Hydrodynamics of River Networks Computed with Two-dimensional Unsteady Flow Equations : Comparison of Numerical Methods

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ABSTRACT

The numerical resolution of fluid dynamic equations expressed in a two-dimensional form may be a useful tool for the dynamic management of hydraulic resources. The aim of this article is the comparison of two computational programs based on different numerical methods. The first one uses a fixed grid method of characteristics. The second one resorts to a Petrov-Galerkin finite element method with special test functions and prediction-correction. The results of codes are compared in examples of computational river hydraulics with a sudden variation of limit conditions and in a network of rivers.

INTRODUCTION

The traditional use of a river as waterway is changing with the technical and social evolution of our industrial societies. Other aims as electric energy production, nuclear power cooling, drinking water management, aquatic sports, have to be considered. In this context, river networks management becomes more and more complicated to come up to the users' expectations.

The introduction of computational fluid dynamics allows to develop mathematical tools which give the manager a clear view of the hydraulic processes involved in the system. Simulations of transient flows allow to predict the results of very disturbed situations with sharp floods or very low water levels, for the working out of an optimal management. The aim of this article is to present the accuracy and the efficiency of two softwares, used as a tool for water resources management. The results of a sudden variation of limit conditions are compared for a single branch of river, with considerations on energy dissipation and parasitic waves with high frequencies. An example of river network with variable limit conditions is also described.

THEORETICAL MODEL

The following two-dimensional equations are used to compute flows in rivers :

$$\frac{\partial \Omega}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial(\varphi U Q)}{\partial x} + g\,\Omega \frac{\partial Z}{\partial x} + gF = 0 \qquad (2)$$

We use the following notations :

x, t	the time and space variables
$\Omega(x,h)$	the wet cross-section
Q(x,t)	the flow
h(x,t)	the height of water
U(x,t)	the average speed in the section (axial speed)
g	the gravity acceleration
Z(x,t)	the elevation of free surface
F	the friction term

Moreover, φ is a parameter of unequal distribution of the axial speed in the section. It is defined by the ratio between the average value over the section of the square of the speed and the square of the mean velocity. In practice, this term is evaluated considering a given distribution of the speed in the section.

This system of equations is derived from the Navier-Stokes equations and the continuity equation. The assumptions that are used were chosen in respect of the phenomena to reproduce. The three main ones are as follows :

- the squares of the ratios between the speeds in the directions perpendicular to the main flow and the axial speed are insignificant. This implies an hydrostatic diagram of pressure, changes the system in a two-dimensional one and allows the integration of the equations in the section,
- the bottom slope is small,
- the no-slip condition is considered along the walls.

NUMERICAL RESOLUTION

Introduction

The numerical resolution of fluid dynamic equations gives results for a wide range of hydrodynamic fields. In another study (Pochet [1]), we developed a model of waves propagation in very distensible tubes applied to a network of arteries. The method of characteristics was used to solve the equations. In the same way, we set up a model based on the finite element method for the simulation of discontinuous flows and in particular for a dam-break flood wave propagation. In the first study, we demonstrated the existence of an analogy between the equations that were used and equations (1) and (2).

In both cases, the linearization of the equations leads to useful theoretical considerations on the stability and convergence of the methods but induces undesirable effects on the simulation results. So, it seems necessary to keep the most common equations in a full non-linear form.

Resolution by the method of characteristics

The method of characteristics is founded on the description of waves propagation. It consists of transforming the system of equations so that the partial derivates disappear thanks to the total derivates, so that you can easily integrate. However, this can only be done if you restrain the domain of validity of the equations, that will only be applied along certain curves of the abscissa-time plane. They are called caracteristics.

