

One-Dimensional Unsteady Flow Simulations in Nets of Variable Cross-Section Arms : Comparison Between Finite Element Method and Method of Characteristics

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ABSTRACT

The numerical resolution of fluid dynamic equations expressed in a quasi two-dimensional form is a major component for a better understanding of most hydrodynamic processes. Two computational programs were developed, handling unsteady flow problems in networks of branches with complex geometries. The first one uses a fixed grid method of characteristics. The second one resorts to a Petrov-Galerkin finite element method with special test functions and prediction-correction.

The results of both codes are compared in examples of tidal waves and computational river hydraulics with a sudden variation of limit conditions. The article first compares the recurring feature of the unsteady phenomena. The influence of different numerical parameters on the numerical dissipation is studied, as well as the presence of parasitic waves with high frequencies. Advantages and shortcomings of both methods and their ideal context of application are deduced from the study, as also the cautions aimed to ensure the best numerical efficiency.

INTRODUCTION

The efficient resolution of fluid dynamic equations is commonly admitted as a fundamental concern in the modern hydraulic engineering. An efficient software based on Saint-Venant equations should be able to modelize most of the physical processes involved in the flow field.

The constant renewal of interest for several years in searching new numerical schemes can be explained by the merit of each of them only in particular situations. The setting up of objective criteria of comparison seems unrealistic, seeing that each code finds its suitable range of application. The numerical resolution of this set of differential equations expressed in a section averaged form gives useful indications in such various topics as hydraulic resources management, dam-break flood wave propagation or blood circulation :

- In river networks, simulations of transient flows allow to predict the results of very disturbed situations with sharp floods or very low water levels, for the working out of an optimal multi-objective management.
- In the strategy of protection of population, the disastrous consequences produced by a sudden dam-break lead the searchers to take interest into the difficult modelization of discontinuous flows in networks of rivers.
- In haemodynamics, simulations of generation, propagation and distortion of blood waves in highly complex networks of distensible arteries allow the understanding of fluid dynamics phenomena involved in normal and pathological situations.

Two softwares were developed, covering together all these three subjects. The first one uses a fixed grid method of characteristics. The second one resorts to a Petrov-Galerkin finite element method with special test functions and prediction-correction.

The general improvement in computational techniques and the advances in computer technology often leads the authors to submit results of applications whose complex character hides the intrinsic behaviour of their numerical method. The aim of this article is to highlight the effects of numerical parameters on both softwares, applied in revealing simulations. The comparison of the temporal evolution of tidal waves or of the unsteady consequences of a sudden variation of limit conditions in a single branch of river points out the practical rules to follow for a fit solution, free of any spurious numerical effect.

THEORETICAL MODEL

The equations used in this study to compute flows in open channels are derived from the Navier-Stokes equations and the continuity equation. The assumptions that are used were chosen in respect of the phenomena to reproduce. The two main ones are as follows :

- the squares of the ratios between the speeds in the directions perpendicular to the main flow and the axial speed are insignificant. This implies a hydrostatic diagram of pressure, changes the system in a quasi two-dimensional one and allows the integration of the equations in the section,
- the no-slip condition is considered along the walls.

They lead to the following system of fully non-linear one-dimensional equations (in reality, quasi two-dimensional) :

$$\frac{\partial \Omega}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad (1)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial(\varphi U Q)}{\partial x} + g \Omega \frac{\partial Z}{\partial x} + gF = 0 \quad (2)$$

We use the following notations :

x, t	the time and space variables
$h(x,t)$	the height of water
$\Omega(x,h)$	the wet cross-section, defined as a function of the space and the height of water
$Q(x,t)$	the flow
$U(x,t)$	the average speed in the section (axial speed)
g	the gravity acceleration
$Z(x,t)$	the elevation of free surface
F	the friction term

Moreover, the parameter ϕ characterizes the unequal distribution of the axial speed in the section. It is defined by the ratio between the average value over the section of the square of the speed and the square of the mean velocity. In practice, this term is evaluated considering a given distribution of the speed in the section.

