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# CONDITIONS FOR ELASTO-PLASTIC STRAIN LOCALIZATION IN UNSATURATED SOILS*

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Abstract. The necessary and sufficient conditions for elasto-plasic strain localization initiation in unsaturated soil massifs are proposed. Unsaturated behaviour is described using net stress and suction as independent stress variables and modifying the basic constitutive surface parameters and hardening rules to consider the role of suction.The process is accepted to be static and isothermal. The conception that the localization band initiates due to the loss of stability of deformation process is used. In that case, the rates of process measures bifurcate. A special example is given to illustrate the application of the obtained conditions.
Key words: elaso-plasic localization, conditions for localization initiation, unsaturated soil.

## 1. Introduction

The landslide events pose serious practical and theoretical problems for the infrastructure growth and environment protection. To overcome the consequences of catastrophic landslide incidents the mechanisms of the soil

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movements leading to instability and failure of landslides need to be investigated. Among the soil failure mechanisms the strain localization is one of the more important because the localization bands prepare crack initiation and growth leading to catastrophic landslide ([10], [14]). In this article the attention is fixed on the following problems:

- Examination of the mechanical properties of typical soils and the deformation and failure mechanisms preparing the landslide incident.
- Establishing the most proper mathematical model in frames of continuum mechanics describing the hydro-mechanical behavior of typical soils encountered in landslides fields.
- Investigation of the initiation and growth of shear strain localization bands as fracture preparing events.

## 2. Mechanical models for soils typically encountered in landslide fields

### 2.1. General remarks

The typical soils for landslides fields are porous materials with complex structure which consists of solid skeleton, liquid or liquid and gas phases in the pores depending if the soil is saturated or unsaturated. For the inelastic behavior of the solid skeleton the existing of several limit surfaces for plasticity and failure is commonly encountered. It is necessary to apply coupled models including skeleton deformation, damage and failure and fluids flow interaction in order to model the soil behavior, see for examples ([3], [11], [13], [15]).

Special attention has to be devoted to the shear failure and pore collapse plastic mechanisms and to the corresponding model parameters and their change during both mechanical and hydraulic loading processes. The suction effect and the diffusion mechanism have to be also considered, [5]. In practical applications usually the unsaturated media discussed is partially saturated with water and gas, [12] or water and oil [5]. Approximately the process could be assumed to be isothermal and that is why no temperature effects are considered here. In this study time dependent effects will not be discussed.

### 2.2 Model measures

#### 2.2.1 Stress state measures

The spatial rectangular coordinate system is denoted as $Ox_i$, $i = 1, 2, 3$. The following stress tensors are used [8]:

- $\sigma_{ij}$ – the total stress tensor.
- $\sigma^*_{ij}$ – the net stress tensor defined as:

$$\sigma^*_{ij} = \sigma_{ij} - p_a\delta_{ij}\,, \qquad i, j = 1, 2, 3\,. \tag{1}$$

- $\sigma'_{ij}$ – the effective stress tensor defined as:

$$\sigma'_{ij} = \sigma_{ij} - p_w \delta_{ij}\,, \qquad i,j = 1,2,3, \tag{2}$$

where $p_a$ is the gas pressure, $p_w$ is the water pressure and $\delta_{ij}$ is the Kronecker symbol.

To take into account the interaction between the fluids in the pores the special measure – suction, $s$ – is introduced by the relation:

$$s = p_a - p_w\,. \tag{3}$$

It is convenient to separate the tensor measures on volumetric and deviatoric parts:

$$s^*_{ij} = s_{ij}\,, \quad s_{ij} = \sigma_{ij} - \frac{1}{3} I_\sigma \delta_{ij}\,, \quad I_\sigma = \sigma_{kk} = -3p\,, \tag{4}$$

$$\sigma^*_{ij} = s_{ij} - (p + p_a)\,\delta_{ij}\,, \quad I^*_\sigma = -3\,(p + p_a)\,,$$

where $s_{ij}$ is the stress deviator and $p$ is the total pressure, positive in compression.

The measures for the stress state in unsaturated soils form the manifold $\{s_{ij}, I^*_\sigma, s\}$. In the case of saturated media $p_w = p_a$ and then $s = 0$. In case the gas is allowed freely to move through the soil, $p_a$ can be considered as a constant and taken to be equal to the atmospheric pressure. Taking the atmospheric pressure as a zero level we have $p_a = 0$ and therefore $s = -p_w$ and $\sigma_{ij} = \sigma^*_{ij}$. The corresponding stress state manifold is $\{s_{ij}, I_\sigma, s\}$. The equation of equilibrium including the mass force $\rho b_i$ is:

$$\sigma_{ij,j} + \rho b_i = 0, \quad (i,j = 1,2,3)\,, \tag{5}$$

where $\rho$ is the total mass density of the unsaturated media.

#### 2.2.2. Strain rate state measures

We introduce the strain rate tensor $\{\lambda_{ij}\}$ according to the next well-known definition:

$$\lambda_{ij} = \frac{1}{2}\,(V_{i,j} + V_{i,j})\,, \tag{6}$$

where $V_i$ is the total velocity of the soil particle and $V_{i,j}$ is the velocity gradient. For the velocity of the liquid in the soil pores we use the notation $V_{(w)i}$.