Let us define the following notation :

$$\lambda^* = c \sqrt{1 - \frac{U^2}{c^2} (1 - \varphi) . \varphi} \qquad (3)$$

$$\text{with} \quad c^2 = \frac{g\Omega}{\frac{\partial \Omega}{\partial h}} \qquad (4)$$

and the factor λ as follows : $\lambda = + \lambda^*$ or $\lambda = - \lambda^*$ (5)

After the usual manipulations, the system of equations (1) and (2) can be transformed into an equivalent system of two pairs of equations, each one defined by one value of λ and composed of the following equation :

$$\frac{g\lambda}{c^2} \left(1 + \frac{U}{\lambda} (1 - \varphi) \right) \frac{dh}{dt} + \frac{dU}{dt} + g \frac{dh'}{dx} + U^2 \frac{\partial \varphi}{\partial x} + \frac{gF}{\Omega} + \lambda \frac{U}{\Omega} \frac{\partial \Omega}{\partial x} = 0 \qquad (6)$$

and of the equation of the characteristics family along which it is valid :

$$\frac{dx}{dt} = U\varphi + \lambda \qquad (7)$$

In equation (6), h' represents the level of the bottom of the river.

Compared to c, λ^* clearly appears to be the generalization of the waves celerity when we take into account the unequal speed distribution. Let us note that λ^* is the waves celerity, not in relation to the speed average, but in relation to its product with φ. On the other hand, the factor φ is algebraically always greater than one, which ensures the existence of a real celerity defined by equation (3) whatever U and c are.

The unsteadiness of the studied phenomena implies a high spatial and temporal variability of the celerities and the speeds, and consequently of the slopes of the characteristic curves. So, the resolution of the equations cannot be based on a simple process. The chosen method is the fixed grid pattern method, which is the most qualified for the treatment of junctions and the control of numerical accuracy. The complete iterative procedure is detailed in Pochet [1].

Resolution by finite element method

The finite element method ensures the spatial discretization of the unknowns Ω, Q. Within each element, every variable is represented as a polynominal function of its values at the nodes of the element. The unknown vector is then conventionally written:

$$X^* = \begin{bmatrix} \Omega^* \\ Q^* \end{bmatrix} = \begin{bmatrix} N_1 & N_j & N_n & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_j & N_n \end{bmatrix} \cdot \begin{bmatrix} \Omega^*_1 \\ \Omega^*_j \\ \Omega^*_n \\ Q^*_1 \\ Q^*_j \\ Q^*_n \end{bmatrix} = N \begin{bmatrix} \Omega^*_1 \\ \Omega^*_j \\ \Omega^*_n \\ Q^*_1 \\ Q^*_j \\ Q^*_n \end{bmatrix} \tag{8}$$

with

*	the indication of approximate values
n	the number of discretization points
Ω_j, Q_j	the nodal values of the unknown quantities
N_j	the shape functions

The Euler equations (1) and (2) can be expressed in the following form :

$$\frac{\partial X}{\partial t} + A \frac{\partial X}{\partial x} + B = 0 \qquad \text{in the domain D} \tag{9}$$

$$\text{where} \qquad A = \begin{bmatrix} 0 & 1 \\ c^2 - \varphi U^2 & 2\varphi U \end{bmatrix} \tag{10}$$

The application of the Petrov-Galerkin technique to the basic equations, along with the use of the approximation (8), leads to the following discretization of the problem :

$$\int_D P^T \left(\frac{\partial X^*}{\partial t} + A \frac{\partial X^*}{\partial x} + B \right) dD = 0 \tag{11}$$

The usual finite element formulation consists of orthogonalizing the error due to approximation of discretization to a weighting function P equal to N. Nevertheless, due to the wide range of applications (management simulations, dam-break flood wave propagation...), the weighting function P is written here in accordance with the works of Katopodes [2].

$$P = N + \varepsilon A^T \frac{\partial N}{\partial x} \tag{12}$$

with ε the parameter setting the degree of dissipation.

