of this lattice, and since for a finite lattice, being modular is equivalent to being both semimodular and dually semimodular, we obtain the characterization of the algebras $\underline{A}$ of $\mathcal{A}$ that have a modular lattice of subalgebras.

Corollary 6.2. *Assume that $\underline{A}$ is a member of $\mathbb{HSP}(\underline{Ł}_n)$. Then, the lattice $\mathsf{Sub}(\underline{A})$ is modular if and only if $\underline{A}$ is isomorphic to one of the following algebras*

- *$\underline{Ł}_m$ where m is a divisor of n,*
- *$\underline{Ł}_m \times \underline{Ł}_{m'}$ where m and m' are relatively prime divisors of n,*
- *$\underline{Ł}_m \times \underline{Ł}_{m'} \times \underline{Ł}_{m''}$ where m, m' and m'' are pairwise relatively prime divisors of n.*

To find the algebras $\underline{A}$ whose lattice of subalgebras is distributive, we have to look among the ones that have a modular lattice of subalgebras.

Corollary 6.3. *Assume that $\underline{A}$ is a member of $\mathbb{HSP}(\underline{Ł}_n)$. Then, the lattice $\mathsf{Sub}(\underline{A})$ is distributive if and only if $\underline{A}$ is isomorphic to one of the following algebras*

- *$\underline{Ł}_m$ where m is a divisor of n,*
- *$\underline{Ł}_m \times \underline{Ł}_{m'}$ where m and m' are relatively prime divisors of n.*

Proof. Since the lattice of subalgebras of the 8-element Boolean algebra is a sublattice of $\mathsf{Sub}(\underline{Ł}_m \times \underline{Ł}_{m'} \times \underline{Ł}_{m''})$ for all divisors m, m' and m'' of n, and since the former is not distributive (it is indeed isomorphic to the diamond), the MV-algebras $\underline{Ł}_m \times \underline{Ł}_{m'} \times \underline{Ł}_{m''}$ do not have a distributive subalgebra lattice.

Then, on the one hand, since the subalgebras of $\underline{Ł}_n$ are exactly the algebras $\underline{Ł}_m$ with $m \in \mathrm{div}(n)$, the lattice $\mathsf{Sub}(\underline{Ł}_n)$ is dually isomorphic to the lattice of the divisors of n. On the other hand, if m and m' are relatively prime, we have already shown in the proof of proposition 5.3 that the lattice $\mathsf{Sub}(\underline{Ł}_m \times \underline{Ł}_{m'})$ is dually isomorphic to the lattice $\mathsf{div}(m) \times \mathsf{div}(m') \cup \{1\}$ (where 1 is an element covering (1,1)), which is distributive. □

As usual now, the two preceding corollaries can be extended to a characterization of the class of finite MV-algebras that have a modular (resp. distributive) lattice of subalgebras.

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