We assume that the strain rate tensor is separated into two additive parts: the first $\left\{\lambda'_{ij}\right\}$ is the strain rate induced by the net stress and the second $\left\{\lambda''_{ij}\right\}$ is the suction induced strain rate. Then:

$$\lambda_{ij} = \lambda'_{ij} + \lambda''_{ij}\,. \tag{7}$$

The second part of the strain rate tensor is only spherical i.e. $\lambda''_{ij} = \frac{1}{3}\lambda''_v\delta_{ij}$, because of material isotopy and due to the fact that it is conjugated with the suction which has a meaning of pressure, i.e. $s = -p_w$.

The soil behavior is elasto-plastic and it is assumed that the strain rate tensor is separated into two additive parts – elastic with upper index $e$ and plastic with upper index $p$. After this decomposition we have:

$$\lambda_{ij} = \lambda^e_{ij} + \lambda^p_{ij}, \quad \lambda^e_{ij} = \lambda'^e_{ij} + \frac{1}{3}\lambda''^e_v\delta_{ij}\,, \tag{8}$$

$$\lambda^p_{ij} = \lambda'^p_{ij} + \frac{1}{3}\lambda''^p_v\delta_{ij}\,.$$

**2.2.3. Mass and volumetric measures**

The internal structure of unsaturated soil leads to the following mass and volumetric measures:

- $\rho_s$ – mass density of the skeleton;
- $\rho_w$ – mass density of the water;
- $V$ – volume of the representative element;
- $V_v$ – volume of the pores;
- $V_s$ – volume of the solid part;
- $V_w$ – volume of water.

The measures for the porosity $n$ and the void ratio $e$ are introduced by definition as it follows:

$$e = \frac{V_v}{V_s}, \quad n = \frac{V_v}{V} = \frac{e}{1+e}, \quad V_s + V_v = V\,. \tag{9}$$

$$e = e_0 \;\text{ is the initial void ratio.} \tag{10}$$

**2.2.4. Internal state measures**

We assume that the loading and deformation histories influence the mechanical response of the soil through the set of internal state measures or internal state variables. For the present state of the investigation it is supposed that the inelastic deformation is the only internal state variable under consideration. Then the internal state variables form the manifold $\left\{\varepsilon_{eq}'^p, \varepsilon_v'^p, \varepsilon_v''^p\right\}$ where:

$$\varepsilon_{eq}'^p = \int_{t_0}^{t} \lambda_{eq}'^p dt, \quad \lambda_{eq}'^p = \sqrt{\frac{3}{2}\bar{\lambda}_{ij}'^p \bar{\lambda}_{ij}'^p}, \quad \bar{\lambda}_{ij}'^p = \lambda_{ij}'^p - \frac{1}{3}\delta_{ij}\lambda_v'^p, \tag{11}$$

$$\varepsilon_v'^p = \int_{t_0}^{t} \lambda_v'^p dt, \quad \lambda_v'^p = \lambda_{kk}'^p, \tag{12}$$

$$\varepsilon_v''^p = \int_{t_0}^{t} \lambda_v''^p dt, \quad \lambda_v''^p = \lambda_{kk}''^p. \tag{13}$$

Here $t_0$ is the time when the process begins.

### 2.2.5. Measures for the liquid phase

For the liquid phase (water) the following measures are introduced:

- $S_{(w)r}$ – degree of water saturation:

$$S_{(w)r} = \frac{V_w}{V}.$$

- $f_{(w)}^*$ – water storage, defined with the relation:

$$f_{(w)}^* = \rho_w\, n\, S_{(w)r}. \tag{14}$$

The corresponding water balance equation reads:

$$\frac{\partial f_{(w)}^*}{\partial t} + \left(\rho_w V_{(w)i}\right)_{,i} = 0. \tag{15}$$

## 3. Mathematical model for the mechanical behavior of non-saturated soils

### 3.1 General assumptions

The basic assumptions commonly accepted in mechanics of unsaturated media are applied in the present study, see for more thorough literature survey references [12] and [13].The assumptions read:

(I) The strain rate tensor is composed by elastic and inelastic parts (See § 2.2.2)

(II) The elastic part of the first strain-rate tensor $\lambda'^{e}_{ij}$ is related to the Jaumann derivative of the net stress tensor, $\tilde{\sigma}^*_{ij}$ as it follows:

$$\lambda'^{e}_{ij} = H^{e}_{ijkl}\,\tilde{\sigma}^*_{kl}, \quad (i,j,k,l = 1,2,3) \tag{16}$$

or

$$\tilde{\sigma}^*_{ij} = C^{e}_{ijkl}\,\lambda'^{e}_{kl}, \qquad \tilde{\sigma}^*_{ij} = \dot{\sigma}^*_{ij} + \sigma^*_{ik}\Omega_{kj} - \Omega_{ik}\sigma^*_{kj}, \qquad \left\{C^{e}_{ijkl}\right\} = \left\{H^{e}_{ijkl}\right\}^{-1},$$

where $H^{e}_{ijkl}$ is the elastic compliance tensor of order four and $C^{e}_{ijkl}$ is the four order tensor of elastic moduli; $\Omega_{ij} = \dfrac{1}{2}\left(V_{i,j} - V_{i,j}\right)$ is the antisymmetric spin tensor.