NUMERICAL RESOLUTION

Introduction

As suggested above, we developed in another study (Pochet [1]) a model of waves propagation in very distensible tubes applied to a network of arteries. The method of characteristics was used to solve the equations. In the same way, we set up a model based on the finite element method for the simulation of discontinuous flows and in particular for a dam-break flood wave propagation. In the first study, we demonstrated the existence of an analogy between the equations that were used and equations (1) and (2).

Resolution by the method of characteristics

The method of characteristics is founded on the description of waves propagation. To facilitate integrations, it consists of transforming the system of equations so that the partial derivatives disappear thanks to the total derivatives. However, this can only be done by restricting the domain of validity of the equations.

Let us define the following notation :

$$\lambda^* = c \sqrt{1 - \frac{U^2}{c^2} (1 - \phi) \cdot \phi} \quad (3)$$

$$\text{with } c^2 = \frac{g\Omega}{\frac{\partial \Omega}{\partial h}} \quad (4)$$

$$\text{and the factor } \lambda \text{ as follows : } \lambda = + \lambda^* \text{ or } \lambda = - \lambda^* \quad (5)$$

After usual manipulations, the system of equations (1) and (2) can be transformed into an equivalent system of two pairs of equations, each one

defined by one value of λ and composed of the following equation :

$$\frac{g\lambda}{c^2} \left(1 + \frac{U}{\lambda} (1 - \varphi) \right) \frac{dh}{dt} + \frac{dU}{dt} + g \frac{dh'}{dx} + U^2 \frac{\partial \varphi}{\partial x} + \frac{gF}{\Omega} + \lambda \frac{U}{\Omega} \frac{\partial \Omega}{\partial x} = 0 \quad (6)$$

and of the equation of the curves family of the abscissa-time plane along which it is valid (these curves are called characteristics) :

$$\frac{dx}{dt} = U\varphi + \lambda \quad (7)$$

In equation (6), h' represents the level of the bottom of the river.

Compared to c , λ^* clearly appears to be the generalization of the waves celerity when we take into account the unequal speed distribution. Let us point out that λ^* is the waves celerity, not in relation to the speed average, but in relation to its product with φ . On the other hand, the factor φ is algebraically always greater than one, which ensures the existence of a real celerity defined by equation (3) whatever U and c are.

The unsteadiness of the studied phenomena implies a high spatial and temporal variability of the celerities and the speeds, and consequently of the slopes of the characteristic curves. So, the resolution of the equations cannot be based on a simple process. The chosen method is the fixed grid pattern method, which is the most qualified for the treatment of junctions and the control of numerical accuracy. Inside of a channel, it relies on the figure 1 diagram.

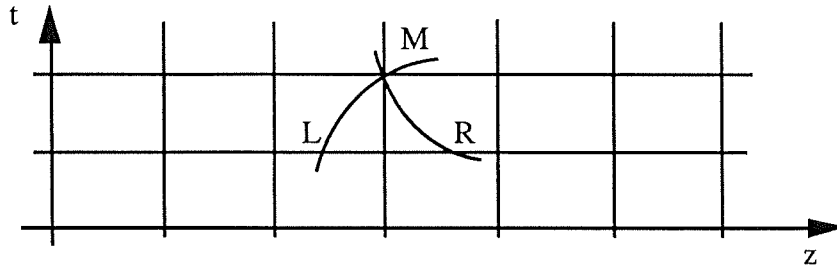


Figure 1. Fundamental diagram of resolution inside a channel with the fixed grid method

When the unknowns have previously been determined at the nodes of the grid corresponding to the preceding time step, their calculation at a point M is carried out as follows. The integration of the characteristics families equations (7) allows to position the piercing points L and R at the preceding time step of the two characteristics that join M. The unknowns at these points are estimated by linear interpolation. Finally, the unknowns at M are deduced from the integration of the equations (6) on the characteristics.