Applied to a regular discretization for a simplified case, this method shows that differences centered on the calculation nodes still appear while introducing dissipation terms. Moreover, these applications show that finite difference schemes known for their ability to modelize discontinuous phenomena are reproduced.

The temporal discretization of the equations is obtained by implicit finite differences on two time steps. The terms are evaluated by weighted average on both steps, that is

$$a = (1 - \theta)\, a^{t} + \theta a^{t + \Delta t} \qquad (13)$$

while the temporal derivative is obtained by difference on two time steps. When that temporal discretization is applied to linearized equations on a net of regular finite elements, a decomposition of the solution in Fourier series shows an unconditional stability of the scheme for $\theta \geq 0.5$ with a stabilizing effect by friction.

For a fast resolution of this system of 2n equations with 2n unknows, the method of resolution proceeds with uncoupling the equations and using a method of prediction - correction. During one iteration, the first system of discretized continuity equations evaluates a new approximation of the nodal sections. The unknown flows are replaced either by the value at the previous iteration, either by a predicted value when it is the first step. The second system composed with discretized momentum equations gives the new value of the nodal flows. We evaluate the discretized sections and non-linear terms in the same way.

Before starting simulations, it seems necessary to point out the model parameters which introduce a dissipative effect (Pirotton [3]).
The first parameter θ comes from temporal discretization. Decentring towards the calculated step ($\theta > 0.5$) appears to make the system much more dissipative. For $\theta = 0.5$, on the other hand, the system is perfectly conservative.
The second parameter ε comes from the particular weighting functions introduced during orthogonalization. Raymond and Garder [4] have studied its very selective action on wavelengths.

NUMERICAL SIMULATION AND DISCUSSION

Sudden variation of upstream condition in a single channel

We consider a 100 m long canal, with a rectangular section and whose width is 2.8 m. The canal has a slope of 10 cm over the 100 m. At the beginning of the simulation, it has a uniform flow, the water level being 0.767 m and the flow 3.4 m^3/s. Suddenly (in one time interval), the upstream level drops by 20 cm and is maintained. The downstream level stays at 0.767 m. We let the system evoluate freely and analyse the new equilibrium towards which it will aim and the way it will follow. The law of losses by friction used in this example is linear. The space interval is 10 m long and the time interval is of 3.3 s.

Figure 1 allows the comparison 30 m downstream of the upstream extremity, of the temporal evolution of the water height.

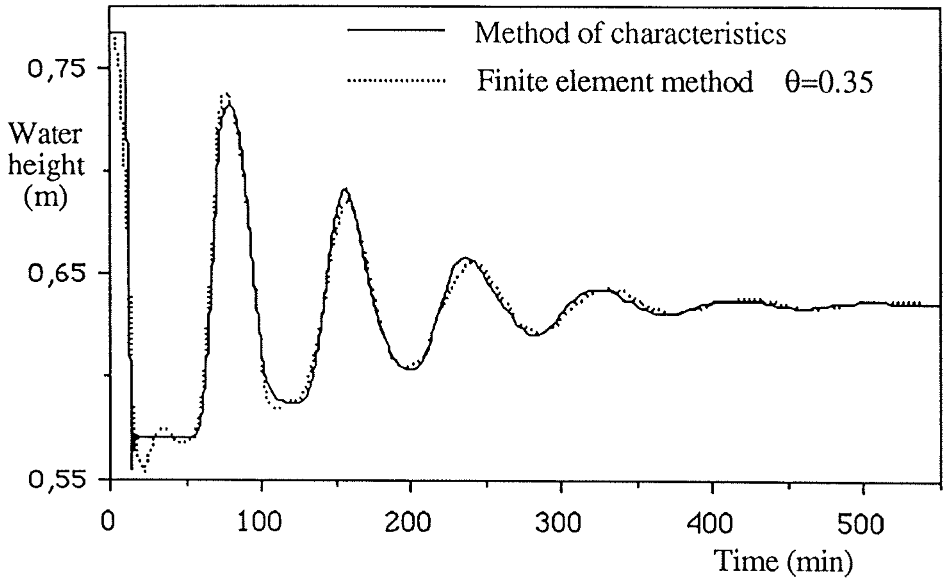

Figure 1. Temporal evolution of the water height 30 m downstream of the upstream extremity

The results of the two methods are close to each other. The amplitudes and the temporal positions of the waves are adequately coherent.