(III) The plastic flow rule is non associate and the multisurface yield condition is explored. The plastic flow is expressed by the following set of equations:

$$\lambda'^{p}_{ij} = \dot{\lambda}'_{p(K)}\,\frac{\partial g_{(K)}}{\partial \sigma_{ij}}, \qquad K = I, II, \ldots, N \tag{17}$$

with $N$ the number of yield surfaces and

$$\dot{\lambda}'_{p(K)} \quad \begin{cases} = 0, & \text{if} \quad f_{(K)} < 0 \quad \text{or} \quad f_{(K)} = 0 \quad \text{but} \quad L_{(K)} \le 0 \\ > 0, & \text{if} \quad f_{(K)} = 0 \quad \text{and} \quad L_{(K)} > 0 \end{cases} \tag{18}$$

$L_{(K)}$ is the loading condition function for the $K$-th yield mechanism and

$$f_{(K)}\left(\sigma_{ij}, s, \varepsilon'^{p}_{ij}, \varepsilon''^{p}_{v}\right) = 0, \quad K = I, II, \ldots, N \tag{19}$$

is the yield function for the $K$-type inelastic deformation mechanism. The plastic potential function $g_{(K)}$ coupled with the yield function $f_{(K)}$ is defined as:

$$g_{(K)} = g_{(K)}\left(\sigma_{ij}, s, \varepsilon'^{p}_{ij}, \varepsilon''^{p}_{v}\right), \quad K = I, II, \ldots, N\ . \tag{20}$$

(IV) The constitutive equations for the suction induced elastic and plastic response are:

$$\lambda''^{e}_{ij} = h^{e}_{ij}\dot{s} = \frac{1}{3}h^{e}\delta_{ij}\dot{s}, \quad \lambda''^{p}_{ij} = h^{p}_{ij}\dot{s} = \frac{1}{3}h^{p}\delta_{ij}\dot{s} \tag{21}$$

$$h^p \begin{cases} = 0, & \text{if} \quad f''(s) < 0 \\ \neq 0 & \text{if} \quad f''(s) = 0 \end{cases}$$

or

$$\lambda''_{ij} = h_{ij}\dot{s} = \frac{1}{3}\delta_{ij} h \dot{s}, \quad h = h^e + h^p \ , \tag{22}$$

where the tensors $h^e_{ij}$ and $h^p_{ij}$ depend on the suction and $f''(s) = 0$ is the yield condition concerning the suction dependent loading paths.

Using the consistency condition $d\, f_{(K)} = 0$ and the constitutive equations (16)–(22), after some algebraic transformations we obtain the following expression for the plastic multiplier:

$$\dot{\lambda}'_{p(K)} = \frac{N_{(K)ij}\,\lambda_{ij}}{G_{(K)}} - \frac{F_{(K)}}{G_{(K)}}\,h\,\dot{s} \ , \tag{23}$$

where

$$M_{(K)} = \sqrt{\frac{\partial g_{(K)}}{\partial \sigma_{ij}}\frac{\partial g_{(K)}}{\partial \sigma_{ij}} - \frac{1}{3}\left(\frac{\partial g_{(K)}}{\partial \sigma_{ij}}\delta_{ij}\right)^2} \ , \tag{24}$$

$$G_{(K)} = \frac{\partial f_{(K)}}{\partial \sigma_{ij}}\, C^e_{ijkl}\frac{\partial g_{(K)}}{\partial \sigma_{kl}} - \frac{\partial f_{(K)}}{\partial \varepsilon'^p_{eq}}\, M_{(K)} - \frac{\partial f_{(K)}}{\partial \varepsilon'^p_v}\,\frac{\partial g_{(K)}}{\partial \sigma_{ij}}\,\delta_{ij}$$

and

$$F_{(K)} = \frac{1}{3}\,\frac{\partial f_{(K)}}{\partial \sigma_{ij}}\, C^e_{ijkl}\,\delta_{kl} - \frac{\partial f_{(K)}}{\partial \varepsilon''^p_v} - \frac{\partial f_{(K)}}{\partial s} \ ,$$

$$N_{(K)ij} = \frac{\partial f_{(K)}}{\partial \sigma_{kl}}\, C^e_{klij} \ . \tag{25}$$

Finally we can write the relation for the connection between stress Jaumann derivative, strain rate and rate of the suction:

$$\tilde{\sigma}^*_{ij} = C^{ep}_{(K)ijkl}\,\lambda_{kl} - V_{(K)ij}\,h\,\dot{s} \ . \tag{26}$$

The notations used in (26) read:

$$C^{ep}_{(K)ijkl} = C^e_{ijkl} - \frac{\partial g_{(K)}}{\partial \sigma_{mn}} C^e_{ijmn}\,\frac{N_{(K)kl}}{G_K} \ , \tag{27}$$

$$V_{(K)ij} = \frac{1}{3}\,C^e_{ijkl}\,\delta_{kl} - \frac{\partial g_{(K)}}{\partial \sigma_{kl}} C^e_{ijkl}\,\frac{F_{(K)}}{G_K} \ . \tag{28}$$

(V) For the water flow in the soil we have the following set of governing equations:

Darcy's law:

$$V_{(w)i} = -\frac{K\left(S_{(w)r}\right)}{\mu_{(w)}}\left[p_{w,i} + g\rho_w y_{,i}\right], \tag{29}$$

where $K$ is the permeability coefficient, which depends on the degree of water saturation $S_{(w)r}$; $\mu_{(w)}$ is the dynamic viscosity of the water; $y$ is the vertical upward directed coordinate; $g$ is the gravity acceleration.