The dependance of the shape of the characteristics on the unknowns at points L, R and M, as also the integration with trapezia of the equations terms that do not correspond to total derivatives, make an iterative calculation procedure

indispensable.

The resolution of the equations at a limit of a channel or at a junction of several channels is either a simplification or a generalization of the procedure described above. The numerical techniques are detailed in Pochet [1].

Resolution by finite element method

In contrast to the method of characteristics wherein the domain of interest is replaced by a set of discrete points, the finite element method divides the domain D into subdomains. In each of these finite elements, every variable is represented as a polynomial function of its values at the nodes of the element. The unknowns are then conventionally written:

$$\Omega^* = \sum_{i=1}^n N_i a_i^* \quad \text{and} \quad Q^* = \sum_{i=1}^n N_i q_i^* \quad (8)$$

with

* the indication of approximate values
n the number of discretization points
 a_i^*, q_i^* the nodal values of the unknown quantities
 N_i the shape functions

Equations (8) can be transformed into a general vectorial expression :

$$X^* = \begin{bmatrix} \Omega^* \\ Q^* \end{bmatrix} = N x^* \quad (9)$$

$$\text{with} \quad x^{*T} = [a_1^* \quad a_j^* \quad a_n^* \quad q_1^* \quad q_j^* \quad q_n^*] \quad (10)$$

The Euler equations (1) and (2) can be expressed in the following form :

$$\frac{\partial X}{\partial t} + A(X) \frac{\partial X}{\partial x} + B(X) = 0 \quad \text{in the domain D} \quad (11)$$

$$\text{where} \quad A(X) = \begin{bmatrix} 0 & 1 \\ c^2 - \varphi U^2 & 2\varphi U \end{bmatrix} \quad (12)$$

The application of the Petrov-Galerkin technique to the basic equations, along with the use of the approximation (9), leads to the following discretization of the problem :

$$\int_D P^T \left(\frac{\partial X^*}{\partial t} + A(X^*) \frac{\partial X^*}{\partial x} + B(X^*) \right) dD = 0 \quad (13)$$

The **usual** finite element formulation consists of an orthogonal projection of the residual error due to discretization approximation to a set of linearly independent complete functions included in a vector **P equal to N**. Nevertheless, due to the wide range of applications (management simulations, dam-break flood wave propagation...), the weighting function **P** is written here in accordance with the works of Katopodes [2] :

$$P = N + \varepsilon A (X^*)^T \frac{\partial N}{\partial x} \quad (14)$$

with ε the parameter setting the degree of dissipation.

The parameter ε , introduced in the particular weighting functions, is used to control undesirable oscillations arising in the simulations with significant convective terms or propagation of sharp fronts (Piroton [3]). Raymond and Garder [4] have studied its very selective action on wavelengths.

Applied to a regular discretization for a simplified case, this method shows that differences centered on the calculation nodes still appear while introducing dissipation terms. Moreover, these applications show that it reproduces and generalizes some finite difference schemes known for their ability to modelize discontinuous phenomena.

The temporal discretization of the equations is obtained by a variable-weighted implicit approximation on two time steps, as follows :

$$r = (1 - \theta) r^t + \theta r^{t + \Delta t} \quad (15)$$

while the temporal derivatives are obtained by difference on two time steps. Written in this form, it is apparent that spatial operator is located at $t + \theta \Delta t$. For $\theta = 0.5$, that discretization reduces to the second order-correct-in-time approximation of Crank-Nicholson.

Applied to linearized equations on a net of regular finite elements with linear basis functions, a decomposition of the solution in Fourier series shows an unconditional stability of the scheme for $\theta \geq 0.5$ with a stabilizing effect by friction.

Moreover, the well-known method of the Lagrange multipliers allows to take into account any junctions of networks.