In Pochet [1], we demonstrated that the method of characteristics has a trend to a behaviour energetically non neutral, without any doubt the most often a trend to the dissipation. The pratical rule that is applied is to choose correctly the relation between time and space intervals by approaching the current numbers (calculated without the velocity) to the unit, rather than to refine the spatial and temporal meshes to the utmost. However, the convergence and the stability impose limits to the usable current numbers. Practically, we are not able to reach the unit. In this example, the current numbers oscillate between 0.78 and 0.91. However, when calculated with the influence of velocity, they temporarily exceed 1.4. The results should still have been slightly improved by using the same space interval and a time interval of 3.45 s. Unfortunately, the comparison was not possible because such an interval did not allow the convergence of the finite element method any more.

An attempt by the finite element method to obtain non dissipation ($\theta=0.5$, $\varepsilon=0$) led to a solution with numerical noise. The temporal decentering strongly improves the simulations and the parasitic oscillations that disrupt simulations are here completely wiped out by the introduction of the particular weighting functions. Besides, the value of the current number does not influence the accuracy of the finite element model.

<u>Example of a network of rivers</u>
The following simulation is inspired from the confluence of the Meuse and the Ourthe rivers, the Meuse derivation and the King Albert canal in Liège, Belgium. This network lies between the dams of Ivoz-Ramet, Chênée and Monsin and the locks of Monsin. Figure 2 presents the studied geometry.

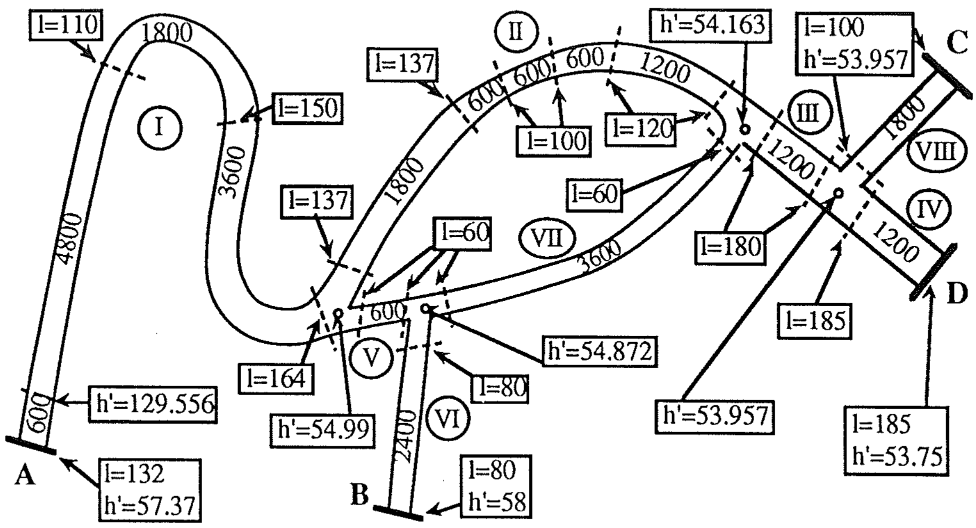

Figure 2. Geometry of the problem. - The Roman numerals designate the branches. Their length is written in the branches (in meters), l designates the width of the sections (in meters) and h' the level of the bottom (in meters).

The sections are rectangular, the width and the level of the bottom vary linearily between given values. The term of friction is here expressed by the well- known Manning law (Pirotton [3]) with a Manning coefficient of 30 (International units). The space interval is 600 m long, except in the branch VI for the method of characteristics, where it is reduced to approach the current numbers to the unit (the time interval used in this method is of 75 s).