Using equation (14) the following relation in rates can be obtained:

$$\dot{f}^*_{(w)} = \dot{\rho}_w n S_{(w)r} + \rho_w \dot{n} S_{(w)r} + \rho_w n \dot{S}_{(w)r}, \tag{30}$$

with

$$\rho_w = \rho_w^0\left(1 + \frac{p_w - p_w^0}{\chi_w}\right) \quad \text{and} \quad \dot{\rho}_w = \rho_w^0 \frac{\dot{p}_w}{\chi_w},$$

where $p_w^0$ is the initial water pressure and $\chi_w$ is the coefficient of the water compressibility.

The relations for the rates of porosity and void ratio are given by:

$$\dot{n} = \frac{1}{(1+e)^2}\dot{e} \quad \text{and} \quad \dot{e} = (1+e_0)\,\lambda_v\ . \tag{31}$$

We introduce a functional relation between the degree of water saturation and suction, which can be obtained experimentally. Therefore:

$$S_{(w)r} = \Psi_w(s)\ . \tag{32}$$

Relation (32) gives after differentiation:

$$\dot{S}_{(w)r} = \Psi'_{(w)}(s)\,\dot{s}\ . \tag{33}$$

The change of suction is related to the change in pore water pressure as follows:

$$\dot{s} = -\dot{p}_w\ . \tag{34}$$

Based on these equations and after differentiation of equation (14) we obtain a new form for the water storage relations in rates:

$$\dot{f}^*_{(w)} = F_{(w)}\,\dot{s} + \Phi_{(w)}\,\lambda_v\ , \tag{35}$$

where

$$F_{(w)} = \rho_w n \left( \Psi'_{(w)}(s) - \rho_w^0 \frac{\Psi_{(w)}(s)}{\rho_w \chi_w} \right) , \tag{36}$$

$$\Phi_{(w)} = \rho_w \frac{(1+e_0)}{(1+e)^2} \Psi_{(w)}(s) .$$

## 4. Conditions for inelastic strain localization in soil massifs

### 4.1. Statement of the problem

Experimentally in soil mechanics it is observed that under special stress and strain state conditions localization bands initiate and grow in the soil massif ([2], [3], [9], [14]). Inside the localization bands it is established that the shear and the volumetric inelastic strains reach values which are much higher compared to values in the neighborhood. With the localization band development shear cracks appear and lead to different fracture events including landslide. That is why the problem discussed is of a major importance from both theoretical and practical points of view. In the literature special attention has been devoted to the conditions of localization band initiation. Nowadays the localization is interpreted as an internal loss of structural stability [18]. Such an approach leads to criteria like Rice necessary conditions for localization band initiation, [16], where the localization is considered as bifurcation of the process measures rates. In reality there are some structural changes which appear when strain localization takes place. Based on this various sufficient conditions have been proposed, see for example [18]. In our case it is assumed that the sufficient condition is satisfied when the stored internal dissipation energy reaches the appropriate critical value. The theory presented hereafter concerns mainly unsaturated soils with water and gas pore content and the constitutive equations from Section 3. are used.

### 4.2. Necessary conditions for strain localization in nonsaturated soil massif

We assume that before initiation of strain localization the basic deformation process is quasistatic and isothermal and the strains are finite. We will apply Rice conception for necessary conditions based on bifurcation criterion for the rates of the process measures inside a narrow band in an unsaturated soil massif. We do not examine the case when the bifurcation appears if the plastic state is on the "corner point" of the yield surface. The process measures rates form the manifold $\left\{ V_i, \lambda_{ij}, \tilde{\sigma}_{ij}, \dot{s}, \dot{\rho}, \dot{f}_w^* \right\}$ and for them the following system of equations is given:

$$
\begin{aligned}
(37)\qquad & \tilde{\sigma}_{ij} = C^{ep}_{(K)ijkl}\lambda_{kl} - V_{(K)ij}\,\dot{s}\,, \quad (K = I, II, \ldots N)\\
& \dot{f}^*_{(w)} = F_{(w)}\,\dot{s} + \Phi_{(w)}\,\lambda_v\,, \qquad \lambda_v = \lambda_{ii}\\
& \dot{\sigma}_{ij,j} + \dot{\rho} b_i = 0\,,\\
& \dot{f}^*_{(w)} - f^*_{(w),i}\,V_{(w)i} + \left(\rho_w V_{(w)i}\right)_{,i} = 0\,,\\
& V_{(w)i} = -\frac{K(s)}{\mu_{(w)}}\left(p_{w,i} + g\rho_w y_{,i}\right)\,,\\
& \dot{\rho} = \left(\rho_w\,\Psi'_{(w)} - \frac{\rho^0_w}{\chi_w}\,\Psi_{(w)}\right) n\,\dot{s} + \left(\rho_w\,\Psi_{(w)} - \rho\right)\frac{1+e_0}{1+e}\,\lambda_v\,.
\end{aligned}
$$

The thickness of the localization band is noted with $2d$. It is small and it can vary. We assume the following (see [9], [19]):

(1) The band is smooth, moves slowly in the massif during the deformation process;

(2) The middle surface of the band does not elongate during the process;

(3) The normal to the middle surface remains normal to it during the changes of the band shape, but the normal longitudes changes.