For a fast resolution of this system of $2n$ equations with $2n$ unknowns, the code proceeds with splitting up the equations and using an iterative method of prediction-correction. During one iteration, the first system of n discretized continuity equations evaluates a new approximation of the nodal sections. The unknown flows are replaced either by the values at the previous iteration, or by predicted values when it is the first time step. The second system composed with n discretized momentum equations gives the new values of the nodal flows. The non-linear terms are estimated either by a linear interpolation from the previous intervals at the first iteration, or, afterwards, by the values at the iteration that precedes.

RESULTS AND DISCUSSION

Both softwares have been developed to study flows in networks (Pirotton and Pochet [5]). Nevertheless, in this paper, they are compared in draconian examples of single branches, especially able to point out their numerical efficiency.

Tidal waves

The first one simulates tidal waves in a horizontal flume, shut at both extremities. At the beginning, the free surface has the shape of an half period of cosinusoid. We leave the system evolve freely in the time without any friction.

If the initial distorted surface has an amplitude sufficiently low to justify the equations linearization, we can theoretically show that the level will then oscillate freely and indefinitely between the initial cosinusoid and the symmetrical curve in relation to the mean horizontal level. A progressive decrease of the calculated amplitude would reveal a simulation process naturally dissipative. On the other hand, if the amplitude increases, the method tends to inject an excess of energy in the system.

The channel is 100 m long and 1 m large. The mean water height is of 1 m and the amplitude of the initial distorted surface of 0.05 m. For the first simulation realized with the method of characteristics, the space interval is 5 m long and the time interval is of 1 s. In this problem, the current number constantly changes in the time and space, but the variations are rather limited. The mean current number (MCN) is 0.63.

Figure 2 shows the temporal evolution of the water level at each end of the channel. It reveals a highly dissipative character.

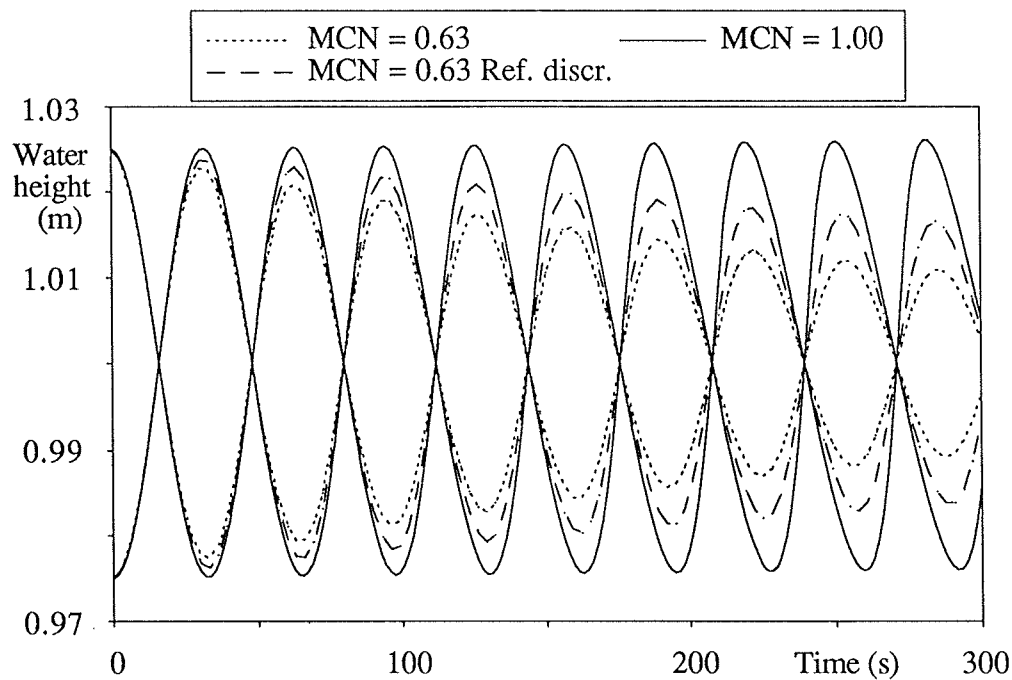


Figure 2. Study of the dissipation introduced by the method of characteristics in the tidal waves problem.
MCN = mean current number Ref. discr. = refined discretizations