At the beginning, there is no flow and the water level is fixed everywhere at 60 meters. During the first 20 minutes, a flow is linearily introduced till the value of 200 m^3/s at the extremity A and 40 m^3/s at B. Then, they are maintained at these values. At the other limits of the network, the following conditions are imposed. In D, the dam is supposed adequately controlled so that the water level is fixed at 60 m and in C, the presence of locks suggests a no-flow condition. After 200 minutes of simulation, an important wave is introduced in B and we impose a linear variation of the flow from 40 m^3/s to 140 m^3/s in 10 minutes. Then, the flow is maintained to 140 m^3/s. The evolution of the waves is studied. In particular, two inversions of flow are observed in the branch V. This is illustrated in figure 3.

CONCLUSIONS

The two applications that we have developed in this article highlight the good quality of the results that are simulated by both methods. The good behaviour of the codes is sustained by a clear understanding of the interpolations and limitations of the numerical resolutions.

The method of characteristics can have a trend to a behaviour energetically non neutral. The practical rule to avoid the dissipation is to correctly choose the relation between time and space intervals by approaching the current numbers to the unit, rather than to refine the spatial and temporal meshes to the upmost.

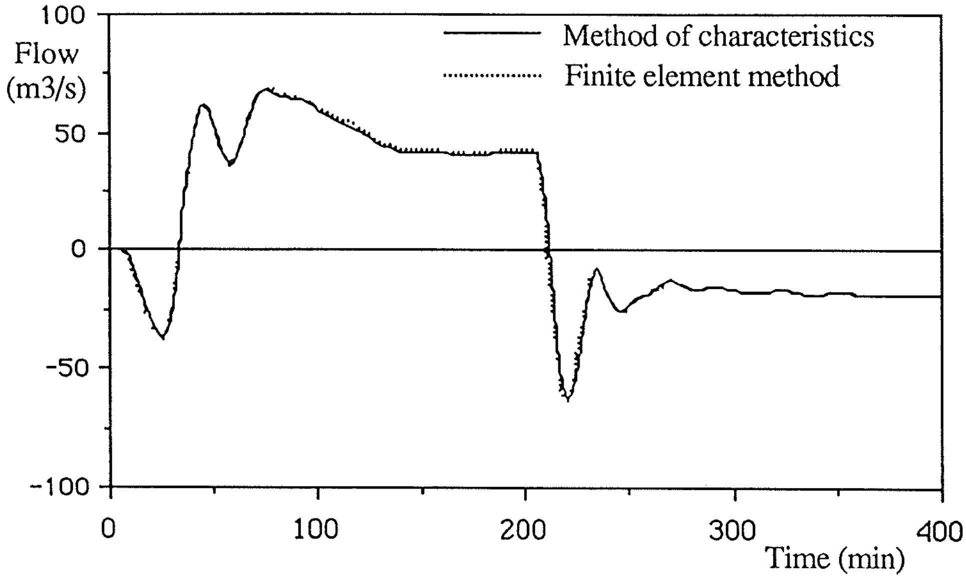

Figure 3. Temporal evolution of the flow at the downstream end of the arm V.

The finite element technique is not influenced by current numbers but its non-dissipative character has restricted for a long time the field of application of the method. The well-known temporal decentering, widely used in the literature, induces a controlled dissipation necessary to make the solution free of most oscillations. In the case of discontinuous flows and sudden variations of limit conditions, the classic formulation where the weighting function is equal to the interpolation function can be advantageously altered. The presented method shows a very selective damping of the high frequency waves.

The general character of the mathematical formulation allows the application of the codes without any restriction to any geometry and network. Each code appears as a powerful software package with a wide range of applications, able to give useful informations for an optimal management of hydraulic resources, in a technological way.

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