We introduce orthogonal curvilinear coordinates along the principle lines of the middle surface $(\xi, \eta)$ and a linear coordinate $\zeta$ along the normal to the surface at the point $(\xi, \eta)$, i.e. $\zeta \subset [-d, d]$, $d = d(\xi, \eta, t)$. In the band region at the time $t_0$ the measures rates bifurcate according to the following definition:

At the moment when the process loses stability there exist at least two admissible solutions, $\Phi^{(1)}(\xi, \eta, \zeta, t_0 + \Delta t)$ and $\Phi^{(2)}(\xi, \eta, \zeta, t_0 + \Delta t)$, of the system of equations (37), such that, [19]:

$$
(38)\quad \dot{\Phi}^{(i)}(\xi, \eta, \zeta, t_0) = \lim_{\triangle t \to 0} \frac{\Phi^{(i)}(\xi, \eta, \zeta, t_0 + \Delta t) - \Phi^{(i)}(\xi, \eta, \zeta, t_0)}{\Delta t}, \ (i = 1, 2)
$$

and

$$
(39)\qquad \Delta\dot{\Phi} = \dot{\Phi}^{(1)} - \dot{\Phi}^{(2)}\,.
$$

At the point $(\xi, \eta)$ of the midle surface the unit normal is $n_\alpha, (n_\alpha n_\alpha = 1)$. Therefore it has the components: $n_\xi = 0, n_\eta = 0$ and $n_\zeta = 1$. Inside the band $\Delta\dot{\Phi} \neq 0$. When rewrite the equation $(37)_5$ in the coordinate system $(\xi, \eta, \zeta)$ the result is:

$$
(40)\qquad V_{(w)\alpha} = -\frac{K(s)}{\mu_{(w)}}\left(p_{w,\alpha} + g\rho_w y_{,\alpha}\right)\,.
$$

From (38), (39) and (40) it follows:

$$\Delta V_{(w)\alpha} = 0, \qquad (\alpha = \xi, \eta, \zeta) \ . \tag{41}$$

In local coordinate system $(37)_4$ becomes:

$$\dot{f}^*_{(w)} - f^*_{(w),\alpha} V_{(w)\alpha} + \left(\rho_w V_{(w)\alpha}\right)_{|\alpha} = 0 \ , \tag{42}$$

where the vertical bar denotes covariant differentiation. From (42) and (41) the conclusion is that:

$$\Delta \dot{f}^*_{(w)} = 0 \ . \tag{43}$$

Taking into account (41) and (43) the following bifurcations remain possible: $\Delta V_\alpha \neq 0$, $\Delta\lambda_{\alpha\beta} \neq 0$, $\Delta\tilde{\sigma}_{\alpha\beta} \neq 0$, $\Delta\dot{s} \neq 0$, $\Delta\dot{\rho} \neq 0$. Following the procedure given in [19] we introduce the bifurcation function $g_\alpha(\zeta)$ such that $g_\alpha(-d) = g_\alpha(d) = 0$ and the following relations take place:

$$\Delta V_{\alpha|\beta} = g_\alpha(\zeta)\; n_\beta(\xi, \eta) \ . \tag{44}$$

Then we have:

$$\Delta\lambda_{\alpha\beta} = \frac{1}{2}\left(\Delta V_{\alpha|\beta} + \Delta V_{\beta|\alpha}\right) = \frac{1}{2}\left(g_\alpha n_\beta + g_\beta n_\alpha\right) \ ,$$

$$\Delta\Omega_{\alpha\beta} = \frac{1}{2}\left(\Delta V_{\alpha|\beta} - \Delta V_{\beta|\alpha}\right) = \frac{1}{2}\left(g_\alpha n_\beta - g_\beta n_\alpha\right) \ . \tag{45}$$

Taking into account that only $n_\zeta \neq 0$ we get the following results concerning the strain rate components:

$$\begin{aligned} &\Delta\lambda_{\xi\xi} = 0\,, \qquad \Delta\lambda_{\eta\eta} = 0\,, \qquad \Delta\lambda_{\xi\eta} = 0 \ , \\ &\Delta\lambda_{\xi\zeta} = \frac{1}{2} g_\xi(\zeta) \neq 0 \ , \\ &\Delta\lambda_{\eta\zeta} = \frac{1}{2} g_\eta(\zeta) \neq 0 \ , \\ &\Delta\lambda_{\zeta\zeta} = \frac{1}{2} g_\zeta(\zeta) \neq 0 \ . \end{aligned} \tag{46}$$