The refinement of the spatial and temporal meshes with constant current numbers, for example as shown on figure 2, by dividing by two the time and space intervals, appears to improve the energetical behaviour.

On the other hand, when the current numbers are approached near the unit, for instance by keeping the space interval constant and increasing the time interval, the dissipation is also reduced. In the example presented on figure 2, with a mean current number equal to one, it completely disappears. This second way to improve the energetical behaviour is surely less expensive in calculation time and appears to be more efficient.

In Pochet [1], we demonstrated that the origin of the non-neutral energetical behaviour of the method of characteristics is located in the linear interpolations used to estimate the unknowns at points L and R (figure 1) during the numerical process. If the mean current number exceeds the unit, the trend reverses : the method injects an excess of energy.

However, from a certain moment, as soon as the mean current number is high, the computed values start to vary without any order, and rapidly the program aborts by non-convergence of the iterative procedure in one point. Furthermore, when extended in the time, the simulation with a mean current number equal to one ends by not converging any more. This suggests that in practice, we are most often not able to reach the unit.

Sensitivity to current number is not a feature of the finite element code. Furthermore, the presence of this diffusive character would be worrying in numerical modelling of dam-break wave propagation where time steps are taken small enough in order to accurately simulate the physical reality.

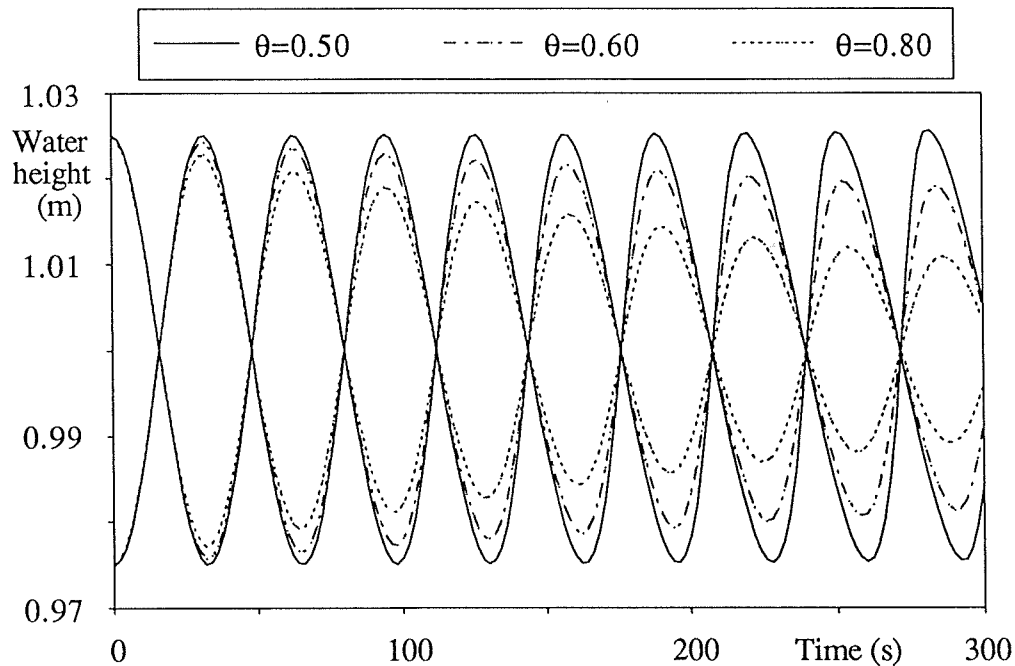


Figure 3. Study of the dissipation introduced by the temporal discretization in the finite element method

However, the success of many numerical schemes, especially in the representation of discontinuous phenomena, lies on a suitable dissipative character which captures the sharp transitions without numerical noise. The numerical dissipation in the classical finite element method of Galerkin can be found in the temporal discretization.