Equations (37) expressed in bifurations read:

$$\Delta\tilde{\sigma}_{\alpha\beta} = C^{ep}_{(K)\alpha\beta\tau\delta}\Delta\lambda_{\tau\delta} + V_{(K)\alpha\beta}\,\Delta\dot{s} \ , \tag{47}$$

with

$$\Delta\tilde{\sigma}_{\alpha\beta} = \Delta\dot{\sigma}_{\alpha\beta} + \sigma_{\alpha\gamma}\,\Delta\Omega_{\gamma\beta} - \Delta\Omega_{\alpha\gamma}\sigma_{\gamma\beta}\ ,$$

(48)
$$\begin{aligned} &\Delta\dot{\sigma}_{\alpha\beta}n_\beta + b_\alpha\Delta\dot{\rho} = 0\ ,\\ &F_{(w)}\Delta\dot{s} + \Phi_{(w)}\Delta\lambda_v = 0\ . \end{aligned}$$

From (43) and (48) it follows:

(49)
$$\Delta\dot{\rho} = \left(-\rho\,\frac{1+e_0}{1+e} + e\,\Phi_{(w)}\right)\,\Delta\lambda_v\ .$$

Equations (45), (46), (47), (48) and (49), after some mathematical transformations give:

(50)
$$L_{(K)\alpha\beta}g_\beta = 0\ ,$$

with

(51)
$$\begin{aligned} L_{(K)\alpha\beta} &= C^{ep}_{(K)\alpha\gamma\beta\delta}n_\delta n_\gamma - V_{(K)\alpha\gamma}n_\gamma\frac{\Phi_{(w)}}{F_{(w)}}n_\beta\ ,\\ &\quad + b_\alpha\Theta_\rho n_\beta + \frac{1}{2}\sigma_{\alpha\beta} - \frac{1}{3}\sigma_M\delta_{\alpha\beta}\ ,\\ 3\sigma_M &= \sigma_{\alpha\beta}n_\alpha n_\beta\ ,\\ \Theta_\rho &= -\rho\,\frac{1+e_0}{1+e} + e\,\Phi_{(w)}\ . \end{aligned}$$

Equation (50) represents a homogeneons system of algebraic equations with respect to $g_\alpha$. The necessary condition for strain localization initiation is the existence of a non trivial solution of the algebric system of equations (50). It means that:

(52)
$$\det\left|L_{(K)\alpha\beta}\right| = 0,\quad \begin{aligned}&(K = I, II \ldots N)\\ &\alpha\beta = \xi, \eta, \zeta\ ,\end{aligned}$$

where:

(53)
$$L_{(K)\xi\xi} = C^{ep}_{(K)\xi\zeta\xi\zeta} + \frac{1}{2}\sigma_{\xi\xi} - \frac{3}{2}\sigma_M \ ,$$
$$L_{(K)\xi\eta} = C^{ep}_{(K)\xi\eta\zeta\zeta} + \frac{1}{2}\sigma_{\xi\eta} = L_{(K)\eta\xi} \ ,$$
$$L_{(K)\xi\zeta} = C^{ep}_{(K)\xi\zeta\zeta\zeta} - V_{(K)\xi\zeta}\frac{\Phi_{(w)}}{F_{(w)}} + \frac{1}{2}\sigma_{\xi\zeta} = L_{(K)\zeta\xi} \ ,$$
$$L_{(K)\zeta\zeta} = C^{ep}_{(K)\zeta\zeta\zeta\zeta} - V_{(K)\zeta\zeta}\frac{\Phi_{(w)}}{F_{(w)}} + b_\zeta\Theta_\rho + \frac{1}{2}\sigma_{\zeta\zeta} - \frac{3}{2}\sigma_M \ ,$$
$$L_{(K)\eta\zeta} = C^{ep}_{(K)\eta\zeta\zeta\zeta} - V_{(K)\eta\zeta}\frac{\Phi_{(w)}}{F_{(w)}} + \frac{1}{2}\sigma_{\eta\zeta} = L_{(K)\zeta\eta} \ ,$$
$$L_{(K)\eta\eta} = C^{ep}_{(K)\eta\zeta\eta\zeta} + \frac{1}{2}\sigma_{\eta\eta} - \frac{3}{2}\sigma_M \ .$$

**4.3. Sufficient condition for inelastic strain localization initiation in unsaturated soil**

According to the approach presented in [18] we propose here the following sufficient condition for inelastic strain localization band initiation:

The inelastic strain localization band initiation occurs when the stored dissipation energy $W^p(x_k,t)$ reaches the critical value $W^p_c$ in a certain point belonging to the domain $\Omega_t$. The domain $\Omega_t$ is defined as the domain occupied by with the soil at the current moment $t$. The critical value $W^p_c$ is a material constant.

The sufficient condition for inelastic strain localization initiation is expressed as follows:

(54)
$$W^p(x_k,t) = \int_{t_0}^{t} \rho\,\dot\gamma_{int}(x_k,\tau)\,d\tau = W^p_c = \text{const},$$

where $\dot\gamma_{int}$ is the rate of the specific internal dissipation. Approximately the rate of the specific internal dissipation can be given by the following relationship:

(55)
$$\rho\dot\gamma_{int} = k\sigma_{ij}\lambda^p_{ij} \geq 0,$$

where according to the second principle of thermodynamics and the energy transformation parameter $k$ is approximately equal to 0.9. The proposed sufficient condition (50) needs to be verified experimentally.

## 5. Example

In this section we give as an example the application of the above described theory to the hydro-mechanical model proposed in [5].

### 5.1. Constitutive functions

The yield function proposed in [5] consists of three functions - one is describing the "cap" inelastic mechanism also called "pore collapse", the other represents the shear failure and the third is for the failure in tension.

"Cap" inelastic mechanism is characterized by significant inelastic volumetric changes. The modified Cam-Clay yield function is used, defined by:

$$f_{(I)} = II_{s_{ij}}^2 + m^2 \left( I_\sigma^* - \frac{3c}{\operatorname{tg} \phi_C} \right) \left( I_\sigma^* + 3p_0 \right) = 0, \tag{56}$$

where $II_{s_{ij}} = \sqrt{\frac{1}{2} s_{ij} s_{ij}}$ ; $\phi_C$ is the friction angle in compression paths; $p_0$ is the preconsolidation pressure; $c$ is the cohesion, see [5].

The coefficient $m$ depends on the third stress invariant and is given by the following expression:

$$m = a \left(1 + b\,\vartheta\right)^{\iota}, \qquad \vartheta = -\frac{3\sqrt{3}}{2} \frac{III_{s_{ij}}}{II_{s_{ij}}^3}, \tag{57}$$

where $III_{s_{ij}} = \frac{1}{3}\, s_{ij} s_{jk} s_{ki}$. Model parameters $a$, $\iota$ and $b$ are defined depending on friction angles in compression and extension. The formulas are given in [4].

The yield condition in case shear failure deformation mechanism is active is given by the expression:

$$f_{(II)} = II_{s_{ij}} + m \left( I_\sigma^* - \frac{3c}{\operatorname{tg} \phi_C} \right) = 0, \tag{58}$$

The yield function for the case of traction paths where $I_\sigma^* > 0$, is of the form:

$$f_{(III)} = I_\sigma^* - 3\sigma_t = 0, \tag{59}$$

where $\sigma_t$ is the traction yield limit.

The yield functions of the mechanism $I$ and $II$ depend on the suction through $p_0 = p_0\,(s)$ and $c = c\,(s)$. According to the Barcelona model, [12] there is an yield condition for inelastic deformation due to changes in suction and in the extended with suction stress space it reads:

$$f_{(IV)} = s - s_0 = 0, \tag{60}$$

Here $s_0$ is an experimentally defined value for the given soil.

The plastic potential functions corresponding to the above given yield functions have the similar forms, i.e.

$$g_{(I)} = II_{s_{ij}}^2 + m'^2 \left( I_\sigma^* - \frac{3c}{\operatorname{tg}\phi_C} \right) \left( I_\sigma^* + 3p_0 \right) ,$$

(61)
$$g_{(II)} = II_{s_{ij}} + m' \left( I_\sigma^* - \frac{3c}{\operatorname{tg}\phi_C} \right)$$

with

(62)
$$m' = a' \left( 1 + b'\vartheta \right)^{\iota'} \ ,$$

where $a'$, $b'$, $\iota'$ are material parameters. Further it is assumed that the cohesion depends on suction and $\varepsilon_{eq}'^p$. The concrete relationship reads:

(63)
$$c = c\,(0) + c_s\, s \ ,$$

where $c_s$ is a material constant and $c\,(0)$ is the cohesion of the saturated media defined as:

(64)
$$c\,(0) = c_0 + \frac{(c_f - c_0)\, \varepsilon_{eq}'^p}{B_c + \varepsilon_{eq}'^p} \ ,$$

where $c_0$ is the initial and $c_f$ is the final cohesion in the considered process; $B_c$ is a model parameter.

Friction angle in compression paths is assumed to be a function of $\varepsilon_{eq}'^p$, given by:

(65)
$$\phi_C = \phi_{C0} + \frac{\left( \phi_{Cf} - \phi_{C0} \right)\, \varepsilon_{eq}'^p}{B_p + \varepsilon_{eq}'^p} \ ,$$

where $\phi_{C0}$ and $\phi_{Cf}$ are initial and final friction angles and $B_p$ is a material constant.

Now as an example we consider the case when inelastic mechanism $II$ is active. Using the above given functions we can obtain the corresponding equations given in § 3 and § 4, e.g.