So, care must be taken about the stability of the temporal diagram. It can lead to a significant damping effect on the oscillations as shown in figure 3. Decentring towards the calculated step ($\theta > 0.5$) appears to make the system much more dissipative. For $\theta = 0.5$, on the other hand, the system is perfectly conservative.

However, it seems undesirable in practice to keep the neutral value of $\theta = 0.5$ in this method due to the non-dissipative character of the spatial discretization. Current values situated between $\theta = 0.6$ and $\theta = 0.8$ are usually mentioned in the literature. This will be illustrated in the second example.

The effect of the second dissipative parameter introduced in the special weighting functions cannot be highlighted here because of the lack of a large wavelengths spectrum. For this application, its value is fixed to zero.

The central oscillating curve of figure 4 illustrates the temporal evolution of the water level at the middle point of the canal. According to the theoretical solution of the linearized equations, it should not move in the time. However, it starts to oscillate with the double fundamental frequency of the phenomenon and the amplitude progressively increases. The perturbation propagates when going away from the centre, but decreases gradually, and up to both extremities, the curves are not perfectly symmetrical. These observations are independent of the numerical method.

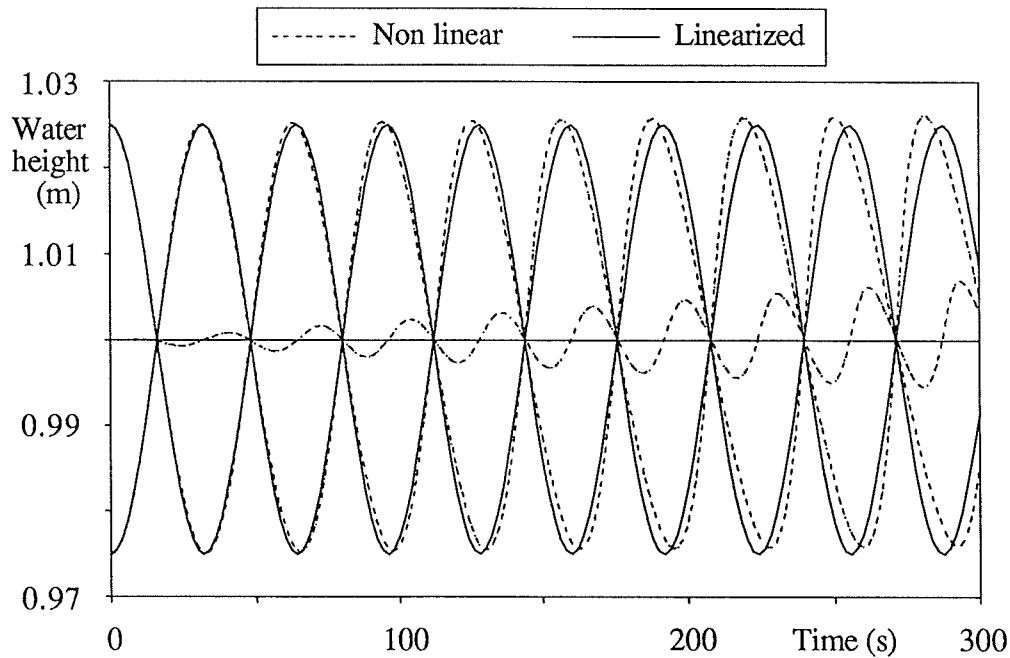


Figure 4. Temporal evolution of the water level at each end and in the middle of the canal with the non-linear and linearized equations.