(66)
$$\begin{aligned} &\frac{\partial f_{(II)}}{\partial s} = -3\, m\, c_s \operatorname{cotg}\phi_C \ , \\ &\frac{\partial f_{(II)}}{\partial \varepsilon_v'^p} = 0 \ , \\ &\frac{\partial f_{(II)}}{\partial \varepsilon_{eq}'^p} = \frac{\partial f_{(II)}}{\partial c} \frac{dc(0)}{d\varepsilon_{eq}'^p} + \frac{\partial f_{(II)}}{\partial \phi_C} \frac{d\phi_C}{d\varepsilon_{eq}'^p} \end{aligned}$$

with

(67)
$$\frac{\partial f_{(II)}}{\partial c} = -3\, m \,\operatorname{cotg} \phi_C \ ,$$
$$\frac{\partial f_{(II)}}{\partial \phi_C} = \frac{3\, m\, c}{\sin^2 \phi_C} \ ,$$
$$\frac{dc(0)}{d\varepsilon'^p_{eq}} = \frac{(c_f - c_0)\ B_c}{\left(B_c + \varepsilon'^p_{eq}\right)^2} \ ,$$
$$\frac{d\phi_C}{d\varepsilon'^p_{eq}} = \frac{(\phi_{Cf} - \phi_{C0})\ B_p}{\left(B_p + \varepsilon'^p_{eq}\right)^2} \ .$$

(68)
$$\frac{\partial f_{(II)}}{\partial \sigma_{ij}} = \frac{\partial f_{(II)}}{\partial I^*_\sigma}\, \delta_{ij} + \frac{\partial f_{(II)}}{\partial II_{s_{ij}}}\, \frac{s_{ij}}{2\, II_{s_{ij}}} + \frac{\partial f_{(II)}}{\partial \vartheta}\, \frac{\partial \vartheta}{\partial \sigma_{ij}} \ ,$$
$$\frac{\partial g_{(II)}}{\partial \sigma_{ij}} = \frac{\partial g_{(II)}}{\partial I^*_\sigma}\, \delta_{ij} + \frac{\partial g_{(II)}}{\partial II_{s_{ij}}}\, \frac{s_{ij}}{2\, II_{s_{ij}}} + \frac{\partial g_{(II)}}{\partial \vartheta}\, \frac{\partial \vartheta}{\partial \sigma_{ij}} \ ,$$

where

(69)
$$\frac{\partial \vartheta}{\partial \sigma_{ij}} = -\frac{3\sqrt{3}}{2\, II^2_{s_{ij}}} \left( s_{ik}\, s_{kj} - \frac{2}{3}\, II^2_{s_{ij}}\, \delta_{ij} - \frac{3}{2}\, \frac{III_{s_{ij}}}{II^2_{s_{ij}}}\, s_{ij} \right) \ ,$$
$$\frac{\partial f_{(II)}}{\partial I^*_\sigma} = m \ , \quad \frac{\partial g_{(II)}}{\partial I^*_\sigma} = m' \ , \quad \frac{\partial f_{(II)}}{\partial II_{s_{ij}}} = \frac{\partial g_{(II)}}{\partial II_{s_{ij}}} = 1 \ ,$$
$$\frac{\partial f_{(II)}}{\partial \vartheta} = a\, b\, \iota\, (1 + b\, \vartheta)^{\iota - 1} \left( I^*_\sigma - \frac{3\, c}{\operatorname{tg} \phi_C} \right) \ ,$$
$$\frac{\partial g_{(II)}}{\partial \vartheta} = a'\, b'\, \iota'\, (1 + b\, \vartheta)^{\iota' - 1} \left( I^*_\sigma - \frac{3\, c}{\operatorname{tg} \phi_C} \right) \ .$$

To complete the definition of all needed functions in the matrix of coefficients in the system (50) we have to give the relationship between suction and degree of water saturation, namely function $\Psi(s)$. The soil-water retention function proposed in [5] reads:

(70)
$$\Psi(s) = \frac{c_3}{\pi}\ \arctan\left( -\frac{s + c_2}{c_1} \right) + \frac{c_3}{2} \ .$$

From (36) and (70) we derive:

(71)
$$\Phi_{(w)} = \rho_w \frac{(1 + e_0)}{(1 + e)^2} \left[ \frac{c_3}{\pi}\ \arctan\left( -\frac{s + c_2}{c_1} \right) + \frac{c_3}{2} \right]$$

and

$$F_{(w)} = \rho_w n \left( -\frac{c_2\, c_3}{\pi\left[c_2^2 + (s+c_1)^2\right]} - \rho_w^0 \frac{\frac{c_3}{\pi}\arctan\left(-\frac{s+c_2}{c_1}\right) + \frac{c_3}{2}}{\rho_w \chi_w} \right). \tag{72}$$

Using equations (66)–(72) the concrete expression for the matrix $L_{(II)\alpha\beta}$ can be readily obtained just by substitution. With the same procedure $L_{(I)\alpha\beta}$ and $L_{(III)\alpha\beta}$ need to be derived in order to use the criterion (52).

## 6. Discussions and conclusions

There are still many problems to be investigated and solved for better understanding the localization band initiation and its influence on the landslide events in real soil massifs.

For numerical procedure where the existence of localization bands has to be taken into account it is advantageous to apply the Charlier's approach given in [1] and [3], where the bands are considered as jump lines (surfaces) and special surface finite elements are used. Whenever more close to the reality description of the catastrophic landslide events is needed it is important to establish and to apply necessary and sufficient conditions for localization band initiation. In present study such conditions are delivered based on the approaches given in [9] and [16].

In a future it will be interesting to consider and to investigate the following topics:

(1) the changes of temperature in the soil massif due to the temperature increase in the localization band because of the internal inelastic energy dissipation in the band [18];

(2) the influence of the strain rate on the soil behaviour and on the elasto-plastic band initiation [17].

The results obtained in the present study encourage such further works.

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