Yet, when the equations are linearized (in practice, by neglecting the convective term using zero for the parameter of unequal speed distribution, instead of one until now, and by considering the coefficient of the water level term in the dynamic equation (2) independent from the unknowns), the oscillations completely vanish (figure 4).

So, though the amplitude of the initial distorted surface could appear to be small, it induces significant non-linear effects. This justifies the efforts to keep the equations in their complete expression.

Sudden variation of the upstream level in a single channel

The second studied case considers a 100 m long flume, with a rectangular section and whose width is 2.8 m. The channel has a slope of 10 cm over the 100 m. At the beginning of the simulation, it has a uniform flow, the water level being 0.767 m and the flow $3.4 \text{ m}^3/\text{s}$.

Suddenly (in one time interval), the upstream level drops by 20 cm and is maintained. The downstream level stays at 0.767 m. We let the system evolve freely and analyse the new equilibrium it will aim to and the way it will follow.

The law of losses by friction used in this example is linear. Such a choice is not ideal for a canal but does not alter the validity of the analysis. The space interval is 10 m long and the time interval is of 3.45 s.

Figures 5 and 6 allow the comparison, 30 m downstream of the upstream extremity, of the temporal evolution of the water height and the flow.

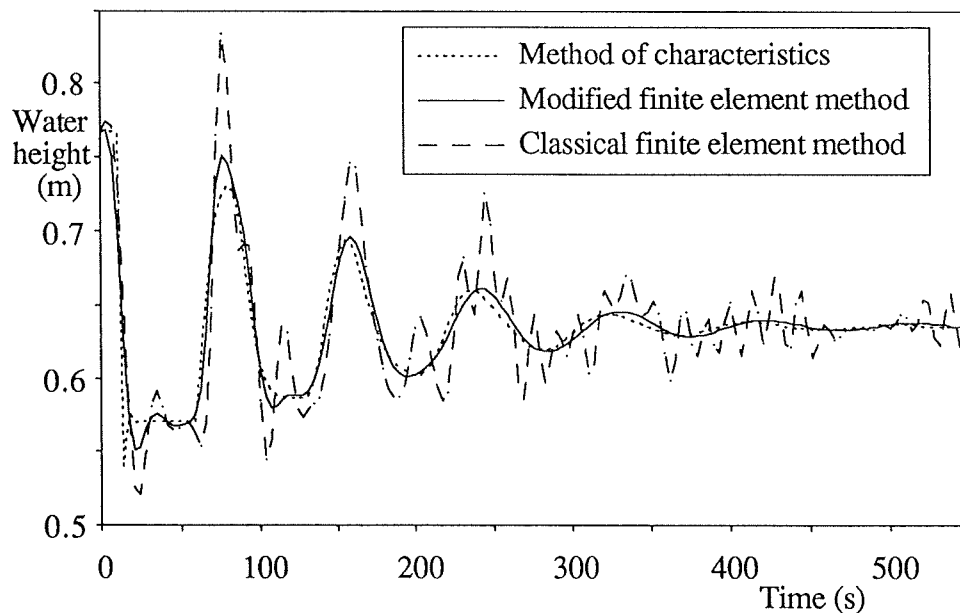


Figure 5. Temporal evolution of the water height
30 m downstream of the upstream extremity

The classical finite element solution based on the Galerkin formulation shows unrealistic results without any physical agreement. The centered Crank-Nicholson temporal discretization coupled with a centered spatial diagram should provide an energetically neutral reference. On the contrary, high frequency waves affecting the flow variables introduce overmeaning oscillations which make any comparison impossible. This spurious behaviour shows the need for a suitable treatment.

A slight temporal decentering ($\theta = 0.60$) strongly improves the simulations and the parasitic oscillations that disrupt simulations are here completely wiped out by the introduction of the upstream-weighted functions.

In this example, the current numbers oscillate between 0.80 and 0.95 to stay as much as possible in the appropriate range of the method of characteristics. However, when calculated with the influence of velocity, they temporarily exceed 1.5.

Figures 5 and 6 illustrate the very similar behaviours of the modified finite element method and the method of characteristics. The amplitudes and the temporal positions of the waves are adequately coherent. These results, compared to the original finite element method ones, demonstrate that the dissipation is maintained at an acceptable level.

Besides, they show the validity of the numerical adjustment introduced in the finite element scheme and point out the stringent numerical stability of recurring applications as tidal waves.

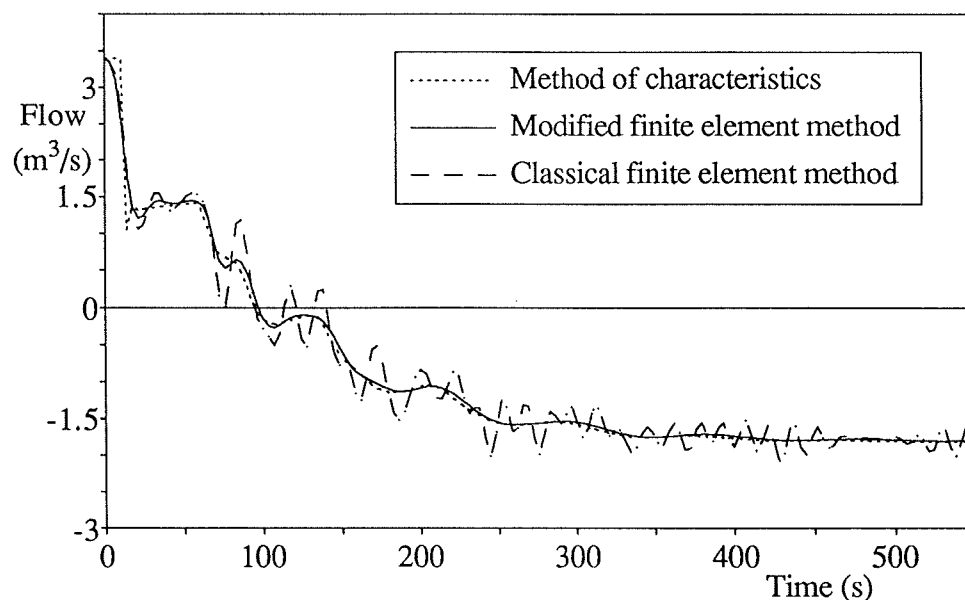


Figure 6. Temporal evolution of the flow
30 m downstream of the upstream extremity

CONCLUSIONS

The two applications we have developed in this article back the hydraulicians' opinion that using a hydraulic software as a "black box" tool is dangerous. The good behaviour of a code must be sustained by a clear understanding of the interpolations and limitations of the numerical resolutions. Gaining a deep knowledge of factors affecting the numerical accuracy and overcoming difficulties to generate a software with good stability features, seems to be the most appropriate solution to produce rational and trustworthy results.

The method of characteristics can have a trend to a behaviour energetically non-neutral. The practical rule to avoid the dissipation is to choose correctly the relation between time and space intervals by approaching the current numbers to the unit, rather than to refine the spatial and temporal meshes to the utmost.

The finite element technique is not influenced by current numbers but its non-dissipative character has restricted for a long time the field of application of the method. The well-known temporal decentering, widely used in the literature, induces a controlled dissipation necessary to make the solution free of most oscillations. In the case of discontinuous flows and sudden variations of limit conditions, the classic formulation where the weighting function is equal to the interpolation function can be advantageously altered. The alternative method shows a very selective damping of the high-frequency waves.

Thus, a discerning choice of simulation parameters leads to results that highlight the good quality of both methods.

The general character of the mathematical formulation allows the use of the codes without any restriction to any geometry and network, making each code a powerful software package within a wide range of applications.

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